Dual-Frame Generalized Harmonic Gauge on Hyperboloidal Slices

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Mathematical and Numerical Aspects of Gravitation
October 2021

- Class.Quant.Grav. 35 (2018) no.5, 055003, with E. Harms, M. Bugner, H. Rüter, B. Brügmann.
- Class.Quant.Grav. 36 (2019) with E. Gasperin.
- Class.Quant.Grav. 37 (2020) with E. Gasperin, S. Gautam, A. Vañó-Viñuales.
- Phys.Rev.D103, 084045 (2021) with S. Gautam, A. Vañó-Viñuales, S Bose.
- Class.Quant.Grav. 38 (2021) with M. Duarte, J. Feng, E. Gasperin.

Wanted: Gravitational Waves at $\mathscr{I}^+$

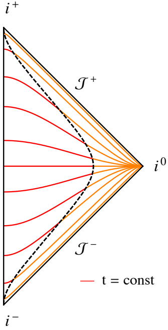
We are concerned with the *first principles* computation of gravitational waves at future null infinity.

State-of-the-art:

- Extrapolation.
- Characteristic-Extraction.

Wishlist:

- Well-posedness. Nice equations and solutions.
- Extension of strong-field setup.
- Proveably good numerics.



Timelike outer boundary. Vañó-Viñuales. 2015.

The weak-field

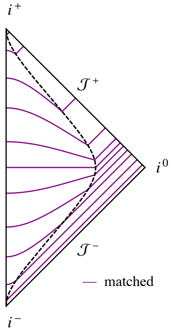
The wavezone is *weak*, so how is it a problem? Infinity is *really* big.

Fundamental ingredients:

- Compactify whilst resolving outgoing waves. Introduces blow-up quantities.
- Asymptotic Flatness: Metric decays near infinity.

Key to any computational strategy is the management of this competition.

Examples: CEFES. CCE/CCM.



CCM Cartoon. Vañó-Viñuales. 2015.

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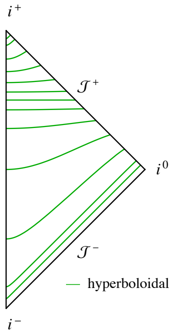


Hyperboloidal foliation. Vañó-Viñuales. 2015.

Compactification: dual-foliation strategy

- Observation: global inertial representation of MK regular.
- Asymptotically flat spacetimes looks like MK plus small terms.
- Choose such a basis carefully, exploit this decay.

Potential strength: developments in nonlinear stability of GR in GHG!

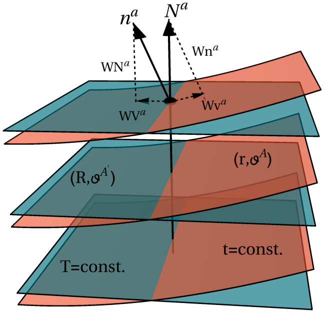


Illustration of DF setup.

Compactification: dual foliation formalism

Relationship between geometry with $X^{\underline{\mu}} = (T, X^{\underline{i}})$ or $x^{\mu} = (t, x^{i})$?

- Parametrize the inverse Jacobian $J^{-1} = \partial_{\underline{\alpha}} x^{\alpha}$ as,

$$J^{-1} = \begin{pmatrix} \alpha^{-1} W (A - B^{\underline{j}} V_{\underline{j}}) & (A - B^{\underline{j}} V_{\underline{j}})\, \Pi^{i} + B^{\underline{j}} (\varphi^{-1})^{i}{}_{\underline{j}} \\ -\alpha^{-1} W V_{\underline{i}} & (\varphi^{-1})^{i}{}_{\underline{i}} - \Pi^{i} V_{\underline{i}} \end{pmatrix}.$$

- Suppose we have a system

$$\partial_T \mathbf{u} = (A \mathbf{A}^{\underline{p}} + B^{\underline{p}} \mathbf{1}) \partial_{\underline{p}} \mathbf{u} + A \mathbf{S}\,,$$

- Then in the lowercase coordinates we have

$$(1 + \mathbf{A}^{V}) \partial_t \mathbf{u} = \alpha W^{-1} (\mathbf{A}^{\underline{p}} (\varphi^{-1})^{p}{}_{\underline{p}} + (1 + \mathbf{A}^{V}) \Pi^{p}) \partial_p \mathbf{u} + \alpha W^{-1} \mathbf{S}.$$

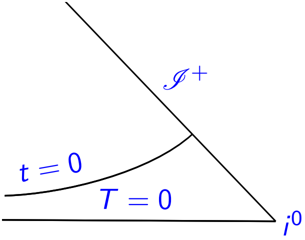
How to choose Jacobian?

Compactification: hyperboloidal initial value problem

$$T = T(t,r) = t + H(R), \qquad R = R(r) = \Omega(r)^{-1} r, \qquad \theta^{\underline{A}} = \theta^{A}\,.$$

- Height function H, compression function Ω.
- Hyperboloidal Jacobian;

$$J_{hyp} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ H'R' & R' & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Rough Idea: $R' \simeq R^n$ and $H' \simeq 1 - 1/R'$, $1 < n \leq 2$ achieves desirable coordinate lightspeeds *whilst* compactifying.

Compactification: blow-up quantities

For systems with wave-equation like principal part (KG, GR in GHG) combining with the J_{hyp} gives;

$$(1+\mathbf{A}^V)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -\gamma_2 W^2 V_{\underline{i}} & {}^{(N)}\mathbf{g}^{\underline{j}}{}_{\underline{i}} & W^2 V_{\underline{i}} \\ -\gamma_2 (W^2-1) & W^2 V^{\underline{j}} & W^2 \end{pmatrix}$$

Observations:

- Composite lower case principal part matrices *regular by construction*; symmetric hyperbolicity invariant.
- On the other hand $R' \sim R^n \Rightarrow W \sim \alpha \sim R^{n/2}$. Therefore need decay in sources $\mathbf{S}$ to absorb growth. But is sufficient decay present?

Asymptotics: The GBU(F)-model I

Consider a toy model for GR in GHG:

$$\Box g = 0, \quad \Box b \simeq \frac{1}{R}\partial_T f + (\partial_T g)^2, \quad \Box u \simeq \frac{2}{R}\partial_T u,$$

with free choice of the equation of motion for f.

Asymptotic system recipe:

- Rescale: $G = Rg, \quad B = Rb, \quad U = Ru, \quad F = Rf.$
- Change coordinates $\mathfrak{u} = T - R, \qquad s = \log(R).$
- Turn krank, collect leading order in R^{-1}.

For the model this gives

$$\partial_s\partial_{\mathfrak{u}} G = 0, \qquad 2\partial_s\partial_{\mathfrak{u}} B = -\partial_{\mathfrak{u}} F - (\partial_{\mathfrak{u}} G)^2, \qquad \partial_s\partial_{\mathfrak{u}} U = -\partial_{\mathfrak{u}} U.$$

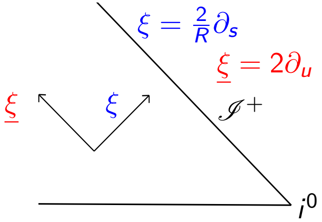
Asymptotics: The GBU(F)-model II

Interesting case I: $f = 0$

$$\partial_s \partial_{\mathfrak{u}} G = 0, \qquad 2\partial_s \partial_{\mathfrak{u}} B = -(\partial_{\mathfrak{u}} G)^2, \qquad \partial_s \partial_{\mathfrak{u}} U = -\partial_{\mathfrak{u}} U.$$

Solution to asymptotic system

- $\partial_s \partial_{\mathfrak{u}} G = 0 \Rightarrow G = \mathcal{F}_g(\mathfrak{u}, \theta^A)$.
- $\partial_s \partial_{\mathfrak{u}} B = -\frac{1}{2}(\partial_{\mathfrak{u}} G)^2 \Rightarrow$
  $\partial_{\mathfrak{u}} B = -\frac{1}{2} s (\partial_{\mathfrak{u}} G)^2$.
- $B = (\ln R)\, \mathcal{F}_b(\mathfrak{u}, \theta^A)$.



Predicts asymptotics of original fields:

$$g = \frac{1}{R}\mathcal{F}_g(\mathfrak{u}, \theta^A), \qquad b = \frac{\log(R)}{R}\mathcal{F}_b(\mathfrak{u}, \theta^A), \qquad u = \frac{1}{R} m_u(\theta^A)\,.$$

Analogy with GR?

Asymptotics: The GBU(F)-model III

Interesting case II: Take

$$\Box f = \tfrac{2}{R}\partial_T f + 2(\partial_T g)^2.$$

Then asymptotically:

$$\partial_s \partial_{\mathfrak{u}} F = -\partial_{\mathfrak{u}} F - (\partial_{\mathfrak{u}} G)^2 \Rightarrow \partial_{\mathfrak{u}} F = -(\partial_{\mathfrak{u}} G)^2,$$

... so now we recover instead:

$$g = \frac{1}{R}\mathcal{F}_g(\mathfrak{u},\theta^A), \qquad b = \frac{1}{R}\mathcal{F}_b(\mathfrak{u},\theta^A), \qquad u = \frac{1}{R}m_u(\theta^A)\,.$$

We have all of this worked out for GR in GHG together with high order asymptotic expansions. *Really* interesting: "TT" gauge!

Spherical GR I

Can we capture stratification without extra structure? Defining suitable null vectors σ and $\underline{\sigma}$, the field equations take the form:

$$-D_\sigma D_\sigma \mathring{R} + \tfrac{1}{\kappa} D_\sigma \mathring{R}(D_\sigma C_+ - D_{\underline{\sigma}} C_+) = 4\pi \mathring{R}\, T_{\sigma\sigma}\,,$$
$$-D_{\underline{\sigma}} D_{\underline{\sigma}} \mathring{R} + \tfrac{1}{\kappa} D_{\underline{\sigma}} \mathring{R}(D_\sigma C_- - D_{\underline{\sigma}} C_-) = 4\pi \mathring{R}\, T_{\underline{\sigma\sigma}}\,,$$

plus

$$-D_a \left(\sigma^a D_{\underline{\sigma}} e^\delta - \tfrac{e^\delta}{\kappa^2}(\sigma^a D_\sigma C_- - \underline{\sigma}^a D_{\underline{\sigma}} C_+)\right) + \tfrac{2}{\mathring{R}^3} M_{\mathrm{MS}}$$
$$+ \tfrac{e^\delta}{\kappa^3}(D_{\underline{\sigma}} C_+ D_\sigma C_- - D_\sigma C_+ D_{\underline{\sigma}} C_-) = \tfrac{8\pi}{\mathring{R}^2} T_{\theta\theta}\,,$$
$$D_a(\tfrac{e^\delta}{\kappa} \sigma^a D_{\underline{\sigma}} \mathring{R}^2) + 1 = 0\,,$$

... which is *surprisingly* pretty!

Spherical GR II

Imposing GHG gives

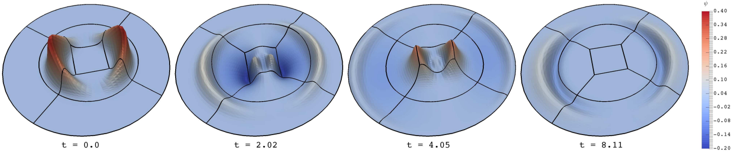
$$\frac{1}{\mathring{R}} D_\sigma\left(\frac{2}{\kappa}\mathring{R}^2 D_{\underline{\sigma}} C_+\right) + D_\sigma(RF^\sigma) - \frac{2}{\kappa}(D_\sigma \mathring{R}) D_\sigma C_+ + 8\pi\mathring{R}\, T_{\sigma\sigma} = 0\,,$$
$$\frac{1}{\mathring{R}} D_{\underline{\sigma}}\left(\frac{2}{\kappa}\mathring{R}^2 D_\sigma C_-\right) - D_{\underline{\sigma}}(RF^{\underline{\sigma}}) - \frac{2}{\kappa}(D_{\underline{\sigma}} \mathring{R}) D_{\underline{\sigma}} C_- - 8\pi\mathring{R}\, T_{\underline{\sigma\sigma}} = 0\,,$$

for the speeds, and

$$D_a\left(\frac{2}{\kappa}\sigma^a D_{\underline{\sigma}} e^\delta - F^a\right) - \frac{2}{\mathring{R}^2}\left(1 - \frac{2M_{\mathrm{MS}}}{\mathring{R}}\right)$$
$$+ \frac{2e^\delta}{\kappa^3}\left(D_{\underline{\sigma}} C_+ D_\sigma C_- - D_\sigma C_+ D_{\underline{\sigma}} C_-\right) = \frac{16\pi}{\mathring{R}^2} T_{\theta\theta}\,,$$
$$D_a\left(\frac{e^\delta}{\kappa}\sigma^a D_{\underline{\sigma}} \mathring{R}^2\right) + 1 = 0\,.$$

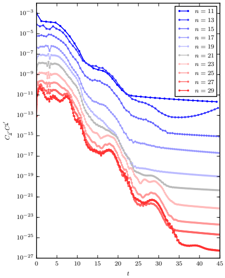
for the det variables... which is *still* surprisingly pretty!

Hyperboloidal numerics with the DF-wave equation



A first numerical sanity check:

- For wave equation **S** small.
- Can even evolve *radiation field* $R\phi$. [Target for GR].
- Respectable pseudospectral convergence achieved.

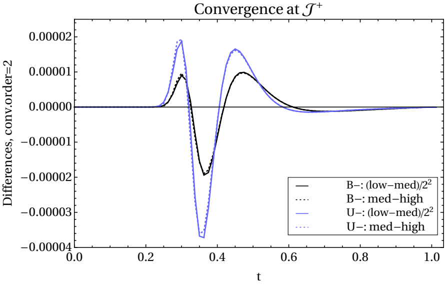


Numerics with the wave equation in bamps.

Hyperboloidal numerics with the GBU-model

Second numerical sanity check:

- 'Radiation fields' evolved.
- Implemented GBU-model in spherical FD code.
- Convergence despite logs.
- (Spectral numerics desirable too; patience needed!)



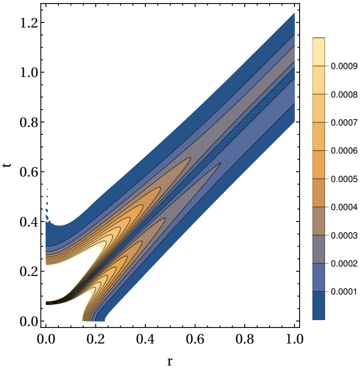
Numerics with GBU-model.

Summation by parts and hyperboloidal slices

SBP methods offer a path to formal stability. Subtleties:

- Singular coefficients.
- Non-degenerate norms.
- Undesirable reflections.

These can be overcome, paving the way for similar discretizations for GR.



WE numerics with SBP.

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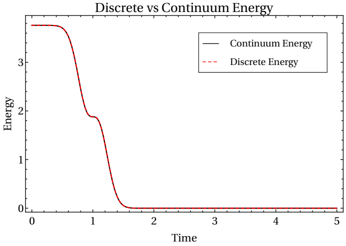


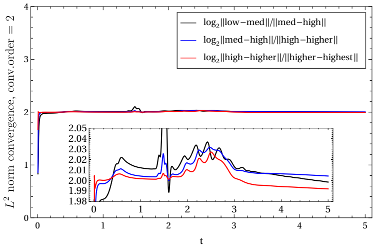
Energy comparison with SBP.

(Preliminary) hyperboloidal numerics for spherical GR

Numerical work with spherical GR ongoing:

- Gauge sources compatible with Schwarzschild in KS coodinates.
- Convergence with constraint violating data promising.
- Initial data (and technicalities) outstanding.

Near-term aspiration: critical collapse.



Norm convergence. Gautam 2021.

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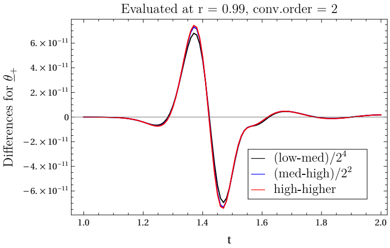

Pointwise convergence near $\mathscr{I}^+$.

Conclusions

Motivated by need for GWs at null infinity we are developing a new regularization using compactified hyperboloids. Features include:

- Dual-foliation formalism.
- Exploiting null-structure for NR.
- Careful choice of gauge and constraint addition to suppress spurious radiation fields (and logs).

Comprehensive GR numerics on the way, stay tuned!