Gravitational singularities, massive fields, and asymptotic localization

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- Recent advances on mathematical modeling in GR
- Benefits to numerical relativity from ideas and techniques
  - asymptotic effects on nonlinear waves
  - new structure of the Einstein equations

– structure-preserving numerical algorithms — simple model problems

complex fluids, multi-scale, shocks, fixed background

– universal scattering maps — gravitational singularities

– global dynamics of massive fields in f(R)-gravity — hyperbolic PDEs

– asymptotic localization method — initial data sets, constraint equations

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Main collaborators: B. Le Floch (Paris), G. Veneziano (Geneva), T.-C. Nguyen (Paris), Y. Ma (Xi'an), F. Beyer (Dunedin)

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1. Structure-preserving algorithms for multi-scale waves

From first principles of continuum physics

Multi-scale wave phenomena

- several parameters — viscosity, surface tension, heat, Hall effect, friction
- competitive effects — several scales (fluid, geometry)
- fine-scale structure — oscillations, turbulence

Massive fields and interfaces

- Klein-Gordon, complex fluids, modified gravity beyond Einstein gravity
- global dynamics of shocks, moving material interfaces, phase boundaries
- impulsive gravitational waves, cosmological singularities

Fluids, gases, plasmas, solid materials

- liquid-vapor flows, thin liquid films, combustion waves, bores in shallow water, astrophysical flows, neutron stars, phase transformations

Scale-sensitive nonlinear waves

- regime where one can extract variables with well-defined limits
- junction laws, scattering laws — under-compressive shocks, determine the dynamics

Diffusive-dispersive nonlinear waves

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}\rho^3 = \varepsilon \frac{\partial^2 \rho}{\partial x^2} + \kappa \frac{\partial^3 \rho}{\partial x^3}$$

- conservation law

  non-convex equation of state

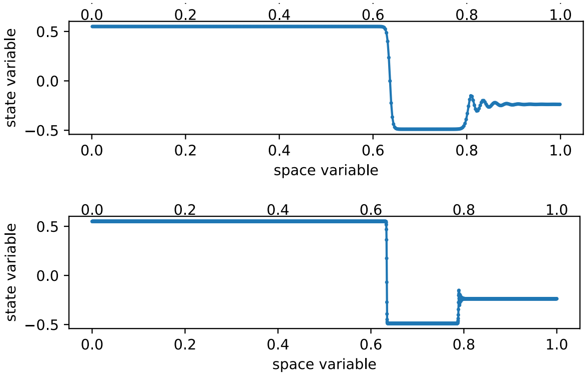
  phase transition phenomena, magnetohydrodynamics

- fluid density $\rho = \rho(t, x)$
- small viscosity coefficient ε and surface tension/capillarity coefficient κ

  Intermolecular forces between a liquid and its surroundings

Riemann problem single initial discontinuity, dam breaking problem

complex wave patterns



Three possible asymptotic regimes $\varepsilon \to 0$

- $\kappa << \varepsilon^2$ (viscosity is dominant) — no oscillations, single limit
- $\kappa = \alpha\,\varepsilon^2$ (balanced regime) — mild oscillations, α-dependent limit
- $\kappa >> \varepsilon^2$ (surface tension is dominant, α fixed) — oscillations, no limit

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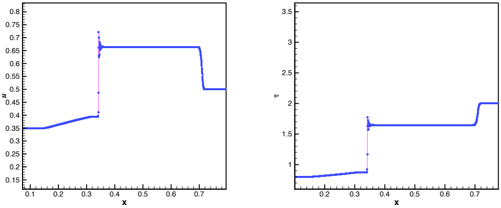
Isothermal compressible fluids

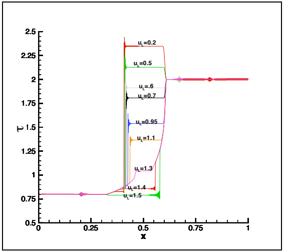
Van der Waals fluid
two coupled conservation laws

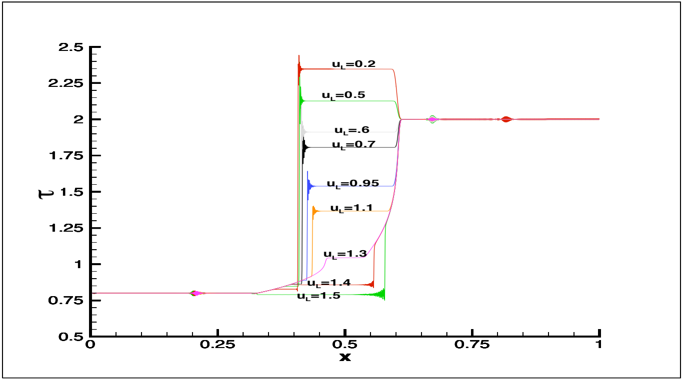
$$\tau_t - u_x = 0, \qquad u_t + p(v)_x = \varepsilon\, u_{xx} - \kappa\, \tau_{xxx}$$

specific volume τ velocity u pressure $p(\tau) = \frac{8T}{3\tau - 1} - 3/\tau^2$

Typical Riemann wave structure







Small-scale dependent nonlinear waves

- varying the ratio surface tension/viscosity $\kappa = \alpha\, \varepsilon^2$
- rules for connecting left- and right-hand state values from both sides
- beyond the standard Rankine-Hugoniot relations!
- notion of a kinetic function/scattering map for interfaces

Need structure-preserving algorithms

- front tracking
- shock capturing with well-controled dissipation

Preserving the asymptotic structure on an inhomogeneous FLRW background

joint with Y. Cao (Paris) and M. Ghazizadeh (Ottawa)
ArXiv:1912.13439

Formulation of the problem

- $2+1$ dim., isothermal, relativistic compressible flow $p(\rho) = k^2 \rho$
- FLRW-type cosmological background, with small inhomogeneities
- future-contracting geometry ($t < 0$ and $t \to 0$) $\rho \to +\infty$

Asymptotic behavior toward the cosmological singularity

- nonlinear hyperbolic systems on a curved geometry
  $\partial_t U + \partial_x F(t, x, U) = H(t, x, U)$
- two competitive effects
  contracting geometry
  shock propagation, nonlinear interactions
- small-scale structure, driven by the background geometry

analogy with phase transition dynamics

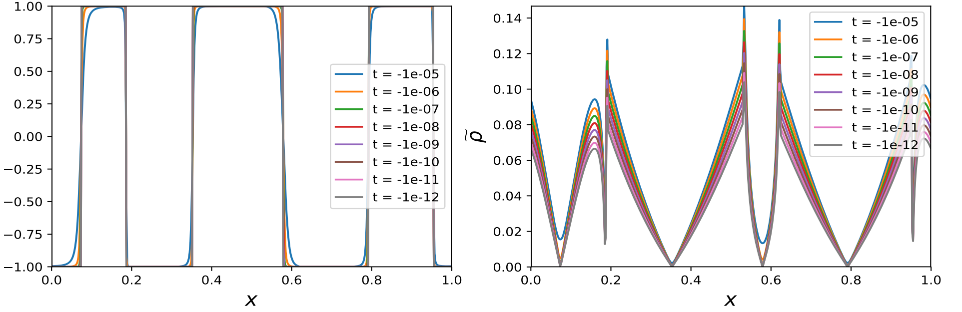
Structure-preserving methodology

- **divergence form:** finite volume scheme, shock-capturing (speed)
- **high accuracy:** 4th-order in time, 2nd-order in space, oscillation-free
- **well-balanced property**
  - introduce suitably rescaled unknowns (Fuchsian PDE method)
  - enforce the asymptotic state equations at the discrete level
  - enforce commutation property

$$\lim_{t \to 0} \lim_{\Delta x \to 0} U = \lim_{\Delta x \to 0} \lim_{t \to 0} U$$

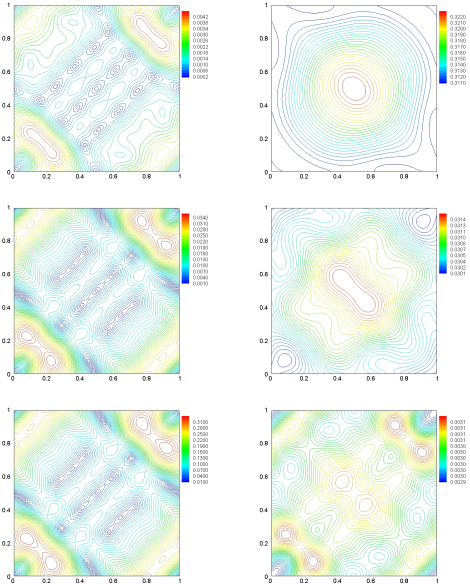
Typical behavior: sharp transitions with spikes

plots of the rescaled velocity component u and rescaled density $\widetilde{\rho}$



Standard algorithm Times $t = -10^{-1}, -10^{-3}, -10^{-5}$

Velocity magnitude V Rescaled density $\widetilde{\rho}$

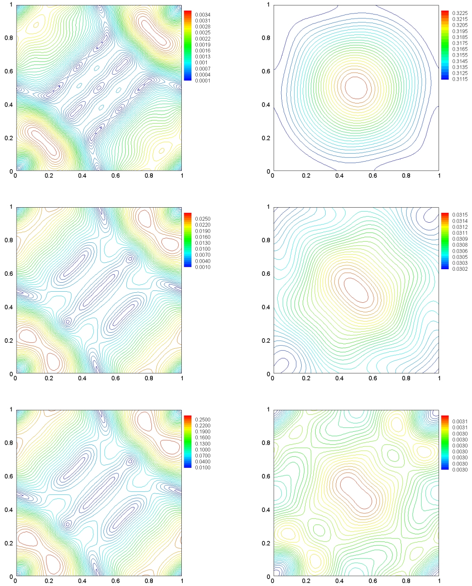


Well-balanced algorithm

Velocity magnitude V

Times $t = -10^{-1}, -10^{-3}, -10^{-5}$

Rescaled density $\widetilde{\rho}$



Preserving the asymptotic structure on a Kasner background: *evolution from the singularity*

joint with F. Beyer (Dunedin) ArXiv:2005.13504

Kasner geometry spatially homogeneous, anisotropic vacuum solution

$$g = t^{(K^2-)/2}\big(- dt^2 + dx^2 + t^{1-K} dy^2 + t^{1+K} dz^2 M = (0, +\infty) \times \mathbb{T}^3$$

with asymptotic velocity $K \in \mathbb{R}$ and Kasner exponents

$$p_1 = \frac{K^2 - 1}{K^2 + 3}, \quad p_2 = \frac{2(1-K)}{K^2+3}, \quad p_3 = \frac{2(1+K)}{K^2+3}$$

Compressible fluid flow with pressure law

$$p = (\gamma - 1)\rho \quad \text{with } \gamma \in (1,2)$$

Characteristic exponent

$$\Gamma = \frac{1}{4}\big(3\gamma - 2 - K^2(2-\gamma)\big) \in (0,1)$$

which compare the geometry and fluid behaviors

- $\Gamma > 0$: sub-critical regime dynamically stable
- $\Gamma \leqslant 0$: super-critical / critical regimes dynamically unstable

Formally, plug an expansion in power of t and attempt to validate it (Fuchsian asymptotics)

Evolution from the cosmological singularity $t = 0$

- formulate a singular initial value problem
$$B^0(U, t, x)\partial_t U + B^1(U, t, x)\partial_x U = f(U, t, x)$$
- suitable "singular initial data" prescribed on $t = 0$
- Fuchsian-type expansions near the cosmological singularity
- sufficiently regular, shock-free regime

Algorithm preserving the Fuchsian structure

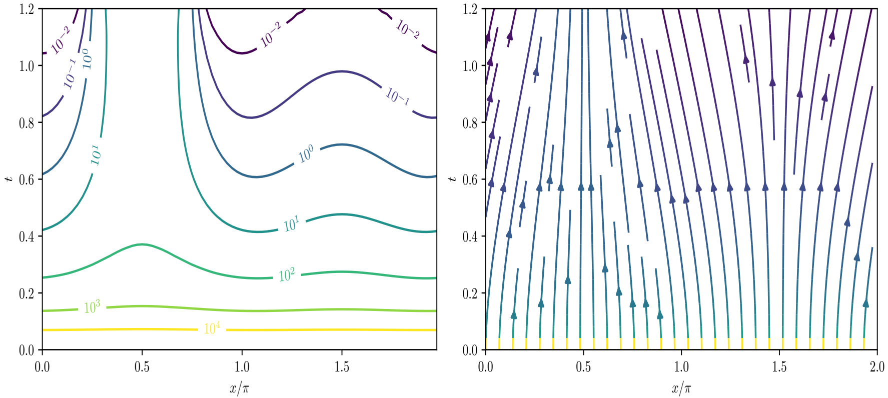
- discretize $U(t, x) \simeq V(t) = (V_j(t))$ by the pseudo-spectral method of lines
$$\partial_t V - AV = h(V, t)$$
- high-order Runge-Kutta discretization in time
- introduce **suitably rescaled variables**
- **careful study of the numerical error**
    - take into account the Fuchsian expansion
    - two sources of approximation error: continuum / discrete
    - our proposal : **keep the two error sources asymptotically in balance**

With this numerical strategy, we demonstrated the **nonlinear stability of the flow** near the cosmological singularity in the sub-critical regime.

Numerical simulations on a Kasner background

– fluid density (contour plot) and velocity field (flow lines)
– time (vertically)

- density ρ unbounded as the time $t \to 0$
- carefully check the numerical error
- reliable and accurate algorithm, despite the solutions being **highly singular**



See ArXiv:2005.13504 fur further numerical results.
Extension to self-gravitating fluids in progress.

2. Universal scattering laws for gravitational singularities

ArXiv:2005.11324 & ArXiv:2005.11324

joint work with B. Le Floch (Paris) and G. Veneziano (Geneva)

– self-gravitating scalar field — also: stiff fluid, compressible fluid

– bouncing cosmologies — contracting/expanding

– vicinity of a singularity hypersurface

– strong fields, Penrose-Hawking singularity theorems (incompleteness), BKL theory, etc.

Large literature — Penrose, Tod, Lübbe, Turok, Barrow, etc.

symmetric spacetimes and special junctions

Formulation of the problem ?

- classes of physically meaningful junction conditions
- degrees of freedom, constraints at the singularity

Main results

- systematic study of (past, future) **singularity data** $(g^{\pm}, K^{\pm}, \phi_0^{\pm}, \phi_1^{\pm})$
- **singularity scattering map** $(g^{-}, K^{-}, \phi_0^{-}, \phi_1^{-}) \mapsto (g^{+}, K^{+}, \phi_0^{+}, \phi_1^{+})$
- fully classify the possible bouncing conditions
- we distinguish between:
  - **universal** scattering laws
  - **model-dependent** scattering maps

Gravitational singularities: main message

Physics literature on bouncing.

- pre-Big Bang scenario — Gasperini, Veneziano, etc.
- modified gravity-matter — Brandenberger, Chamseddine, Cotsakis, Mukhanov, Peter, Steinhardt, Turok, etc.
- loop quantum cosmology — Asthekar, de Cesare, Gupt, Pawlowski, Singh, Wilson-Ewing, etc.

Our standpoint.

- Scattering maps associated with specific theories
- Flexible framework and classification
  - uncovered *all possible* classes of junction

    geometrically / physically meaningful

    conformal/non-conformal spacelike/null/timelike

    scalar field stiff fluid compressible fluid

    a complete classification
  - discovered three universal laws

    constrain macroscopic aspects of spacetime bounces

    regardless of their origin from different microscopic corrections

    a guide to uncover relevant structures

Proposed formulation of the problem

Local ADM formulation near a singularity hypersurface

- Gaussian foliation (local patch)

$$g^{(4)} = \left(g^{(4)}_{\alpha\beta}\right) = -d\tau^2 + g(\tau) \qquad g(\tau) = g_{ij}(\tau)dx^i dx^j$$

- Einstein's evolution equations: unknowns g and K

$$\partial_\tau g_{ij} = -2\, K_{ij} \qquad \partial_\tau K^i_j = \mathrm{Tr}(K)K^i_j + R^i_j - 8\pi\, M^i_j$$

$$M^i_j = \frac{1}{2}\rho g^i_j + T^i_j - \frac{1}{2}\mathrm{Tr}(T)g^i_j$$

- Einstein's constraint equations

$$R + |K|^2 - \mathrm{Tr}(K^2) = 16\pi\rho \qquad \nabla_i K^i_j - \nabla_j(\mathrm{Tr} K) = 8\pi J_j$$

- coupled to the wave equation $\Box_{g^{(4)}}\phi = 0$ for a scalar field

Fuchsian approach (Rendall, Isenberg, Moncrief, etc.)

– solve from $\tau = 0$ toward the past ($\tau < 0$) or the future ($\tau > 0$)

– based on the so-called "velocity dominated" Ansatz

– derive an ODE system from the full Einstein equations

Singularity data and asymptotic profiles

a 3-manifold $\mathcal{H}$

Definition

1. **Asymptotic profile**

$$\tau \in (-\infty, 0) \mapsto \big(g^*, K^*, \phi^*\big)(\tau) \qquad g^*(\tau) = |\tau|^{2K^-} g^-$$

$$K^*(\tau) = \frac{-1}{\tau} K^- \qquad \phi^*(\tau) = \phi_0^- \log|\tau| + \phi_1^-$$

2. **Singularity initial data set** $(g^-, K^-, \phi_0^-, \phi_1^-)$, consisting of two tensor fields (rescaled metric and extrinsic curvature) and two scalar fields defined on $\mathcal{H}$

3. **Asymptotic version of the Einstein constraints**

Riemannian metric	g^-
CMC symmetric $(1,1)$-tensor	K^- with $\mathrm{Tr}(K^-) = 1$
Hamiltonian constraint	$1 - \lvert K^-\rvert^2 = 8\pi\,(\phi_0^-)^2$
momentum constraints	$\mathrm{Div}_{g^-}(K^-) = 8\pi\,\phi_0^-\,d\phi_1^-$

$\mathbf{I}(\mathcal{H})$: space of all singularity data

Bounce based on a singularity scattering map

- Singularity hypersurface as a (fluid-like) **interface** between two "phases", across which the geometry and the matter field encounter a "jump"
- Fluid dynamics and material science with phase transitions
  - when some (micro-scale) parameters (like viscosity, surface tension, heat conduction, etc.,) are neglected in the modeling
  - macro-scale effects are captured by jump conditions

    Rankine-Hugoniot, kinetic relations
  - kinetic relations in material science

    martensite-austenite

    two-phase liquid-valor flows

Definition

A (past-to-future) **singularity scattering map** on $\mathcal{H}$:

a diffeomorphism-covariant map on $\mathbf{I}(\mathcal{H})$

$$\mathbf{S} : \mathbf{I}(\mathcal{H}) \ni \left(g^-, K^-, \phi_0^-, \phi_1^-\right) \mapsto \left(g^+, K^+, \phi_0^+, \phi_1^+\right) \in \mathbf{I}(\mathcal{H})$$

satisfying the **ultra-locality property**: for all $x \in \mathcal{H}$

$$\mathbf{S}(g^-, K^-, \phi_0^-, \phi_1^-)(x) \text{ depends only on } (g^-, K^-, \phi_0^-, \phi_1^-)(x)$$

- **S** is a **tame-preserving map** if it preserves positivity:

  if $K^- > 0$ then $K^+ > 0$, where K^+ is defined from the image of **S**.
- **S** is a **rigidly-conformal map** if g^+ and g^- only differ by a conformal factor.

Observations

- Quiescent singularities $K > 0$: motivated by the absence of BKL oscillations in this case (named after Belinsky, Khalatnikov, and Lifshitz)

  quiescent regime, monotone behavior : Rendall, Andersson

  Lott, Fournodavlos, Luk, Rodnianksi, Speck
- Asymptotic profiles with $K^-, K^+ > 0$ describe a "bounce":
  - volume element decreases to zero as $\tau \to 0^-$
  - then increases back to finite values for $\tau > 0$

- Further notions and constructions: **cyclic spacetimes** with many singularity hypersurfaces
- In the present lecture, we focus on the **junction at the bouncing.**

Main classification

Theorem 5.2. Rigidly conformal maps

Only two classes of ultra-local spacelike *rigidly conformal* singularity scattering maps for self-gravitating scalar fields:

- **Isotropic rigidly conformal bounce $\mathbf{S}^{\text{iso, conf}}_{\lambda,\varphi}$**

$$g^+ = \lambda^2 g^- \qquad K^+ = \delta/3 \qquad \phi_0^+ = 1/\sqrt{12\pi} \qquad \phi_1^+ = \varphi$$

  parametrized by a conformal factor $\lambda = \lambda(\phi_0^-, \phi_1^-, \det K^-) > 0$ and a constant φ

- **Non-isotropic rigidly conformal bounce $\mathbf{S}^{\text{ani, conf}}_{f,c}$**

$$g^+ = c^2\mu^2 g^- \qquad K^+ = \mu^{-3}(K^- - \delta/3) + \delta/3$$

$$\phi_0^+ = \mu^{-3}\phi_0^-/F'(\phi_1^-) \qquad \phi_1^+ = F(\phi_1^-)$$

  parametrized by a constant $c > 0$ and a function $f : \mathbb{R} \to [0, +\infty)$

$$\mu(\phi_0, \phi_1) = \left(1 + 12\pi(\phi_0)^2 f(\phi_1)\right)^{1/6} \qquad F(\phi_1) = \int_0^{\phi_1} (1 + f(\varphi))^{-1/2} d\varphi$$

Theorem 5.3. General classification

Only two classes of ultra-local spacelike singularity scattering maps for self-gravitating scalar fields:

- **Isotropic bounce $\mathbf{S}^{\text{iso}}_{\lambda,\varphi}$**
- **Non-isotropic bounce $\mathbf{S}^{\text{ani}}_{\Phi,c}$**

where now λ is a two-tensor, Φ a "canonical transformation", c a constant.

Theorem 5.4. Three universal laws of quiescent bouncing cosmology.

- **First law: scaling of Kasner exponents**

There exists a (dissipation) constant $\gamma \in \mathbb{R}$ such that

$$|g^+|^{1/2} \mathring{K}^+ = -\gamma \, |g^-|^{1/2} \mathring{K}^-$$

spatial metric g in synchronous gauge, volume factor $|g|^{1/2}$

traceless part $\mathring{K}$ of the extrinsic curvature (as a $(1,1)$ tensor)

- **Second law: canonical transformation of matter**

conjugate matter momentum $\pi_\phi \sim \phi_0$

– there exists a nonlinear map $\Phi \colon (\pi_\phi, \phi)^- \mapsto (\pi_\phi, \phi)^+$

– preserving the volume form in the phase space $d\pi_\phi \wedge d\phi$

– depending solely on the scalar invariant $\mathbf{det}(\mathring{K}_-)$

- **Third law: directional metric scaling**

$$g^+ = \exp\big(\sigma_0 + \sigma_1 K + \sigma_2 K^2\big) g^-$$

nonlinear scaling in each proper direction of K

$\gamma = 0$: isotropic scattering, no restriction $\sigma_0, \sigma_1, \sigma_2$

$\gamma \neq 0$: non-isotropic scattering, explicit formulas in terms of Φ, γ

Follow-up work

- **Scattering maps associated with specific theories**
  - small-scale physical modeling
- **Flexible framework and classification**
  - uncovered *all possible* classes of junction

    geometrically / physically meaningful

    conformal/non-conformal spacelike/null/timelike

    scalar field stiff fluid compressible fluid

    a complete classification

    Work with general spacetimes. Earlier approaches: symmetric spacetimes & special junctions
  - discovered three universal laws

    constrain macroscopic aspects of spacetime bounces

    regardless of their origin from different microscopic corrections

    a guide to uncover relevant structures
- **Numerical simulations of cyclic spacetimes with a bounce**

3. Dynamics near Minkowski spacetime in f(R)-gravity

joint with Yue Ma (Xi'an)

Einstein gravity theory

- minimally coupled, Klein-Gordon field ϕ $\qquad U(\phi) = (c^2/2)\phi^2$
- Einstein-Klein-Gordon system $\qquad \Box_g \phi = U'(\phi)$

$$R_{\alpha\beta} = 8\pi \Big(\nabla_\alpha \phi \nabla_\beta \phi + U(\phi)\, g_{\alpha\beta} \Big)$$

f(R)-gravity theory

- generalized action $\int_M f(R)\, dV_g$
- $f(R) = R + \frac{\kappa}{2} R^2$ $\qquad$ a large (mass) parameter $1/\kappa$
- field equations of modified gravity $\qquad M_{\alpha\beta} = 8\pi T_{\alpha\beta}$

$$M_{\alpha\beta} = f'(R)\, G_{\alpha\beta} - \frac{1}{2}\Big(f(R) - Rf'(R) \Big) g_{\alpha\beta} + \Big(g_{\alpha\beta} \Box_g - \nabla_\alpha \nabla_\beta \Big) \Big(f'(R) \Big)$$

- up to fourth-order terms in g, but second-order after suitable transformations, self-gravitating massive fields

Global dynamics near the Minkowski regime

- rich and complex dynamics
- small perturbations
- global stability in the dispersive regime

Spherically symmetric collapse of a massive field

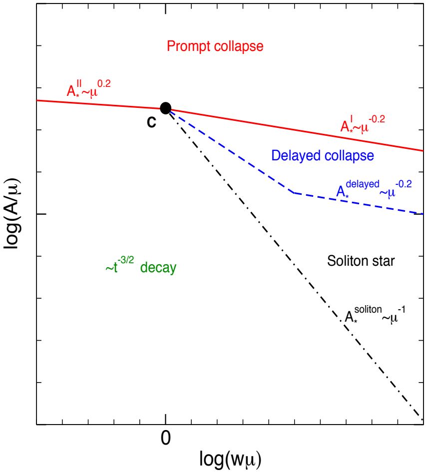
- asymptotically-flat massive matter spacetimes, stability/instability
- dispersion of the matter toward the infinite future
- or gravitational collapse and formation of a black hole, or rather "oscillating soliton stars" ?

Goncalves et al. (1997)

Okawa, Cardoso, and Pani (2014)

- highly accurate, multi-scale simulation over a long time
- 'total mass' vs. 'amplitude'

Other extensive numerical investigations of Klein-Gordon fields: J.A. Font and collaborators



Massive scalar field

two independent proofs: Ionescu-Pausader

PLF–Yue Ma

Theorem 1.2. Stability of self-gravitating massive fields PLF-Yue Ma ArXiv:171210045

- Einstein equations coupled to a Klein-Gordon field $-\Box_g \phi + m^2 \phi = 0$
- initial data set $(M_0 \simeq \mathbb{R}^3, g_0, k_0, \phi_0, \phi_1)$ sufficiently close to Minkowski data
- decay conditions at spacelike infinity possibly non-spherically symmetric
- Einstein's constraint equations `discussed next`

The initial value problem admits a globally hyperbolic Cauchy development.

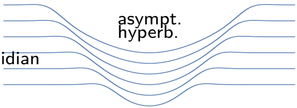
asymptotically close to Minkowski spacetime

future causally geodesically complete

Euclidian-hyperboloidal foliation method.

(1) FOLIATION

asympt. Euclidian



(2) (approximate) SYMMETRIES of Minkowski spacetime **except** the scaling field

(3) SHARP energy, pointwise decay timelike, null, spacelike infinity

(4) nonlinear geometry/matter INTERACTIONS coupled wave-Klein-Gordon equations

Formulation of the governing equations of f(R) gravity

- rigorous proof for the regime of dispersion
- wave gauge $\Box_g x^\alpha = 0$, coupled wave-Klein-Gordon, second-order PDEs

f(R)-gravity for a self-gravitating massive field

$$\Box_{g^\dagger} g^\dagger_{\alpha\beta} = F_{\alpha\beta}(g^\dagger, \partial g^\dagger) + 8\pi \Big(-2e^{-\kappa\rho}\partial_\alpha\phi\partial_\beta\phi + c^2\phi^2 e^{-2\kappa\rho} {g^\dagger}_{\alpha\beta}\Big) - 3\kappa^2\partial_\alpha\rho\partial_\beta\rho + \kappa\, \mathcal{O}(\rho^2) g^\dagger_{\alpha\beta}$$

$$\Box_{g^\dagger}\phi - c^2\phi = c^2\big(e^{-\kappa\rho} - 1\big)\phi + \kappa g^{\dagger\alpha\beta}\partial_\alpha\phi\partial_\beta\rho$$

$$3\kappa\,\Box_{g^\dagger}\rho - \rho = \kappa\,\mathcal{O}(\rho^2) - 8\pi e^{-\kappa\rho}\Big(g^{\dagger\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + 2c^2\, e^{-\kappa\rho}\phi^2\Big)$$

- wave gauge conditions $g^{\dagger\alpha\beta}{\Gamma^\dagger}^\lambda_{\alpha\beta} = 0$
- curvature compatibility $e^{\kappa\rho} = f'(R_{e^{-\kappa\rho}g^\dagger})$
- Hamiltonian and momentum constraints of modified gravity propagate from any given Cauchy hypersurface

- global stability theorem
- gravitational radiation, time and space decay

Structure relevant to numerical relativity and new challenges

– **Euclidian-hyperboloidal spacetime foliation**

- hyperboloidal slices in the light cone interior *capture the decay in time*
- asymptotically Euclidian slices in the exterior *capture the decay in space*
- merged together near the light cone

Numerical investigations: Rinne, Zenginoglu, Hilditch

– **Weighted energy norms based on symmetries** of Minkowski spacetime

- "asymptotic" Killing fields (translations, spatial rotations, boost)
- exclude the scaling field $S = t\partial_t + r\partial_r$ lack of scale invariance
- frames of vector fields *semi-hyperboloidal, semi-null*
- hierarchy and geometry-matter coupling *nonlinear interaction terms*

- provide weighted energy norms and quantitative error estimates
- handle asymptotic behaviors that are not spherically symmetric

  directional/angular effects into account
- study the dependency in κ: passage from f(R) gravity to standard gravity

  singular perturbation problem

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4. Asymptotic localization method

joint work with T.C. Nguyen (Paris) ArXiv: 1903.00243

Einstein's constraint equations $(\mathbf{M}, g, k)$

- from the extrinsic curvature k we define $h := k - \mathrm{Tr}(k) g$
- matter content scalar field $H_\star$, vector field $J_\star$
- Hamiltonian and momentum constraints

$$\mathcal{G} = (\mathcal{H}, \mathcal{M}) \qquad \begin{aligned} \mathcal{H}(g,h) &= R_g + \frac{1}{2}\big(\mathrm{Tr}(h)\big)^2 - |h|^2 &= H_\star \\ \mathcal{M}(g,h) &= \mathrm{Div}_g h &= J_\star \end{aligned}$$

Many mathematical works

- nonlinear elliptic system of partial differential equations
- Lichnerowicz, Choquet-Bruhat, ..., Corvino, Chrusciel, Delay, Dilts, Galloway, Holst, Isenberg, Maxwell, Mazzeo, Miao, Pollack, ... Carlotto and Schoen.

A new analytical approach: the seed-to-solution method

- a seed data/approximate solution $(\mathbf{M}, g_1, h_1)$
- prescribe the asymptotic behavior at infinity

Theorem. The seed-to-solution method (case of vacuum data) —— (LeFloch & Nguyen, 2019)

Given a seed data set $(\mathbf{M}, g_1, h_1)$ on a manifold (with one asymptotic end) consisting of a Riemannian metric g_1 and a symmetric two-tensor h_1:

$$1/2 < p_G \leqslant \min(1, p_M) \text{ and } 1/2 < p_M < +\infty$$

$$g_1 = g_{\text{Eucl}} + \mathcal{O}(r^{-p_G}) \qquad h_1 = \mathcal{O}(r^{-p_G-1})$$

$$\mathcal{H}(g_1, h_1) = \mathcal{O}(r^{-p_M-2}) \qquad \mathcal{M}(g_1, h_1) = \mathcal{O}(r^{-p_M-2})$$

there exists a solution to Einstein's constraint equations $\mathcal{G}(g, h) = 0$.

- **sub-critical decay:** $p_M < 1$

$$g = g_1 + \mathcal{O}(r^{-p_M}) \qquad h = h_1 + \mathcal{O}(r^{-p_M-1})$$

- **critical decay:** $p_M = 1$ with $\mathcal{H}(g_1, h_1)$ and $\mathcal{M}(g_1, h_1)$ in $L^1(M)$

$$g = g_1 + \widetilde{m}/r + o(r^{-1}) \qquad h = h_1 + \mathcal{O}(r^{-2})$$

- **super-critical decay:** $p_M > 1$ $\qquad p = \min(p_G + 1, p_M, 2)$

$$g = g_1 + \widetilde{m}/r + \mathcal{O}(r^{-p}) \qquad h = h_1 + \mathcal{O}(r^{-2}).$$

in which the "mass corrector" is

$$\widetilde{m} = \widetilde{m}(g_1, h_1) = -\frac{1}{8\pi} \int_{\mathbf{M}} \mathcal{H}(g_1, h_1)\, dV_{g_1} + \mathcal{O}(\mathcal{G}(g_1, h_1)^2)$$

Iterative construction scheme

- approximation based on the seed data
- a fixed-point strategy for nonlinear elliptic equations
- **converging sequence** of approximate solutions
- **stability property**: continuous dependence w.r.t. the Einstein operator

$$\|g - g_1\|_{L^2 C_p^{2,\alpha}(\mathbf{M})} \lesssim \|\mathcal{H}(g_1, h_1) - H_\star\|_{L^2 C_{p+2}^{\alpha}(\mathbf{M})} + \varepsilon_G \|\mathcal{M}(g_1, h_1) - M_\star\|_{L^2 C_{q+1}^{1,\alpha}(\mathbf{M})}$$

$$\|h - h_1\|_{L^2 C_q^{2,\alpha}(\mathbf{M})} \lesssim \varepsilon_G \|\mathcal{H}(g_1, h_1) - H_\star\|_{L^2 C_{p+2}^{\alpha}(\mathbf{M})} + \|\mathcal{M}(g_1, h_1) - M_\star\|_{L^2 C_{q+1}^{1,\alpha}(\mathbf{M})}$$

Structure relevant for numerical relativity

- construct solutions $(\mathbf{M}, g, h)$ with *prescribed behavior at infinity*
- control the mass corrector

$$\widetilde{m} = \widetilde{m}(g_1, h_1) = -\frac{1}{8\pi} \int_{\mathbf{M}} \mathcal{H}(g_1, h_1)\, dV_{g_1} + \mathcal{O}(\mathcal{G}(g_1, h_1)^2)$$

"spurious wave" propagating to infinity

- allow for free parameters to be fitted, produce realistic initial data sets
- for instance *asymptotically localized in angular directions*
- *quantitative error bounds* in specific weighted norms

Einstein constraints

- Carlotto-Schoen : localization at spacelike infinity $g - \delta = 1/r^{1-\varepsilon}$

  non-sphericallly symmetric with decay $1/r$ *?*

- LeFloch-Nguyen: *relax the requirement at infinity* *also physically natural*

Theorem 1.1. The asymptotic localization problem PLF-Nguyen ArXiv:1903.00243

– Einstein's constraint equations on a 3-manifold with one asymptotic end

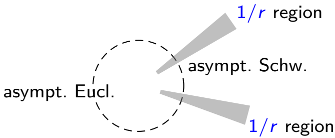
– two asymptotic, disjoint angular regions, say $\mathscr{C}_{\text{Eucl}}$ and $\mathscr{C}_{\text{Sch}}$

Given (for instance) the Euclidean and Schwarzschild metrics, there exists a solution to Einstein's constraint equations such that for some $q \in (1, 2)$

$$g = g_{\text{Eucl}} + \mathcal{O}(r^{-1}) \text{ everywhere}$$

$$g = g_{\text{Eucl}} + \mathcal{O}(r^{-q}) \quad \text{in } \mathscr{C}_{\text{Eucl}} \qquad g = g_{\text{Sch}} + \mathcal{O}(r^{-q}) \quad \text{in } \mathscr{C}_{\text{Sch}}$$

Technique of construction.



(1) PRESCRIBE a "seed metric" at infinity

(2) DESIGN seed data with free parameters

(3) MASS CORRECTORS determined implicitly

References

- **Structure-preserving methods**

2014 Structure-preserving shock-capturing methods: late-time asymptotics, curved geometry, small-scale dissipation, and nonconservative products
Lectures for the J.-L. Lions's Spanish-French School

2005.13504 A numerical algorithm for Fuchsian equations and fluid flows on cosmological spacetimes with F. Beyer

1912.13439 Asymptotic structure of cosmological fluid flows in one and two space dimensions with Y.-Y. Cao and M.A. Ghazizadeh

- **Gravitational singularities and scattering maps** B. Le Floch and G. Veneziano

2006.08620 Universal scattering laws for bouncing cosmology

2005.11324 Cyclic spacetimes through singularity scattering maps Phys. Rev. D 2021

2020 On the global evolution of self-gravitating matter. Phase boundaries, scattering maps, and causality

- **Foliations and spacetime decay**

2018 The global nonlinear stability of Minkowski space for self-gravitating massive fields, World Scientific Press, Singapore with Y. Ma

1903.00243 The seed-to-solution method for the Einstein equations and the asymptotic localization problem with T.-C. Nguyen