# A tale of two tails

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Joint work with Sung-Jin Oh (Berkeley)

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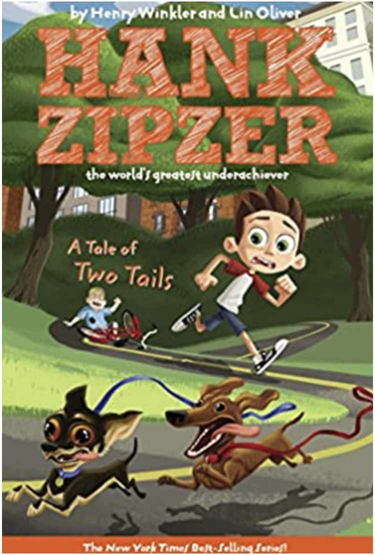

# Schwarzshild and Kerr black holes

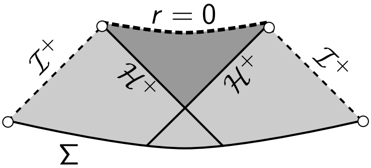


Figure: Schwarzschild

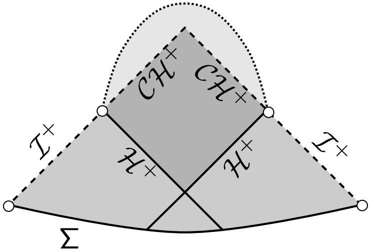


Figure: Kerr $(0 < |a| < M)$

# Strong cosmic censorship

**Conjecture**

*For* **generic** *asymptotically flat initial data, the maximal Cauchy development solving the Einstein vacuum equations is* **inextendible as a suitably regular Lorentzian manifold**.

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- The Schwarzschild case, not the rotating Kerr case, is expected to be generic.
- Small perturbations of Kerr data are expected to lead to singularities in the black holes.

# Stability of the Kerr Cauchy horizon

**Theorem (Dafermos–L. (2017))**

*If the exterior region to the black hole converges to Kerr with $0 < |a| < M$ (sufficiently fast), then*

- *the black hole interior has a null Cauchy horizon.*
- *Moreover, the metric is continuously extendible to the Cauchy horizon, and*
- *(in appropriate coordinate systems) is close to the Kerr metric in amplitude.*

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- *(in appropriate coordinate systems) is close to the Kerr metric in amplitude.*

- If the Kerr exterior is stable (as is widely expected), then small perturbations of Kerr data lead to Cauchy developments which are $C^0$ extendible.

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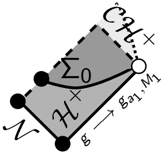


Figure: Stability of $\mathcal{CH}^+$ from data on $\mathcal{H}^+ \cup \mathcal{N}$

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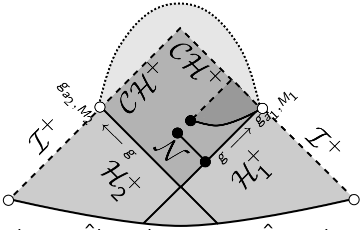


Figure: The global stability of the Kerr Penrose diagram

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## Theorem

*Consider the Einstein–Maxwell–scalar field system in* **spherical symmetry** *with two-ended asymptotically flat admissible smooth initial data.*

1. *(Dafermos, Dafermos–Rodnianski (2005)) The black hole interior has a Cauchy horizon across which the metric extends in $C^0$.*
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- A key step of the proof is to show that generically, the scalar field obeys a **lower bound** in the exterior of the black hole.

# Price's law

PHYSICAL REVIEW D VOLUME 5, NUMBER 10 15 MAY 1972

**Nonspherical Perturbations of Relativistic Gravitational Collapse.
I. Scalar and Gravitational Perturbations***

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Price suggested that

- $\Box_g \phi = 0$,
- $\phi$ initially compactly supported,
- $\phi$ supported on the spherical harmonics of degree $\ell$

then, on a finite $r$ region

$$|\phi|(t, r) \sim (1+t)^{-2\ell-3}.$$

# Price's law

$\Box_g \phi = 0$ on Schwarzschild spacetime, $\phi$ initially compactly supported.

## Theorem (Price's law)

*The following bounds hold on a finite $r$ region:*

1. *(Dafermos–Rodnianski (2005), Tataru (2013), Donninger–Schlag–Soffer (2012), Metcalfe–Tataru–Tohaneanu (2012))*

$$|\phi| \lesssim (1+t)^{-3}.$$

2. *(Donninger–Schlag–Soffer (2011)) If $\phi$ is supported on $\ell$-th spherical harmonics,*

$$|\phi| \lesssim (1+t)^{-2-2\ell}.$$

3. *(Angelopoulos–Aretakis–Gajic (2018, 2021), Hintz (2020)) Generic $\phi$ supported on spherical harmonics $\geq \ell$ obeys,*

$$(1+t)^{-3-2\ell} \lesssim |\phi| \lesssim (1+t)^{-3-2\ell}.$$

# Remarks on Price's law

1. Upper bound results known for more general — dynamical or stationary — spacetimes (Tataru, Metcalfe–Tataru–Tohaneanu).
2. Precise asymptotics known for more generally on subextremal Reissner–Nordström and Kerr (Hintz, Angelopoulos–Aretakis–Gajic).
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3. See Angelopoulos–Aretakis–Gajic for more precise information as well as the extremal cases.
4. See also Barack–Ori, Bičák, Bizoń–Chmaj–Rostworowski, Blaksley–Burko, Burko–Khanna, Casals–Ottewill, Gómez–Winicour–Schmidt, Gundlach–Price–Pullin, Hod, Krivan–Laguna–Papadopoulos–Andersson, Leaver, Lucietti–Murata–Reall–Tanahashi, Marsa–Choptuik, Poisson, Szpak, Zenginoğlu–Khanna–Burko, ..., Aretakis, Baskin–Vasy–Wunsch, Gajic, Guillarmou–Hassell–Sikora, Kehrberger, Looi, L.–Oh, Ma, Ma–Zhang, Morgan, Morgan–Wunsch, Moschidis, Oliver–Sterbenz, Schlue, ...

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**Lemma**

*Solutions to the wave equation on $(3+1)$-dimensional Minkowski spacetime with compactly supported data decay $O(t^{-\infty})$ on any compact set $\{|x| \leq R\}$.*

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- Late-time tails on Minkowski can only arise if the initial data are not compactly supported.
  - See blackboard.

# Why is Price's law true?

We try to understand Price's law as follows:

- Justify $r\phi = \Phi_0 + r^{-1}\Phi_1 + r^{-2}\Phi_2 + \dots$, with $\Phi_i = \Phi_i(u, \vartheta)$.
- Analyze the $\Phi_i$ near null infinity (for $r \gtrsim u$ large).
- Find the first $i$ such that
  - $\Phi_{i,\infty}(\vartheta) := \lim_{u\to\infty} \Phi_i(u, \vartheta) \neq 0$, and
  - $r\phi \restriction_{C_0} = cr^{-i}\Phi_{i,\infty}(\vartheta)$ gives a non-trivial late-time tail on **Minkowski**.
- Prove that the late time tail is exactly the late-time Minkowski tail for $r\phi \restriction_{C_0} = cr^{-i}\Phi_{i,\infty}(\vartheta)$, at least in $\{r \geq R\}$, $R$ large.

# Wave equation on Schwarzschild

The wave equation on Schwarzschild with fixed $\ell$ mode is given in Bondi–Sachs coordinates by

$$[2\partial_u - (1-\frac{2M}{r})\partial_r][(1-\frac{2m}{r})\partial_r(r\phi)] + \frac{\ell(\ell+1)}{r^2} r\phi + \frac{2M}{r^2}(1-\frac{2M}{r})\phi = 0.$$

- Assume an expansion $r\phi = \Phi_0 + r^{-1}\Phi_1 + r^{-2}\Phi_2 \dots$ with $\Phi_i = \Phi_i(u,\vartheta)$.

# Schwarzschild in the case $\ell = 0$

Assume an expansion $r\phi = \Phi_0 + r^{-1}\Phi_1 + r^{-2}\Phi_2 \dots$ with $\Phi_i = \Phi_i(u, \vartheta)$.

- When $\ell = 0$, only $\Phi_0$ does not contribute to the late-time tail
- The wave equation gives

$$\partial_u \Phi_1 = 0, \quad \partial_u(\Phi_2 - M\Phi_1) = \frac{1}{2} M\Phi_0.$$

- Generically $\mathfrak{L} := \int_{-\infty}^{\infty} \frac{1}{2} M\Phi_0 \, du \neq 0$.
- Hence, $r\phi = \Phi_0 + \mathfrak{L}r^{-2} + \dots$, which gives a late time tail $\phi \sim t^{-3}$ on compact $r$ region.

# Schwarzschild in the case $\ell = 1$

- When $\ell = 1$, $\Phi_0$ and $\Phi_1$ do not contribute to late time tails, so we expect $\Phi_0$, $\Phi_1 \to 0$.
- Using the wave equation, we obtain the recursion relations:

$$\partial_u \Phi_1 = \Phi_0,$$
$$\partial_u(\Phi_2 - M\Phi_1) = \frac{1}{2} M\Phi_0,$$
$$\partial_u(3\Phi_3 - 4M\Phi_2) = -2\Phi_2 + 14M\Phi_1 - 2M^2\Phi_0.$$

- The first two equations can be combined to

$$\partial_u(\Phi_2 - \frac{3M}{2}\Phi_1) = 0,$$

  so that $\Phi_2 \to 0$ as well!
- $\Phi_3$ in general $\not\to 0$. Hence, the main contribution far-away comes from $r\phi \sim r^{-3}$, which gives a late time tail $\phi \sim t^{-5}$ on compact $r$ region.

# What if we turn to dynamical spacetimes?

Spherically symmetric spacetime

$$g = -\Omega^2 du\, dv + r^2 \sigma_{\mathbb{S}^2(1)}.$$

Define $m(u,v) = \frac{r}{2}(1 + \frac{4(\partial_v r)(\partial_u r)}{\Omega^2})$, and the Bondi mass

$$M(u) = \lim_{v\to\infty} m(u,v).$$

The previous condition suggests if

$$\mathfrak{L} = \int_{-\infty}^{\infty} M(u)[\lim_{r\to\infty}(r\phi)\partial_u r(u,r)]\, du \neq 0,$$

then $|\phi|$ gets a $(1+t)^{-3}$ tail.

# Nonlinear but spherically symmetric tails

Consider the Einstein–(Maxwell)–scalar field system in spherical symmetry.

**Theorem**

1. *(L.–Oh 2015) For dispersive solutions (with zero charge) converging to Minkowski, if* $\mathfrak{L} \neq 0$*, then on* $\{r \leq R\}$*,*

$$(1+t)^{-3} \lesssim |\partial_u (r\phi)|, |\partial_v (r\phi)| \lesssim (1+t)^{-3}.$$

2. *(L.–Oh 2019) For black hole solutions with non-zero charge, if* $\mathfrak{L} \neq 0$*, then* $|\partial_v \phi|$ *obeys an averaged lower bound on the event horizon, and the Cauchy horizon is singular.*

# Precise tails on Ultimately Schwarzschildean spacetimes

**Theorem (L.–Oh, in preparation)**

*Let $(\mathcal{M}, g)$ be a spacetime settling down to the Schwarzschild exterior suitably.*
*Then*

1. *given a solution $\phi$ to $\Box_g \phi = 0$ with compactly supported data, if $\mathfrak{L} \neq 0$, then*

   $$(1+t)^{-3} \lesssim |\phi| \lesssim (1+t)^{-3}$$

   *on $\{r \leq R\}$.*
2. *Moreover, generic solutions satisfy $\mathfrak{L} \neq 0$.*

# The $\ell = 1$ mode in the dynamical case

- Suppose we now have a spherically symmetric dynamical spacetime converging to Schwarzschild.
- For the $\ell = 1$ mode, we revisit the recurrence relations:

$$\partial_u \Phi_1 = \Phi_0,$$
$$\partial_u(\Phi_2 - M\Phi_1) = \frac{1}{2} M\Phi_0.$$

- If $M(u)$ is **not** a constant, then in general

$$\int_{-\infty}^{\infty} M(u)\Phi_0(u)\, du \neq 0!$$

# Instability of Price's law under dynamical perturbations

**Theorem (L.–Oh, in preparation)**

*Consider a spherically symmetric spacetime $(\mathcal{M}, g)$ converging to Schwarzschild exterior suitably, but the Bondi mass is non-constant. Then, for generic compactly supported initial data supported on $\ell \geq 1$, the solution to the wave equation $\Box_g \phi = 0$ satisfies*

$$(1+t)^{-4} \lesssim |\phi| \lesssim (1+t)^{-4}$$

*on a region $\{r \leq R\}$.*

- Contrast this with the $t^{-5}$ Price's law tail!

# General spacetimes, higher dimensions, etc.

- The theorem applies to more general (not necessarily ultimately Schwarzschildean) spacetimes.
  - Need only (1) (strong) asymptotic flatness, (2) asymptotic stationarity and (3) some integrated local energy decay estimate.
  - We obtain the sharp tail at least in $\{R \leq r \leq R'\}$, for $R$ large.

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  - Need only (1) (strong) asymptotic flatness, (2) asymptotic stationarity and (3) some integrated local energy decay estimate.
  - We obtain the sharp tail at least in $\{R \leq r \leq R'\}$, for $R$ large.
- Similarly phenomenon for higher $\ell$'s and higher dimensions on dynamical spacetimes.
- The phenomenon of different late-time tails in the dynamical setting in $(5+1)$ dimensions was previously observed numerically by Bizoń–Chmaj–Rostworowski.

# Returning to the strong cosmic censorship problem

The higher $\ell$ problems can be viewed as model problems for electromagnetic and gravitational perturbations.

**Conjecture**

1. *Generic solutions to the Maxwell equation on a dynamical spacetime settling down to Kerr (sufficiently fast) decays with an exact rate of* $(1+t)^{-4}$.
2. *Generic small perturbations of Kerr initial data lead to solutions to the Einstein vacuum equations which converge to Kerr with an exact rate of* $(1+t)^{-6}$.

- Again, contrast this with Price's law.

# Failure of peeling and corrections to late-time tails

- If initial characteristic data have a slower tail, then by the recurrence relations it will dominate the late-time asymptotics.
- In the spherically symmetric setting, this observation gives an easy proof of strong cosmic censorship with slow initial tail (L.–Oh (2019)).

# Failure of peeling and corrections to late-time tails

- If initial characteristic data have a slower tail, then by the recurrence relations it will dominate the late-time asymptotics.
- In the spherically symmetric setting, this observation gives an easy proof of strong cosmic censorship with slow initial tail (L.–Oh (2019)).
- Christodoulou (2002) argued that slower tails on characteristic hypersurfaces with logarithmic terms arise naturally in physical situations.
- The corresponding corrections to late-time tails for (1) spherically symmetric waves on spherically symmetric dynamical spacetimes, (2) higher $\ell$-modes on Schwarzschild are recently proven by Kehrberger (2021).

**Problem**

*Understand the effect of the extra logarithmic terms in dynamical spacetimes.*

# Nonlinear example: wave maps

Our method is sufficiently general to consider other settings.

- $\Sigma$ a 2-dimensional Riemannian manifold
- $\phi : \mathbb{R}^{3+1} \to \Sigma$ a wave map

$$\Box\phi^I = \Gamma^I_{JK}(\phi) m^{\alpha\beta} \partial_\alpha \phi^J \partial_\beta \phi^K.$$

**Theorem (Christodoulou, Klainerman (80s))**

*Let $p \in \Sigma$. Suppose the initial data $(\phi, \partial_t \phi)$ are smooth, agree with $(p, 0)$ on $\mathbb{R}^3 \setminus B(0, R)$ and are sufficiently close to $(p, 0)$ everywhere.*
*Then there exists a global-in-time solution to the wave map problem. Moreover, the map converges pointwise to the constant map $\mathbb{R}^3 \mapsto p$.*

# Polynomial tails for wave maps

**Theorem (L.–Oh, in preparation)**

1. *The global-in-time wave map in the previous theorem obeys*

$$\sup_{x:|x|\leq R} \mathrm{dist}_{\Sigma}(\phi(t,x),p) \lesssim (1+t)^{-3}.$$

2. *Suppose the Gauss curvature $K(p) \neq 0$. Then, for an open and dense subclass of small initial data, the solution obeys*

$$\inf_{x:|x|\leq R} \mathrm{dist}(\phi(t,x),p) \gtrsim (1+t)^{-3}.$$

# Idea of the proof

0. Basic structure of the proof
   - Divide the spacetime into (1) wave zone, (2) intermediate/near region.
   - Based on an iteration argument, starting with some slower decay rate.
   - Use vector field commutators $T$, rotations and scaling.

# Idea of the proof

1. Justify an expansion $r\phi = \Phi_0 + r^{-1}\Phi_1 + r^{-2}\Phi_2 \dots$ near null infinity
   - $\Phi_i$ satisfies recursion relations determined by wave equation.
   - Differentiate with the (conformally regular) commutator $\mathbf{K} = r^2\partial_v$.
   - For lower angular modes, estimate this using the method of characteristics.
   - For higher angular modes, use the Dafermos–Rodnianski $r^p$ method.
   - Inspired by Angelopoulos–Aretakis–Gajic.

# Idea of the proof

2. In intermediate region $\{R \leq r \leq (1-\eta)t\}$ ($R$ large)
   - Show that "first terms" do not contribute by introducing "correctors".
   - Write $\Box_g = \Box_{Minkowski} + \text{better}$.
     - This allows use to use strong Huygen's principle (!) to control the error terms.

# Idea of the proof

3. Proof of the lower bound.
   - Justify a more precise expansion
     $r\phi = \Phi_0 + r^{-1}\Phi_1 + r^{-2}\Phi_2 + \cdots + \mathfrak{L} r^{-i}\Phi_i + \mathrm{error}.$
   - $\mathfrak{L} r^{-i}\Phi_i$ is the first contributing term: if $\mathfrak{L} \neq 0$, this determines the late time asymptotics.
   - Finally, prove that $\mathfrak{L} \neq 0$ generically.
     - This is an open condition by stability argument.
     - If $\mathfrak{L} = 0$, small perturbations near infinity give a non-trivial contribution to $\mathfrak{L}$.

Thank you!