The mysterious nature of the Big Bang singularity

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Einstein's equations

Cosmological setting

- Main unknown: A metric $\boldsymbol{g}$ of signature $(-1, 1, \ldots, 1)$ defined over $I \times \Sigma$, for Σ closed of $\dim\Sigma = D \geq 3$.
- Vacuum or massless-scalar field:

$$Ric_{\mu\nu}(\boldsymbol{g}) = \partial_\mu \psi \partial_\nu \psi$$
$$\Box_{\boldsymbol{g}} \psi = 0$$

- Evolutionary part: wave-type system $1 + D$ dimensions.
- Initial data: $(\Sigma, g, k, \psi, \varphi)$
- Constraint equations (elliptic):

$$R(g) - |k|^2 + (\mathrm{tr}k)^2 = \varphi^2 + |\nabla \psi|^2$$
$$\mathrm{div}k - \nabla \mathrm{tr}k = -\varphi \nabla \psi$$

- Well-posed initial value problem, there exists a unique maximal solution (Choquet-Bruhat–Geroch '69).

Generalized Kasner solutions

Homogeneous, anisotropic

- Defined for all $(t, x) \in (0, +\infty) \times \mathbb{T}^D$:

$$\boldsymbol{g}_{Kasner} = -dt^2 + \sum_{i=1}^{D} t^{2p_i} (dx^i)^2, \qquad \psi = B \log t.$$

- Kasner relations:

$$\sum_{i=1}^{D} p_i = 1, \qquad \sum_{i=1}^{D} p_i^2 = 1 - B^2, \qquad p_i < 1.$$

- There is a Big Bang singularity at $\{t = 0\}$ (spacelike):

$$V(\Sigma_t) \sim t, \qquad |Riem(\boldsymbol{g})|^2 \sim t^{-4}, \qquad \text{as } t \to 0.$$

- Is this due to the symmetries or a general new phenomenon?

"Singularity" theorems

Cosmological setting

- Generic conditions on initial data that lead to a breakdown of solutions.
- Valid for perturbations of exact Big Bang solutions.

Theorem (Hawking '65)

Let $(\mathcal{M}, \mathbf{g})$ be a (maximal) cosmological solution with initially negative past mean curvature:

$$\mathrm{tr}k\big|_{\Sigma} < 0.$$

Then it is past causally geodesically incomplete.

Q: Is this breakdown related to a singularity formation?

A: Not always. The Taub-NUT solution possesses a smoothly extendible Cauchy horizon. Failure of determinism. Unstable?

General solutions

Predictions

Strong cosmic censorship (Penrose '69)
Maximal cosmological solutions arising from generic initial data are inextendible as suitably regular Lorentzian manifolds.

- Inextendibility for Hawking's theorem: Singularity formation.

Spacelike or null?

- Big Bang generically expected in the cosmological setting of Hawking's theorem.
- Cf. Black hole interior, generic Cauchy horizon near timelike infinity (Dafermos-Luk '17).

Kasner-like Big Bang singularity or of other type?

- Kasner-like Big Bangs in the subcritical regime, stability (F.-Rodnianski-Speck '20).
- Oscillatory scenario (Belinskii-Khalatnikov-Lifshitz '70).
- Spikes? (Ringström '09)

Kasner-like singularities

Asymptotic profile

The asymptotic profile is given by a metric of the form

$$\boldsymbol{g} = -dt^2 + \sum_{I=1}^{D} t^{2p_i(x)} \omega^i \otimes \omega^i, \qquad \psi = B(x) \log t,$$

for $(t, x) \in (0, T] \times \mathbb{T}^D$, $T > 0$, where

$$\omega^i = \sum_{j=1}^{D} c^i_j(x) dx^j$$

and

$$\sum_{i=1}^{D} p_i(x) = 1, \qquad \sum_{i=1}^{D} p_i^2(x) = 1 - B^2(x), \qquad p_i(x) < 1.$$

Heuristics

KL '63, BK '72, Demaret-Henneaux-Spindel '85

- Consider the spatial frame $e_i = t^{-p_i(x)}(\omega^i)^\#$ and $k_{ij} = \boldsymbol{g}(D_{e_i} e_j, \partial_t)$.
- Using the form of $\boldsymbol{g}$: $k_{ij} = -\delta_{ij}\frac{p_i(x)}{t}$ (no sum), $\mathrm{tr}k = -\frac{1}{t}$.
- k_{ij} satisfies the evolution equation:

$$\partial_t k_{ij} + \frac{1}{t}k_{ij} = Ric_{ij}(g) - Ric_{ij}(\boldsymbol{g})$$

- On the other hand, the form of $\boldsymbol{g}$ gives:

$$|Ric(g)|(x) \sim t^{-2\max_{i,j,b}\{p_i(x)+p_j(x)-p_b(x)\}}, \qquad i \neq j \neq b \neq i.$$

- For consistency we need the subcriticality condition:

$$|Ric(g)|(x) \leq Ct^{-2+\sigma} \Leftrightarrow \max_{i\neq j\neq b\neq i}\{p_i(x)+p_j(x)-p_b(x)\} < 1,$$

for some $\sigma > 0$.

Heuristics

KL '63, BK '72, Demaret-Henneaux-Spindel '85

The subcritical regime includes:

- Scalar field, $D \geq 3$, e.g. $D = 3$ and $p_1, p_2, p_3 > 0$: $p_i + p_j - p_b = 1 - 2p_b < 1$.
- Higher dimensional vacuum, $D \geq 10$.

1+3 vacuum violates the subcriticality condition:

- Let $p_1 < 0$. Then $p_2 + p_3 - p_1 = 1 - 2p_1 > 1$, for all Kasner exponents.
- The crucial estimate $|Ric(g)| \leq Ct^{-2+\sigma}$ is valid if

$$\omega^1 \wedge \mathrm{d}\omega^1 = 0.$$

  This eliminates one degree of freedom (one of the c^i_j's).
- Kasner-like singularities in $1 + 3$ vacuum should be non-generic (instabilities).

Heuristics

BKL '69

Oscillatory scenario in 1+3 vacuum:

- Going back to Einstein's equations:

$$\partial_t k_{ii} + \frac{1}{t} k_{ii} = Ric_{ii}(g) \sim t^{-2+4p_1}$$

- There exists a $t_c \ll 1$ where t^{-2+4p_1} dominates.
- For $t \gg t_c$, the original Kasner-like behavior is valid. For $t \sim t_c$, a different evolution will follow.
- The solution will remain Kasner-like, but with new Kasner exponents:

$$p_1' = -\frac{p_1}{1+2p_1}, \qquad p_2' = \frac{p_2+2p_1}{1+2p_1} < 0, \qquad p_3' = \frac{p_3+2p_1}{1+2p_1}$$

- Then there exists a $t_c' \ll t_c$ where the Kasner exponents will change again in the same fashion etc.
- Infinite oscillations of the Kasner exponents, chaotic behavior.

What is known?

Oscillations: Part of the picture confirmed for homogeneous solutions (Bianchi VIII and IX), Ringström '01, Heinzle-Uggla '09, Reiterer-Trubowitz '10.

Numerics: In accordance with heuristics, apart from some "spiky" behavior at certain points, not predicted by BKL! (Berger, Garfinkle, Grubišić, Isenberg, Moncrief, Weaver, . . .)

Gowdy symmetry:

- 1+3 vacuum with two surface orthogonal Killing fields.
- Kasner-like behavior in the polarized case (Chruściel-Isenberg-Moncrief '90).
- Kasner-like behavior in the general case, apart from finitely many spikes (Ringström '09).
- Discontinuity of the asymptotic data, rate of blowup.
- Construction of spikes (Rendall-Weaver '08).

More general spikes: Heinzle-Uggla-Lim '12, Moughal-Lim '21.

What is known?

Constructions of Kasner-like singularities:

- Prescribe the asymptotic data at $t = 0$ and solve a singular initial value problem in $(0, T] \times \mathbb{T}^D$.
- Ames, Andersson, Beyer, Damour, Elery, Fournodavlos, Henneaux, Isenberg, Kichenassamy, Klinger, LeFloch, Luk, Moncrief, Nützi, Reiterer, Rendall, Trubowitz, Weaver, ...

Stability of generalized Kasner singularities:

- Prescribe near-Kasner initial data at $t = 1$ and prove a Kasner-like singularity formation towards the past.
  - Subcritical regime.
  - 1+3 vacuum with polarized symmetry.
- Rodnianski-Speck '14-'18, F.-Rodnianski-Speck '20, Ames-Beyer-Isenberg-Oliynyk '21.

Conditions for Kasner-like behavior:

- Scale invariant curvature bounds (Lott '20).
- Bounds on the normalized Weingarten map (Ringström '21).

Construction of smooth Kasner-like singularities

1+3 vacuum, without symmetries

Theorem (w/ Jonathan Luk '20)

Let $c_{ij}, p_i : \mathbb{T}^3 \to \mathbb{R}$ be smooth functions satisfying:

1. $\sum_{i=1}^3 p_i(x) = \sum_{i=1}^3 p_i^2(x) = 1$, $p_1(x) < p_2(x) < p_3(x) < 1$,
2. $c_{ij}(x) = c_{ji}(x)$, $c_{11}(x), c_{22}(x), c_{33}(x) > 0$,
3. *The differential constraints*[1]

$$\sum_{\ell=1}^{3} \frac{1}{2}\frac{\partial_i c_{\ell\ell}}{c_{\ell\ell}}(p_\ell - p_i) + \sum_{\ell>i} \frac{\partial_\ell(\sqrt{c_{11}c_{22}c_{33}}{\kappa_i}^\ell)}{\sqrt{c_{11}c_{22}c_{33}}} = \partial_i p_i.$$

Then there exists a smooth solution $(\boldsymbol{g}, (0, T] \times \mathbb{T}^3)$ to the Einstein vacuum equations with the asymptotic profile:

$$\boldsymbol{g} = -\,dt^2 + \sum_{i,j=1}^3 c_{ij}(x) t^{2\max\{p_i(x),p_j(x)\}} dx_i dx_j + \mathrm{l.o.t.}$$

[1] for ${\kappa_1}^2 = (p_1 - p_2)\frac{c_{12}}{c_{22}}$, ${\kappa_2}^3 = (p_2 - p_3)\frac{c_{23}}{c_{33}}$, ${\kappa_1}^3 = -{\kappa_1}^2\frac{c_{23}}{c_{33}} + (p_1 - p_3)\frac{c_{13}}{c_{33}}$

Method of proof

Two main steps

- Compute asymptotic expansions to all orders, using the ADM equations

$$\partial_t g_{ij} = -2 g_{aj} k_i{}^a, \qquad \partial_t k_i{}^j - \mathrm{tr} k k_i{}^j = Ric_i{}^j(g),$$

- Derive weighted energy estimates for the remainder of a truncated series, using the second order equation:

$$\partial_t^2 k_i{}^j - \Delta_g k_i{}^j = -\nabla_i \nabla^j k_\ell{}^\ell + \mathcal{N}(k, \partial_t k)_i{}^j.$$

Main difficulties:

- Approximate propagation of constraints for the expansions (Bianchi equations).
- The second order equation for $k_i{}^j$ is not exactly wave type, loss of derivatives (need for elliptic estimates).

Stability of Kasner singularities

Subcritical regime

Let $\boldsymbol{g}_{Kasner} = -dt^2 + \sum_{i=1}^{D} t^{2p_i}(dx^i)^2$, $\psi = B \log t$, be a generalized Kasner solution, satisfying the subcriticality condition

$$\max_{i \neq j \neq b \neq i}\{p_i + p_j - p_b\} < 1,$$

either for the scalar field $D \geq 3$ or in vacuum $D \geq 10$.

Theorem (w/ Igor Rodnianski, Jared Speck '20)

Solutions arising from sufficiently small perturbations of the above Kasner initial data at $t = 1$, in high-order Sobolev spaces, satisfy:

$$|tk_{ij} + p_i\delta_{ij}| + |t\partial_t\psi - B| \leq C_{initial.data}$$
$$|Ric(g)| \leq C_{initial.data} t^{-2+\sigma},$$

relative to the CMC foliation, $\mathrm{tr}k = -t^{-1}$, for all $t \in (0, 1]$.

Key ingredients

- Synchronization of the singularity at $t = 0$, using a CMC gauge (elliptic, infinite speed of propagation).
- Key variables: The structure coefficients of a Fermi-propagated frame $\{e_i\}_1^D$ (from $t = 1$)

$$S_{ijb} = g([e_i, e_j], e_b), \qquad i \neq j \neq b \neq i$$

- Asymptotic diagonalization of their ODE part:

$$\partial_t S_{ijb} + \frac{p_i + p_j - p_b}{t} S_{ijb} = \text{l.o.t.} \quad \Rightarrow \quad |S_{ijb}| \leq C t^{p_b - p_i - p_j} \leq C t^{-1+\varepsilon}.$$

- This is the only place where the subcritical condition is used!

Thank you!