Mathematical and Numerical Aspects of Gravitation

Seeing Through Space-Time

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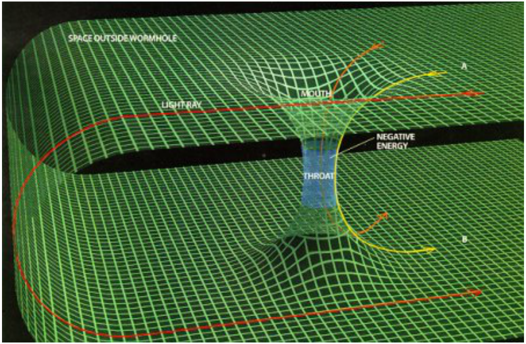
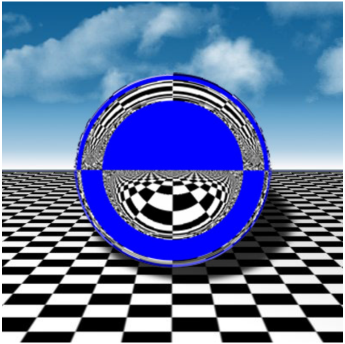
IPAM, October 26, 2021

Goal: To Determine the Topology and Metric of Space-Time





How can we determine the topology and metric of complicated structures in space-time with a radar-like device?

Figures: Anderson institute and Greenleaf-Kurylev-Lassas-U.

Non-linearity Helps

We will consider inverse problems for non-linear wave equations, e.g.

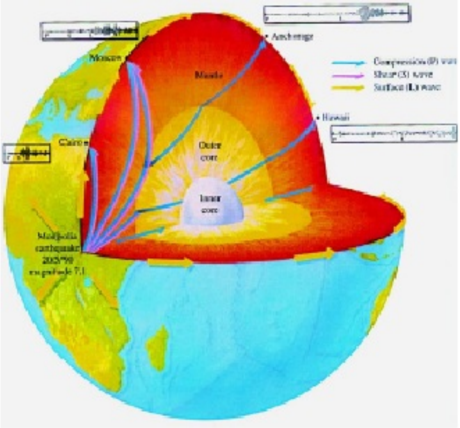
$$\frac{\partial^2}{\partial t^2}u(t,y) - c(t,y)^2\Delta u(t,y) + a(t,y)u(t,y)^2 = f(t,y).$$

We will show that:

-Non-linearity helps to solve the inverse problem,

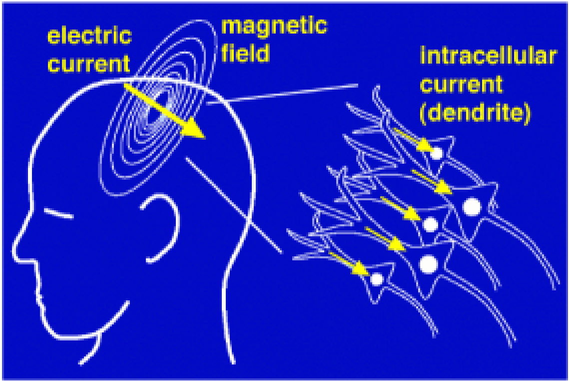
-“Scattering” from the interacting wave packets determines the structure of the spacetime.

Passive Measurements



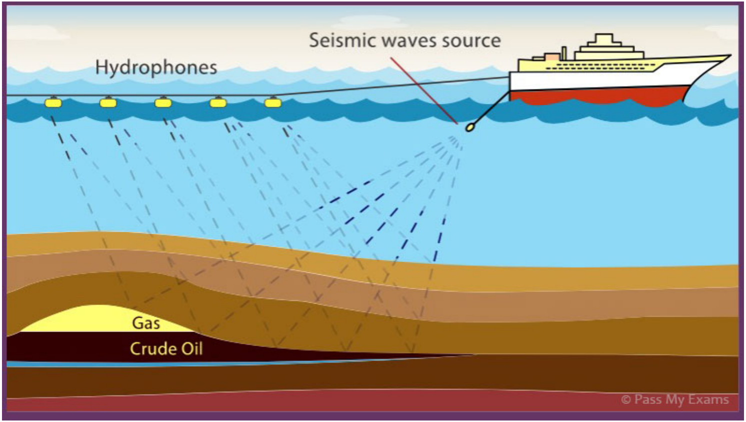
Passive Measurements

In Magnetoencephalography (MEG) brain activity is imaged by measuring magnetic fields produced by electrical currents in the brain.



Active Measurements

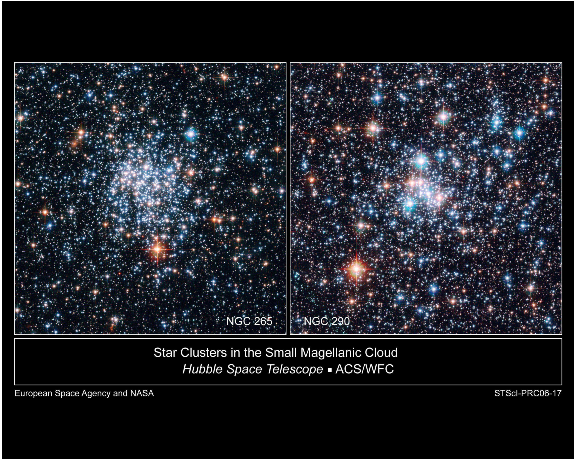
Oil Exploration



Active Measurements

Ultrasound

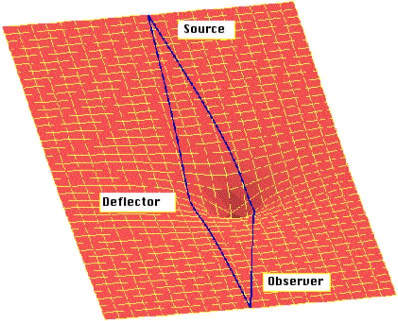
Inverse Problems in Space-Time: Passive Measurements

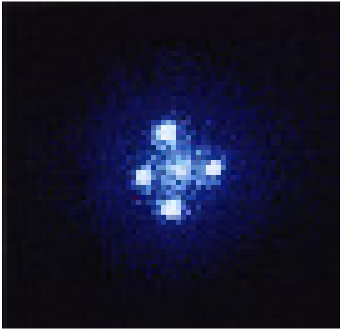


Can we determine the structure of space-time when we see light coming from many point sources varying in time? We can also observe gravitational waves.

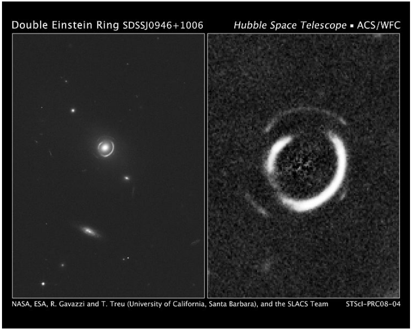
Gravitational Lensing

We consider e.g. light or X-ray observations or measurements of gravitational waves.

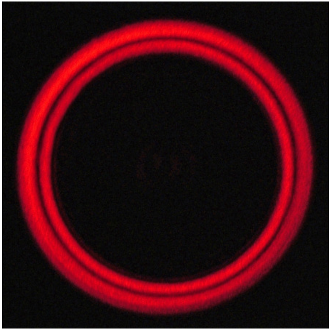




Gravitational Lensing



Double Einstein Ring

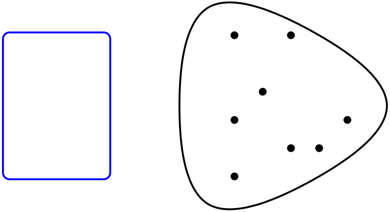


Conical Refraction

Passive Measurements: Gravitational Waves

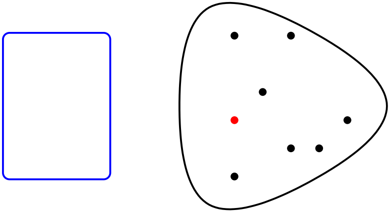
NSF Announcement, Feb 11, 2015

Inverse Problem for Passive Measurements



Can we determine the structure of space-time when we observe wavefronts produced by point sources?

Inverse Problem for Passive Measurements



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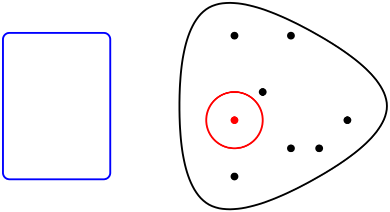

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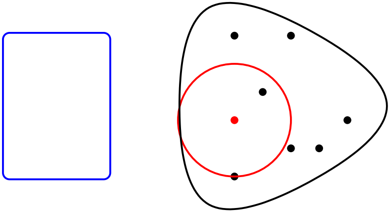

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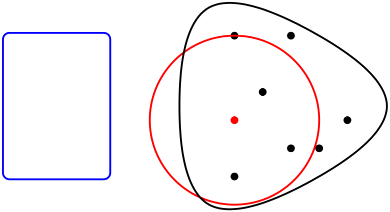

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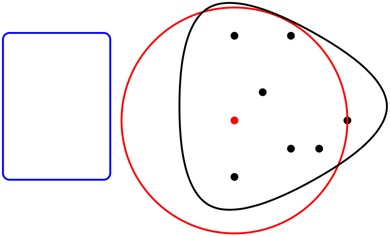

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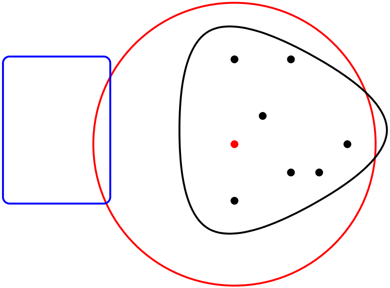

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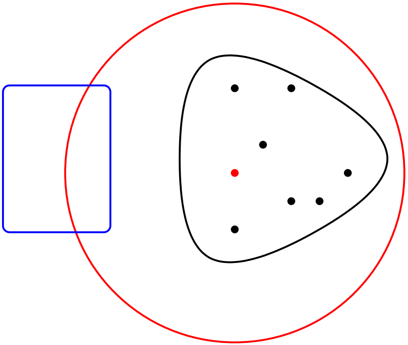

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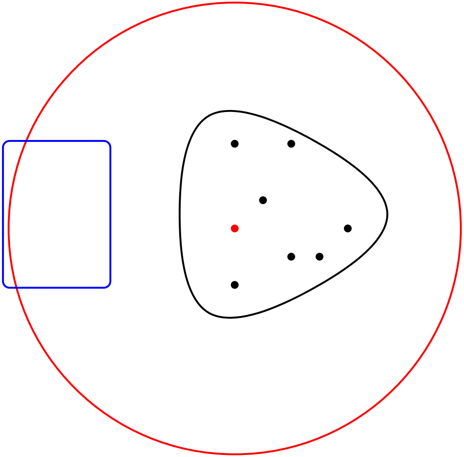

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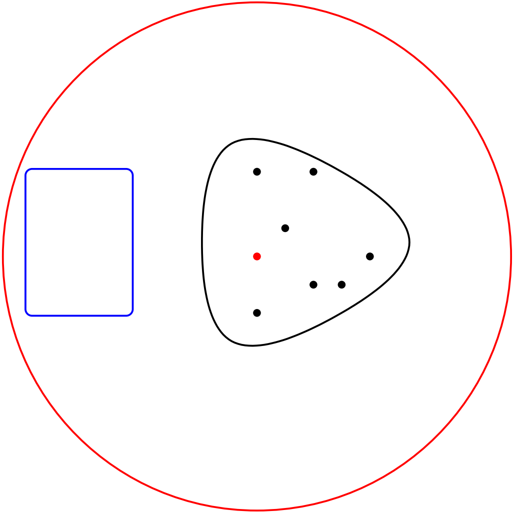

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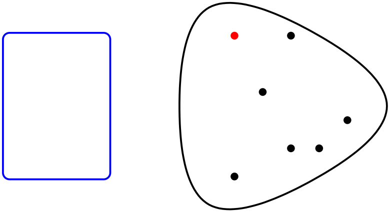

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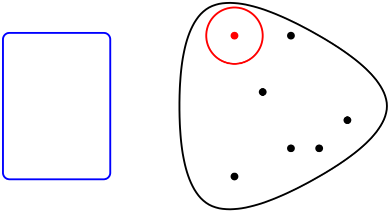

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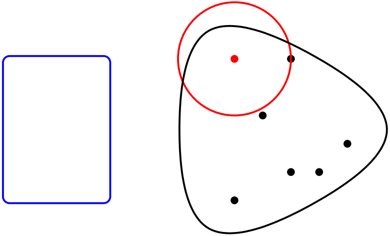

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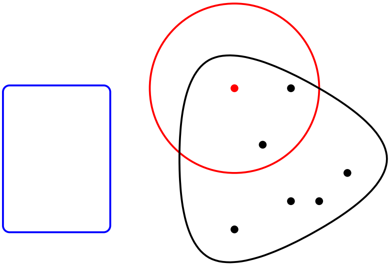

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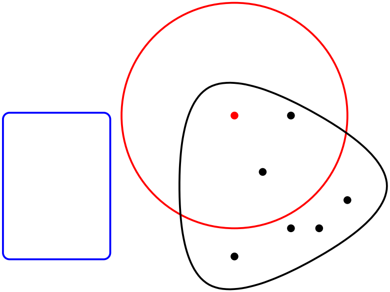

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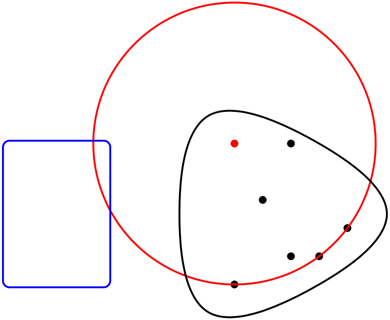

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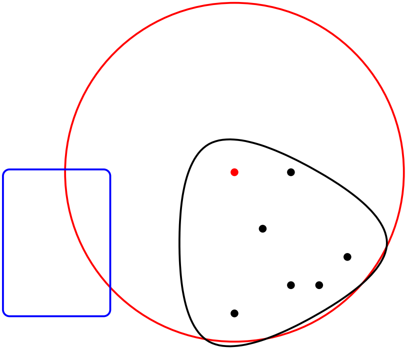

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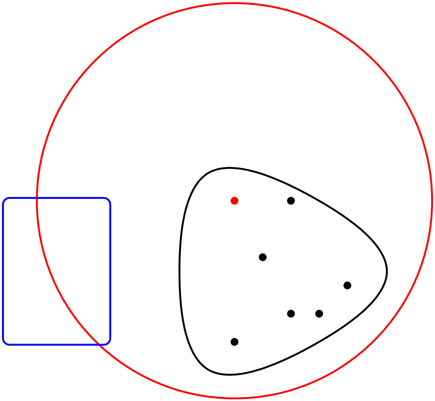


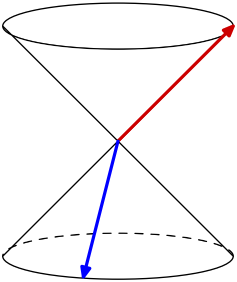
Can we determine the structure of space-time when we observe wavefronts produced by point sources?

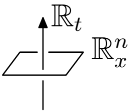
Lorentzian Geometry

$(n+1)$-dimensional Minkowski space: (M, g)

$$M = \mathbb{R}^{1+n} = \mathbb{R}_t \times \mathbb{R}^n_x, \quad \text{metric: } g = -dt^2 + dx^2.$$

Null/lightlike vectors: $V \in T_q M$ with $g(V, V) = 0$.





$L^\pm_q M$: future/past null vectors

Lorentzian Geometry

In general:

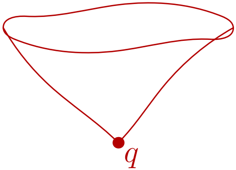
$$M = (n+1)\text{-dimensional manifold},\ g \text{ Lorentzian } (-,+,\dots,+).$$

Assume: existence of time orientation.

$$T_qM \cong (\mathbb{R}^{1+n}, \text{Minkowski metric}).$$

Null-geodesics: $\gamma(s) = \exp_q(sV)$, $V \in T_qM$ null.

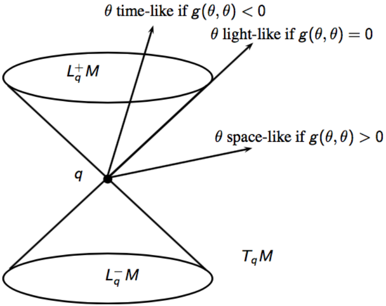
Future light cone: $\mathcal{L}_q^+ = \{\exp_q(V) \colon V \text{ future null}\}$



Lorentzian Manifolds

Let (M, g) be a $1+3$ dimensional time oriented Lorentzian manifold.
The signature of g is $(-,+,+,+)$.
Example: Minkowski space-time $(\mathbb{R}^4, g_m)$, $g_m = -dt^2 + dx^2 + dy^2 + dz^2$.

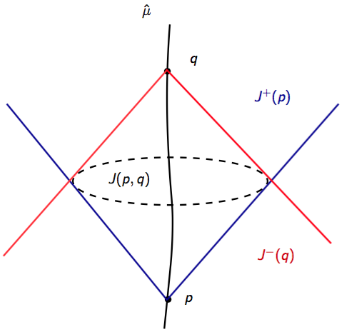
- $L_q^{\pm} M$ is the set of future (past) pointing light like vectors at q.
- Casual vectors are the collection of time-like and light-like vectors.
- A curve γ is time-like (light-like, causal) if the tangent vectors are time-like (light-like, causal).



Causal Relations

Let $\widehat{\mu}$ be a time-like geodesic, which corresponds to the world-line of an observer in general relativity. For $p, q \in M$, $p \ll q$ means p, q can be joined by future pointing time-like curves, and $p < q$ means p, q can be joined by future pointing causal curves.

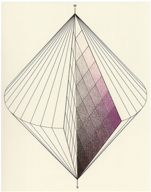
- The chronological future of $p \in M$ is $I^+(p) = \{q \in M : p \ll q\}$.
- The causal future of $p \in M$ is $J^+(p) = \{q \in M : q < p\}$.
- $J(p, q) = J^+(p) \cap J^-(q)$, $I(p, q) = I^+(p) \cap I^-(q)$.



Global Hyperbolicity

A Lorentzian manifold (M, g) is globally hyperbolic if

- there is no closed causal paths in M;
- for any $p, q \in M$
  and $p < q$, the set $J(p, q)$ is compact.



Then hyperbolic equations are well-posed on (M, g)
Also, (M, g) is isometric to the product manifold

$$\mathbb{R} \times N \text{ with } g = -\beta(t, y)dt^2 + \kappa(t, y).$$

Here $\beta : \mathbb{R} \times N \to \mathbb{R}_+$ is smooth, N is a 3 dimensional manifold and κ is a Riemannian metric on N and smooth in t.
We shall use $x = (t, y) = (x_0, x_1, x_2, x_3)$ as the local coordinates on M.

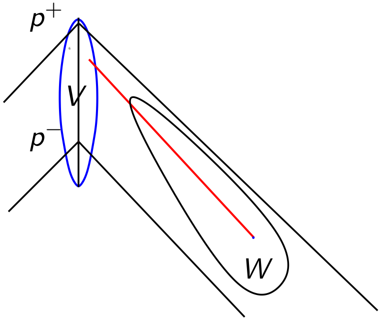
Light Observation Set

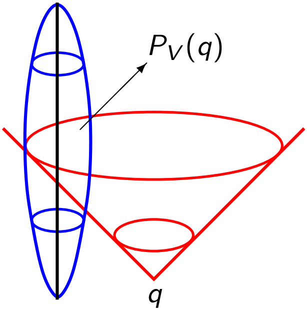
Let $\mu = \mu([-1,1]) \subset M$ be time-like geodesics containing p^- and p^+. We consider observations in a neighborhood $V \subset M$ of μ.

Let $W \subset I^-(p^+) \setminus J^-(p^-)$ be relatively compact and open set.

The light observation set for $q \in W$ is

$$P_V(q) := \{\gamma_{q,\xi}(r) \in V;\ r \geq 0,\ \xi \in L_q^+ M\}.$$





Inverse Problems with Passive Measurements

The earliest light observation set of $q \in M$ in V is

$$\mathcal{E}_V(q) = \{x \in \mathcal{P}_V(q) : \text{ there is no } y \in \mathcal{P}_V(q) \text{ and future pointing time like path } \alpha \text{ such that } \alpha(0) = y \text{ and } \alpha(1) = x\} \subset V.$$

In the physics literature the light observation sets are called light-cone cuts (Engelhardt-Horowitz, arXiv 2016)

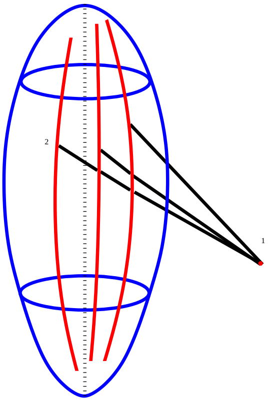
Theorem (Kurylev-Lassas-U 2018, arXiv 2014)

Let (M, g) be an open smooth globally hyperbolic Lorentzian manifold of dimension $n \geq 3$ and let $p^+, p^- \in M$ be the points of a time-like geodesic $\widehat{\mu}([-1,1]) \subset M$, $p^\pm = \widehat{\mu}(s_\pm)$. Let $V \subset M$ be a neighborhood of $\widehat{\mu}([-1,1])$ and $W \subset M$ be a relatively compact set. Assume that we know

$$\mathcal{E}_V(W).$$

Then we can determine the topological structure, the differential structure, and the conformal structure of W, up to diffeomorphism.

Conformal Class of g



1. First observation times from q on some freely falling observers μ_j, $j = 1, 2, 3, 4$ form coordinates $q \mapsto X(q) \in \mathbb{R}^4$ near q_0.
2. Given $P_V(q_0)$, find $(x, \xi) \in L^+V$ so that

   $$\gamma_{x,\xi}((-2\varepsilon, 0)) \subset \text{earliest}(P_V(q_0)).$$

   Then, find $P_V(q) \in P_V(U)$ such that

   $$\gamma_{x,\xi}((-2\varepsilon, -\varepsilon)) \subset \text{earliest}(P_V(q)).$$

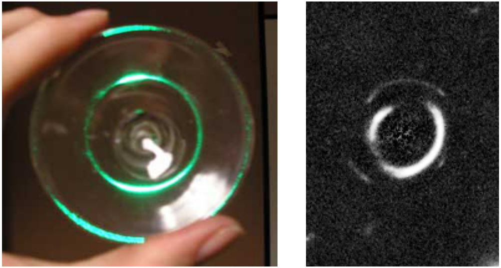
   Then q is on a geodesic $\gamma_{x,\xi}(\mathbb{R}_-)$.

   Thus we can determine many light-like geodesics in the X-coordinates.
3. The light cone $L^+_{q_0}M \subset T_{q_0}M$ is a quadratic variety and thus its open subset determines the whole light cone.

Determination of Conformal Type

The light cone $L_x^+ M \subset T_x M$ is a quadratic variety and thus real-analytic. When we are given an open subset of it, the whole surface can be determined. This determines the conformal type of the metric g at any $x \in U$.

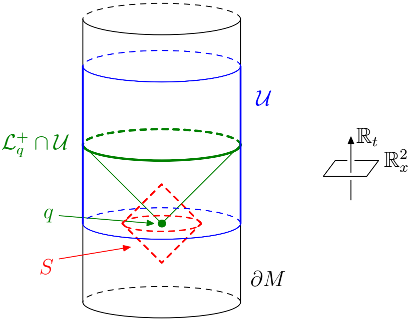
Due to caustics, there are many exceptional cases.



Figures: Wineglass by P. Doherty and Einstein's ring by R. Gavazzi and T. Treu.

Boundary Light Observation Set

$M = \{(t, x)\colon |x| < 1\} \subset \mathbb{R}^{1+2}$.



Set of sources $S \subset M^\circ$.

Observations in $\mathcal{U} \subset \partial M$.

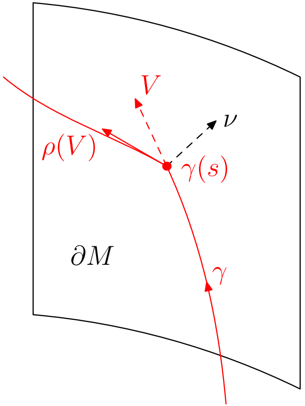
Data: $\mathscr{S} = \{\mathcal{L}_q^+ \cap \mathcal{U}\colon q \in S\}$

Theorem

The collection $\mathscr{S}$ determines the topological, differentiable, and conformal structure $[g|_S] = \{fg|_S\colon f > 0\}$ of S.

Reflection at the Boundary

γ null-geodesic until $\gamma(s) \in \partial M$.



$\rho(V) =$ reflection of V across ∂M. (Snell's law.)

$\rightarrow$ continuation of γ as broken null-geodesic

Null-convexity

Simplest case:

$$\text{All null-geodesics starting in } M^\circ \text{ hit } \partial M \text{ transversally.} \qquad (1)$$

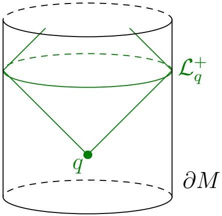
Proposition

(1) *is equivalent to null-convexity of ∂M:*

$$II(W,W) = g(\nabla_W \nu, W) \geq 0, \quad W \in T\partial M \text{ null.}$$

Stronger notion: strict null-convexity. ($II(W,W) > 0$, $W \neq 0$.)

Define light cones $\mathcal{L}_q^+$ using broken null-geodesics.



Main Result

Setup:

- (M, g) Lorentzian, dim ≥ 2, strictly null-convex boundary
- existence of $t \colon M \to \mathbb{R}$ proper, timelike
- sources: $S \subset M^\circ$ with $\bar{S}$ compact
- observations in $\mathcal{U} \subset \partial M$ open

Assumptions:

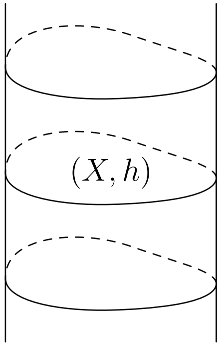
1. $\mathcal{L}^+_{q_1} \cap \mathcal{U} \neq \mathcal{L}^+_{q_2} \cap \mathcal{U}$ for $q_1 \neq q_2 \in \bar{S}$
2. points in S and $\mathcal{U}$ are not (null-)conjugate

Theorem (Hintz–U, 2019)

The smooth manifold $\mathcal{U}$ and the unlabelled collection $\mathscr{S} = \{\mathcal{L}^+_q \cap \mathcal{U} \colon q \in S\} \subset 2^{\mathcal{U}}$ uniquely determine $(S, [g|_S])$ (topologically, differentiably, and conformally).

Example for (M, g)

(X, h) compact Riemannian manifold with boundary.

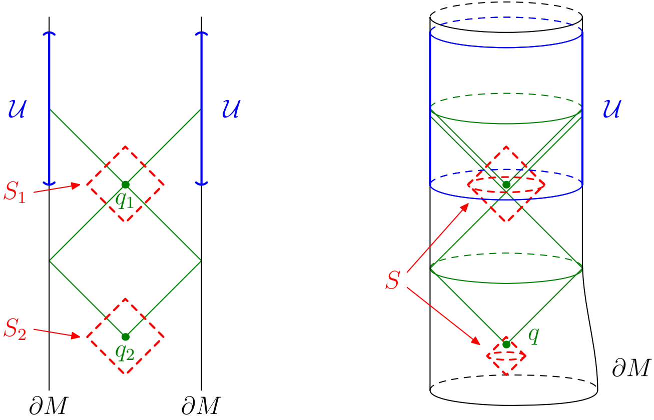


$$M = \mathbb{R}_t \times X, \quad g = -dt^2 + h.$$

(Strict) null-convexity of $\partial M \iff$ (strict) convexity of ∂X

‘Counterexamples’

Necessity of assumption 1. ($\mathcal{L}^+_{q_1} \cap \mathcal{U} \neq \mathcal{L}^+_{q_2} \cap \mathcal{U}$ for $q_1 \neq q_2 \in \bar{S}$)



S_1 and $S_1 \cup S_2$ are indistinguishable from $\mathcal{U}$.

Active and Passive Measurements

(M,g) $(2+1)$-dimensional, $\Box_g u = a(x)u^3 + f$, $a \neq 0$.

Idea (Kurylev-Lassas-U 2018, arXiv 2014): Using nonlinearity to create point sources in $I(p_-, p_+)$.

$$f = \sum_{i=1}^{3} \epsilon_i f_i, \quad u_i := \Box_g^{-1} f_i.$$

Take f_i = conormal distribution, e.g.

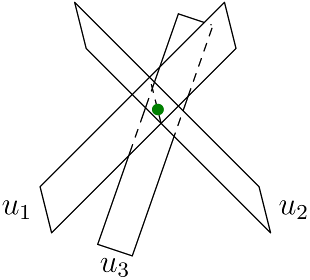
$$f_1(t,x) = (t - x_1)_+^{11} \chi(t,x), \quad \chi \in C_c^\infty(\mathbb{R}^{1+2}).$$

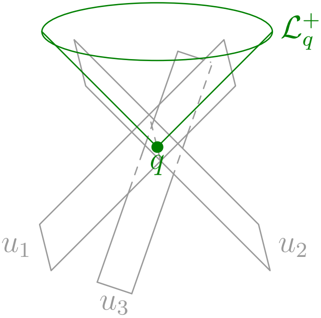
Then

$$u \approx \sum \epsilon_i u_i + 6\epsilon_1\epsilon_2\epsilon_3 \Box_g^{-1}(u_1 u_2 u_3).$$

Generating Point Sources

non-linear interaction of conormal waves $u_i = \Box_g^{-1} f_i$: $\Box_g^{-1}(u_1 u_2 u_3)$





$$q = \bigcap_{i=1}^{3} \mathit{sing\ supp}\ u_i, \quad \mathcal{L}_q^+ = \mathit{sing\ supp}\ \Box_g^{-1}(u_1 u_2 u_3)$$

$\Rightarrow$ singularities of $\partial^3_{\epsilon_1 \epsilon_2 \epsilon_3} u$ give light observation sets $\mathcal{L}_q^+$

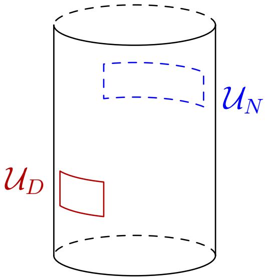
Active Measurements for Boundary Value Problems

Theorem (Hintz-U-Zhai, 2021)

Model (in $\dim M = 1 + 2$)

$$\Box_g u = a(x)u^3, \; a \neq 0, \quad u|_{\mathcal{U}_D} = u_0 \in \mathcal{C}_c^{10}(\mathcal{U}_D).$$

Measure $L\colon u_0 \mapsto \partial_\nu u|_{\mathcal{U}_N}$. Recover a and g from L.



(Special case: $\mathcal{U}_N = \mathcal{U}_D$.)

Propagation of singularities: (strict) null-convexity assumption simplifies structure of null-geodesic flow. (Taylor '75, '76, Melrose–Sjöstrand '78, '82.)

Main Result

Setup:

- (M, g) Lorentzian with strictly null-convex boundary
- existence of $t\colon M \to \mathbb{R}$ proper, timelike
- sources: $S \subset M^\circ$ with $\bar{S}$ compact
- observations in $\mathcal{U} \subset \partial M$ open

Assumptions:

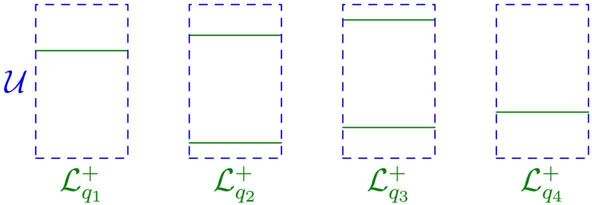
1. $\mathcal{L}^+_{q_1} \neq \mathcal{L}^+_{q_2}$ for $q_1 \neq q_2 \in \bar{S}$
2. points in S and $\mathcal{U}$ are not (null-)conjugate

Theorem (Hintz–U, 2019)

The smooth manifold $\mathcal{U}$ and the unlabelled collection $\mathscr{S} = \{\mathcal{L}^+_q \cap \mathcal{U}\colon q \in S\} \subset 2^{\mathcal{U}}$ uniquely determine $(S, [g|_S])$ (topologically, differentiably, and conformally).

Generalities

- reference structures on $\mathscr{S}$ via $S \ni q \mapsto \mathcal{L}_q^+ \cap \mathcal{U} \in \mathscr{S}$
- Goal: describe reference structures only using $\mathscr{S}$ and $\mathcal{U}$
- Difficulty: no good intrinsically defined global embedding $\mathscr{S} \hookrightarrow$ Banach space



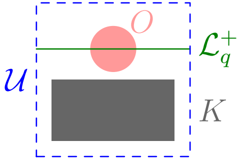
$\Rightarrow$ reconstruct reference structures more directly/locally

Topology

Define subbasis of a topology $\mathcal{T}$ on $\mathscr{S}$ by declaring

$$\{L \in \mathscr{S} : L \cap O \neq \emptyset\}, \quad O \subset \mathcal{U} \text{ open},$$
$$\{L \in \mathscr{S} : L \cap K = \emptyset\}, \quad K \subset \mathcal{U} \text{ compact},$$

to be open.



Proposition

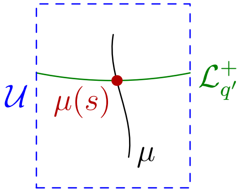
$\mathcal{T}$ is equal to the reference topology on $\mathscr{S}$ (from $S \cong \mathscr{S}$).

Smooth Structure

Define smooth functions on neighborhoods of $L = \mathcal{L}_q^+ \cap \mathcal{U}$ in $\mathscr{S}$:

For $\mu\colon (-1,1) \to \mathcal{U}$, $\mathcal{C}^\infty$ curve, transversal to L, $\#(\mu \cap L) = 1$, define 'observation time' for q' near q:

$$x^\mu(\mathcal{L}_{q'}^+ \cap \mathcal{U}) = s \in (-1,1) \ \text{ s.t. } \{\mu(s)\} = \mu \cap \mathcal{L}_{q'}^+.$$



x^μ is smooth in a maximal connected open subset $O_\mu \ni L$ of $\mathscr{S}$.
(Uses assumption: no conjugate points.)

Smooth Structure

Proposition

For all $q \in S$, there exist $n+1$ curves μ_i, $i = 0, \ldots, n$, as above such that

$$dx^{\mu_0}|_q, \ldots, dx^{\mu_n}|_q \text{ are linearly independent.}$$

Define subalgebra $C \subset \mathcal{C}^0(\mathscr{S})$: $f \in C$ iff near each $L \in \mathscr{S}$, f is a smooth function of some finite collection of functions x^{μ_i}.

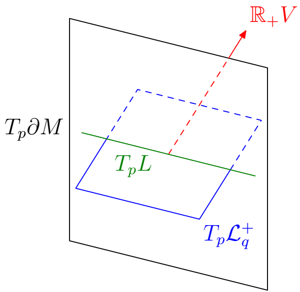
Corollary

C is equal to the reference structure $\mathcal{C}^\infty(\mathscr{S})$ (from $S \cong \mathscr{S}$).

(This also determines local coordinate systems.)

Conformal Structure

At $p \in \mathcal{U}$ where $L = \mathcal{L}_q^+ \cap \mathcal{U}$ is $\mathcal{C}^\infty$, L uniquely determines the ray of future-directed outward pointing null vectors $\mathbb{R}_+ V \subset T_p M$.

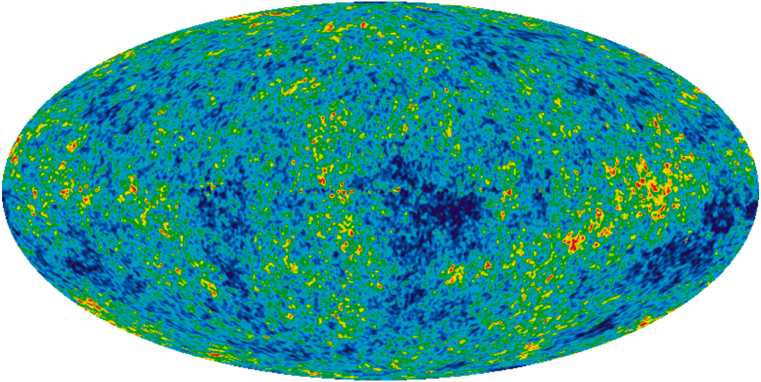


The broken null-geodesic $\gamma(s) = \exp_p(-sV)$ contains $q = \gamma(s_0)$.

$\Rightarrow$ Null-geodesic segments $(-1, 1) \ni s \mapsto \mu(s) \in S$ with $(\mu(0), \mu'(0)) = (q, \gamma'(s_0))$ have constant $T_p(\mathcal{L}_{\mu(s)}^+ \cap \mathcal{U})$.

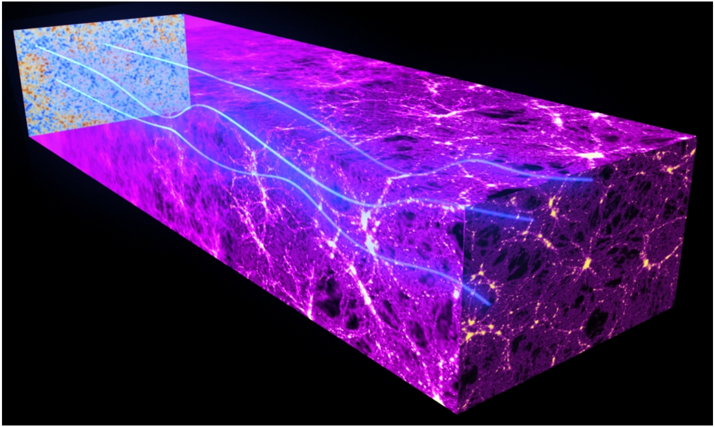
$\Rightarrow$ Varying p, reconstruct open subset of analytic submanifold $L_q M$, $q \in S$. Uniquely determines conformal class of $g|_S$.

Cosmic Microwave Background (CMB)



When the universe cooled enough due to expansion, around 380,000 years after the Big Bang, protons and electrons combined to form atoms. These atoms could no longer absorb the thermal radiation, and the photons that existed at that time have been propagating ever since.

Gravitational Lensing



CMB photons are deflected by the gravitational lensing effect of massive cosmic structures as they travel across the Universe. The gravitational lensing effect is seen e.g. in the data from ESA's Planck satellite.

Photons in the Theory of General Relativity

Let (M, g) be a $1+3$ dimensional Lorentzian manifold. On each tangent plane T_pM, $p \in M$, there is a basis $V_0, \ldots, V_3$ such that in this basis

$$g(p) = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

We suppose that (M, g) is time-oriented, that is, there is a vector field X on M satisfying $(X, X)_g < 0$ everywhere. We say that X is timelike and gives the direction of the future.

A geodesic $\beta : [0, \ell] \to M$ models a *photon* if it is lightlike and future pointing, that is, $(\dot\beta(\tau), \dot\beta(\tau))_g = 0$ and $(\dot\beta(\tau), X)_g < 0$ for one and hence for all $\tau \in [0, \ell]$.

Observers and Energy Measurements

A point $(p, Z) \in TM$ is called an *observer* if Z is future pointing and $(Z, Z)_g = -1$.

If $\beta : [0, \ell] \to M$ is a photon and $\beta(\ell) = p$. Then the *energy* E and *Newtonian velocity* V of β as measured by (p, Z) are

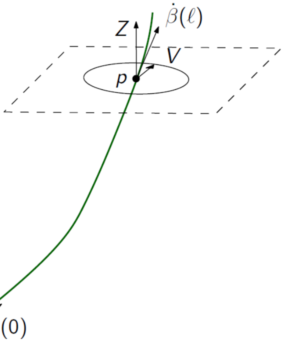
$$E = -(\dot{\beta}(\ell), Z)_g, \qquad V = \frac{\dot{\beta}(\ell)}{E} - Z.$$

The energies of CMB photons as measured by (Z, p) can be parametrized by the velocities V. The velocity V satisfies

$$(V, Z)_g = 0, \quad (V, V)_g = 1. \tag{2}$$

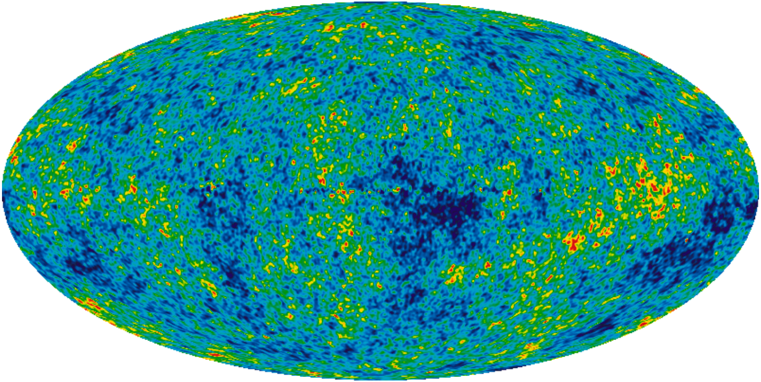
The equations (2) define the *celestial sphere* of (p, Z). The physical meaning of the celestial sphere is "all the directions in the sky of (p, Z)".

A Photon and an Observer



The observer (p, Z) measures the energy $E = -(\dot\beta(\ell), Z)_g$ of the CMB photon β coming from the direction V in the celestial sphere.

“Energies” on the Celestial Sphere of the Planck Satellite



A map of CMB radiation on the celestial sphere of the Planck satellite. Color indicates the temperature of the black body corresponding to the spectrum of the radiation. CMB is the most perfect black body ever measured in nature.

Gauge Invariances of Energy Measurements

If $\beta : [0, \ell] \to M$ is a photon and $\beta(\ell) = p$. Then the *energy* E and *Newtonian velocity* V of β as measured by (p, Z) are

$$E = -(\dot\beta(\ell), Z)_g, \qquad V = \frac{\dot\beta(\ell)}{E} - Z.$$

The measurement depends on g, the initial conditions $(\beta(0), \dot\beta(0))$, and the observer (p, Z). If $(\beta(0), \dot\beta(0))$ and (p, Z) are fixed, then the measurement is invariant under the following two transformations:

1. $g \mapsto \Phi^* g$ where $\Phi : M \to M$ is a diffeomorphism fixing the end points $\beta(0)$ and $\beta(\ell) = p$.
2. $g \mapsto cg$ where c is a strictly positive conformal scaling satisfying $c = 1$ at the end points $\beta(0)$ and $\beta(\ell) = p$.

The conformal invariance follows from the fact that lightlike geodesics of g and cg coincide up to a reparametrization.

A Simple Mathematical Model for CMB Measurements

- $M = (0, \infty) \times \mathbb{R}^3$ and the metric tensor g is close to

$$g_0(t, y) = -dt^2 + a^2(t)dy^2, \quad (t, y) \in (0, \infty) \times \mathbb{R}^3,$$

  where the warping factor $a(t)$ is strictly positive. For example, $a(t) = t^{2/3}$ gives the Einstein-de Sitter cosmological model.

- The CMB photons are emitted with a fixed energy E_0 uniformly in all future pointing lightlike directions on

$$\Sigma = \{(t_0, y);\ y \in \mathbb{R}^3\}.$$

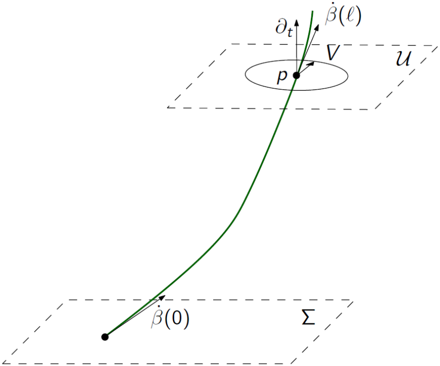
  The physical meaning of $t_0 > 0$ is "380,000 years after the Big Bang".

- The CMB photons are observed by (p, ∂_t), $p \in \mathcal{U}$, where

$$\mathcal{U} = \{t_1\} \times \mathcal{U}_1.$$

  Here $t_1 > t_0$ and $\mathcal{U}_1 \subset \mathbb{R}^3$ is open.

Parametrization of the CMB Measurements



The observer (p, ∂_t) measures the energy $E_g(p, V)$ of the CMB photon β coming from the direction V in the celestial sphere. Here $p = (t_1, y)$ and $y \in \mathcal{U}_1 \subset \mathbb{R}^3$.

Linearization of the CMB Measurements

Let g_ϵ, $\epsilon \in [0,1]$, be a one parameter family of Lorentzian metric tensor on $(0,\infty) \times \mathbb{R}^3$, and suppose that

$$g_0(t,y) = -dt^2 + a^2(t)dy^2, \quad (t,y) \in (0,\infty) \times \mathbb{R}^3.$$

We define the *redshift* R_ϵ, $\epsilon \in [0,1]$, by

$$R_\epsilon(p,V) = \frac{E_0}{E_{g_\epsilon}(p,V)} - 1,$$

where $E_{g_\epsilon}(p,V)$ is the energy of the CMB photon with respect to g_ϵ, coming from the direction V, as measured by (p,∂_t).

Linearized CMB inverse problem. Given $\partial_\epsilon R_\epsilon|_{\epsilon=0} = E_0 \partial_\epsilon E_{g_\epsilon}^{-1}|_{\epsilon=0}$ determine $\partial_\epsilon g_\epsilon|_{\epsilon=0}$ up to natural invariances.

The Light Ray Transform

We define the *light ray transform* $Lf(p,V)$ of a 2-tensor f by

$$Lf(p,V)=\int_{\mathbb{R}} f_{jk}(\gamma(\tau))\dot\gamma^j(\tau)\dot\gamma^k(\tau)d\tau,$$

where V is a vector in the celestial sphere S^2_p of (p,∂_t), and γ is the geodesic on (M,g_0) with the initial data $\gamma(0)=p$, $\dot\gamma(0)=V+\partial_t$.

Theorem [Lassas-Oksanen-Stefanov-U, 2019; Sachs-Wolfe, 1967].
Let g_ϵ, $\epsilon\in[0,1]$, be a one parameter family[1] of Lorentzian metric tensors on $(0,\infty)\times\mathbb{R}^3$. Suppose that $g_0(t,y)=-dt^2+a^2(t)dy^2$, and that $g_\epsilon=g_0$ in $\Sigma\cup\mathcal{U}$. Then

$$\partial_\epsilon R_\epsilon|_{\epsilon=0}(p,V)=Lf(p,V),\quad p\in\mathcal{U},\ V\in S^2_p,$$

where $f=(2a(t_0))^{-1}a^2\mathcal{L}_{a\partial_t}a^{-2}\partial_\epsilon g_\epsilon|_{\epsilon=0}$ on $M_1=(t_0,t_1)\times\mathbb{R}^3$ and $f=0$ elsewhere. Here $\mathcal{L}$ is Lie derivative, $\Sigma=\{t_0\}\times\mathbb{R}^3$ and $\mathcal{U}\subset\{t_1\}\times\mathbb{R}^3$.

[1] C^1 in ϵ and C^2 in (t,y)

Light Ray Transform

Integral of a function (or distribution) over light rays

$$Lf(x,\theta) = \int f(s, x + s\theta)\,ds, \quad (x,\theta) \in \mathbb{R}^n \times S^{n-1}$$

Fourier Slice Theorem: for any $f \in \mathcal{S}(\mathbb{R}^{n+1})$,

$$\widehat{f}(\zeta) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} Lf(x,\theta)\,dx, \quad \text{when } (1,\theta) \perp \zeta, \ \theta \in S^{n-1},$$

where $\zeta = (\tau, \xi) \in T^*_{(t,x)}\mathbb{R}^{1+n}$.

- knowing $Lf(\cdot,\theta)$ for some $\theta \in S^{n-1}$, then we know all $\widehat{f}(\zeta)$ for ζ on the plane $\tau + \xi \cdot \theta = 0$
- if $Lf = 0$, then $\widehat{f}(\zeta) = 0$ for all ζ satisfying $|\tau| \leq |\xi|$

Injectivity on Light Ray Transform

The light ray transform L is injective on $C_0^\infty(\mathbb{R}^{1+n})$.

- $f \in C_0^\infty(\mathbb{R}^{1+n}) \Rightarrow \widehat{f}$ is real analytic
- if $Lf = 0$, then $\widehat{f} = 0$ in the cone $|\tau| \le |\xi|$
- real analytic functions vanishing in an open set $\Rightarrow f = 0$

The light ray transform L is not injective on $\mathcal{S}(\mathbb{R}^{1+n})$.

- let $\psi \in C_0^\infty(\mathbb{R}^{1+n})$ supported in the cone $|\tau| > |\xi|$
- set $f = \check{\psi}$, then $\widehat{f}(-\theta \cdot \xi, \xi) = 0$, for all $\theta \in S^{n-1}$ and all ξ
  $\Rightarrow \quad Lf = 0$
- thus, we can construct $f \neq 0$ but $Lf = 0$

Normal Operator

$$L^*Lf(t,x) = \int_{S^{n-1}} \int_{\mathbb{R}} f(s, x - t\theta + s\theta)\, d s\, d\theta$$
$$= \int_{\mathbb{R}^n} \frac{f(t - |x - x'|, x') + f(t + |x - x'|, x')}{|x - x'|^{n-1}}\, d x',$$

which has the Schwartz kernel

$$N(t,x;t',x') = \frac{\delta(t - t' - |x - x'|, x') + \delta(t - t' + |x - x'|, x')}{|x - x'|^{n-1}}.$$

Here we define $\delta(t \mp |x|)/|x|^{n-1}$ as the linear functional

$$\phi(t,x) \mapsto \int \frac{\phi(\pm|x|, x)}{|x|^{n-1}}\, d x,$$

for $\phi \in C_0^\infty(\mathbb{R}^{1+n})$.

Normal Operator

L^*L is a convolution:

$$L^*L = \mathcal{N} * f, \quad \mathcal{N}(t,x) = \frac{\delta(t-|x|)+\delta(t+|x|)}{|x|^{n-1}}$$

- $\mathcal{N}$ is a tempered distribution homogeneous of order $-n$

If we denote by $\mathcal{F}f$ ($\mathcal{F}^{-1}f$) the FT (inverse FT) of f, then

$$L^*Lf = 2\pi|S^{n-2}|\mathcal{F}^{-1}\frac{(|\xi|^2-\tau^2)_+^{\frac{3}{2}}}{|\xi|^{n-2}}\mathcal{F}f, \quad f \in \mathcal{S}(\mathbb{R}^{1+n})$$

- the FT $\mathcal{F}f$ can be constructed stably in the timelike cone
- the estimate deteriorates at the light cone
- no stable inversion can be done in the space-like cone

Linearized Gauge Invariances

We recall that $\Sigma = \{t_0\} \times \mathbb{R}^3$, $\mathcal{U} \subset \{t_1\} \times \mathbb{R}^3$ and $M_1 = (t_0, t_1) \times \mathbb{R}^3$.

The energy $E_g(p, V), p \in \mathcal{U}$, is invariant under

1. diffeomorphisms $g \mapsto \Phi^* g$ fixing $\Sigma \cup \mathcal{U}$.
2. conformal scalings $g \mapsto cg$ with $c = 1$ on $\Sigma \cup \mathcal{U}$.

These correspond to subspaces in the null space of the light ray transform

1. $L(d^s \omega) = 0$ for 1-forms ω supported on M_1.
2. $L(cg_0) = 0$ for functions c supported on M_1.

Here d^s is the symmetric differential defined in coordinates as follows

$$(d^s \omega)_{ij} = \frac{(\nabla_i \omega)_j + (\nabla_j \omega)_i}{2},$$

and $\nabla_i = \nabla_{\partial_{x^i}}$, $x = (t, y)$, is the covariant derivative with respect to g_0.

Microlocal Inversion of the Light Ray Transform

We write $g = g_0$ where $g_0(t, y) = -dt^2 + a^2(t)dy^2$ as before.
We recall that $\Sigma = \{t_0\} \times \mathbb{R}^3$, $\mathcal{U} \subset \{t_1\} \times \mathbb{R}^3$ and
$M_1 = (t_0, t_1) \times \mathbb{R}^3$.

Theorem [Lassas-Oksanen-Stefanov-U, 2019].
Let $(x, \xi) \in T^*M_1$ be spacelike, that is, $(\xi, \xi)_g > 0$. Suppose that there is a lightlike geodesic γ of (M, g) and $\tau_1, \tau_2 \in \mathbb{R}$ such that

$$\gamma(\tau_1) = x, \quad \xi(\dot\gamma(\tau_1)) = 0, \quad \gamma(\tau_2) \in \mathcal{U}. \tag{3}$$

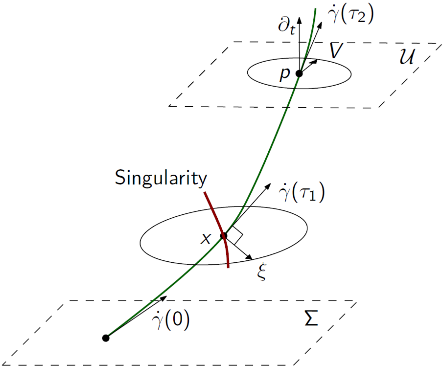
Then there is a microlocal cutoff χ vanishing outside $\mathcal{U}$ such that for all 2-tensors f the following are equivalent

(i) $(x, \xi) \in \mathrm{WF}(L^*\chi Lf)$,

(ii) $(x, \xi) \in \mathrm{WF}(f + h)$ for all h of the form $h = d^s\omega + cg$.

Moreover, L is smoothing on timelike covectors in T^*M_1, and also on the spacelike covectors that do not satisfy the visibility condition (3).

The Visibility Condition



The visibility condition for spacelike (x, ξ): there is a lightlike geodesic γ and $\tau_1, \tau_2 \in \mathbb{R}$ such that $\gamma(\tau_1) = x$, $\xi(\dot{\gamma}(\tau_1)) = 0$ and $\gamma(\tau_2) \in \mathcal{U}$.

Note the analogy with the limited angle X-ray tomography.

Microlocal Inversion of the Light Ray Transform

Theorem [Lassas-Oksanen-Stefanov-U, 2019].
Let $(x, \xi) \in T^*M_1$ be spacelike and suppose that it satisfies the visibility condition. Then there is a microlocal cutoff χ vanishing outside $\mathcal{U}$ such that for all 2-tensors f the following are equivalent

(i) $(x, \xi) \in \mathrm{WF}(L^*\chi L f)$,

(ii) $(x, \xi) \in \mathrm{WF}(f + h)$ for all h of the form $h = d^s\omega + cg$.

Moreover, L is smoothing on timelike covectors in T^*M_1, and also on the spacelike covectors that do not satisfy the visibility condition.

- The theorem is sharp except that it doesn't cover the case of lightlike $(x, \xi) \in T^*M_1$, that is, $(\xi, \xi)_g = 0$. This is an open question.
- The cutoff χ can be chosen so that $L^*\chi L$ is a pseudodifferential operator of order -1 and its principal symbol can be given explicitly.
- Theorem is invariant under diffeomorphisms and conformal scalings.

The Principal Symbol of the Normal Operator

After a change of coordinates and a conformal transformation we have $g = -dt^2 + dy^2$. We write $x = (x^0, x') \in \mathbb{R}^{1+3}$ and $\xi = (\xi_0, \xi') \in \mathbb{R}^{1+3}$. The cutoff χ can be chosen so that $L^*\chi L$ has the principal symbol

$$\sigma(x,\xi) = \frac{2\pi\chi_1(|\xi_0|/|\xi'|)}{\sqrt{|\xi'|^2 - |\xi_0|^2}} N^{jklm}, \quad N^{jklm} = \int_{S^1_\xi} \chi_2(x' - x^0 v)\theta^j\theta^k\theta^l\theta^m dv,$$

where $\chi_1 \in C_0^\infty(-1,1)$, $\chi_2 \in C_0^\infty(\mathcal{U}_1)$, $\mathcal{U} = \{0\} \times \mathcal{U}_1$, $\theta = (1, v) \in \mathbb{R}^{1+3}$, $S^1_\xi = \{v \in S^2;\ \xi_0 + \xi' v = 0\}$, and $j,k,l,m \in \{0,1,2,3\}$.

- χ_1 localizes on spacelike covectors, and χ_2 localizes on the set where we have data.
- N^{jklm} is homogeneous of degree zero in ξ.
- If $\chi_2(x' - x^0 v) \geq 0$ and does not vanish identically then the null space of the linear map $N : f_{lm} \mapsto N^{jklm} f_{lm}$ on 2-tensors f is

$$\{c g_{lm} + \xi_l \omega_m + \xi_m \omega_l;\ c \in \mathbb{R},\ \omega \in \mathbb{R}^4\}.$$

This corresponds to the gauge invariance.

On the Microlocal Structure of the Light Ray Transform

- L is a Fourier integral operator of order $-3/4$ whose canonical relation is the (twisted) conormal bundle of the point-line relation corresponding to the light rays.
- The canonical relation satisfies the Bolker condition away from the lightlike covectors, and the clean intersection calculus implies that $L^*\chi L$ is a pseudodifferential operator when χ cuts off the lightlike covectors.
- When the lightlike covectors are *not* cut off, the canonical relation is a $(1,2)$-fibered folding canonical relation in the sense of [Greenleaf-U, 1991], and $L^*\chi L$ is an IPL class operator.

Backprojection in the "Full Angle" Case

Let us consider the case $g = -dt^2 + dy^2$, and suppose (unrealistically) that we have data the whole slice $\{0\} \times \mathbb{R}^3$. We parametrize the light ray transform

$$Lf(y, v) = \int_{\mathbb{R}} f_{lm}(s, y + sv)\theta^l \theta^m ds, \quad y \in \mathbb{R}^3, \ v \in S^2,$$

where $\theta = (1, v) \in \mathbb{R}^{1+3}$. Then the normal operator L^*L is the convolution $K^{jklm} * f_{lm}$, with the kernel

$$(K^{jklm}, \phi)_{\mathcal{D}' \times \mathcal{D}(\mathbb{R}^4)} = \int_{S^2} \int_{\mathbb{R}} \theta^j \theta^k \theta^l \theta^m \phi(\rho\theta) d\rho dv.$$

The kernel is supported on the light cone $\{(x^0, x') \in \mathbb{R}^{1+3}; \ |x^0| = |x'|\}$.

Fourier Transform of the Backprojection

We write $a(v) = \theta^j\theta^k\theta^l\theta^m$, $\theta = (1, v)$, and have

$$(\widehat{K}^{jklm}, \phi) = \int_{S^2}\int_{\mathbb{R}} a(v)\int_{\mathbb{R}^4} e^{-i\xi(\rho\theta)}\phi(\xi)d\xi d\rho dv$$
$$= 2\pi\int_{\mathbb{R}^4}\int_{S^2}\delta(\xi\theta)a(v)dv\phi(\xi)d\xi.$$

The equation $\xi\theta = 0$ for v defines the affine plane $\xi^0 + \xi' v = 0$.

- If ξ is timelike (that is, $|\xi_0| > |\xi'|$) then the affine plane does not intersect with the unit sphere $S^2 \subset \mathbb{R}^3$. Hence $\widehat{K}^{jklm}(\xi) = 0$.
- If ξ is spacelike (that is, $|\xi_0| < |\xi'|$) then

$$\widehat{K}^{jklm}(\xi) = 2\pi\int_{S^2}\delta(\xi\theta)a(v)dv = \frac{2\pi}{\sqrt{|\xi'|^2 - |\xi_0|^2}}\int_{S^1_\xi} a(v)dv.$$

Here $S^1_\xi = \{v \in S^2;\ \xi\theta = 0\}$ is a circle of radius $|\xi'|^{-1}\sqrt{|\xi'|^2 - |\xi_0|^2}$.