Extensions to GR: roadblocks and potential way through

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[based on results with: Juan Cayuso, Gwyneth Allwright, Raimon Luna, Ramiro Cayuso, Guillaume Dideron, *Nestor Ortiz & Laura Bernard, Franca, Figueras*]

Motivation

- Many arguments point to considering extensions from GR (singularity resoln, BH entropy, DM, DE)

- *Important* data coming through: in *gravitational waves, EHT*, binary pulsars, etc.

- Especially in the context of compact binary mergers → tests reach the highly dynamical ($v/c \sim 0.5$), strongly gravitating ($M/R \sim 1$), 'strong' curvature regimes (M/R^2 $\sim 10^{-9}$-$1\ M_O^{-1}$)
  - Search is suboptimal without guidance
  - Analysis is incomplete without guidance [phenomenological approach?]

- *Not a goal here*: to advocate for/analyze a particular theory

- *Goals:*
  - Discuss issues to bear in mind when considering *any extension*
  - Illustrate such issues do arise & consequences
  - Discuss options to plow forward

Why any particular beyond Einstein theory for compact object mergers?

- Are we simply inventing problems to solve?

- On the other hand, one motivation is of theoretical interest even if GR turns to be correct up to any level GW observations can probe *:*

  To learn what mergers events could conceivably look like in a generic metric theory of gravity that has black hole solutions

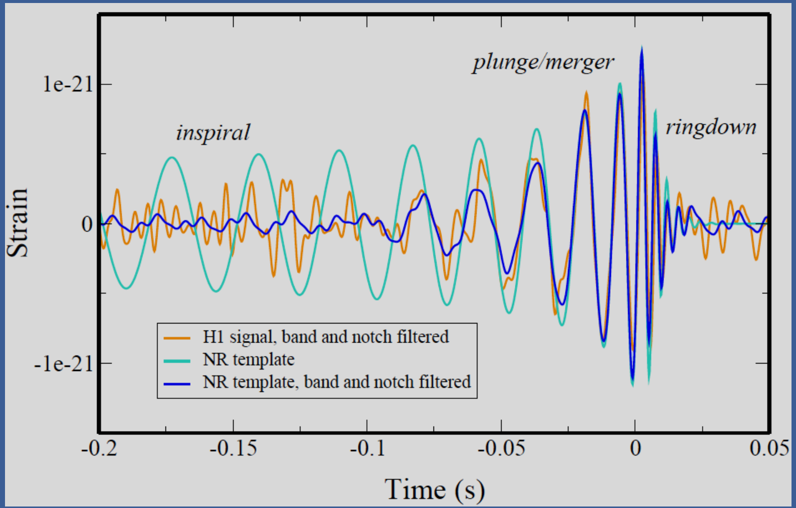
  – For instance, generic metric theories of gravity allow 6 propagating degrees of freedom, GR only has 2 (+ and x); *what is the analogous statement with respect to black holes solutions and horizon coalescence?*
  – Cosmic censorship? Ultimate state conjecture?
  – Efficiency of mass-to-radiation conversion

Terminology

- An *interesting* modified gravity theory : consistent with existing tests/experiments of general relativity in the weak-field, but may offer observable differences in the strong-field

- A *viable* modified gravity theory: it possesses a well-posed initial value formulation that can be solved to make concrete predictions to confront with data

  – *i.e., looking for viable, interesting modifications to GR*



What should one consider modifying?

- modified coupling between matter/geometry
- modified geometry (LHS) [e.g. further degrees of freedom, relaxing symmetries…]
- modified matter (RHS): “exotic” alternatives to black holes/neutron stars
- We have strong anchors! GR/BHs/NSs → EFT approach, corrections to Einstein comes with further operators/higher derivatives and some *key* assumptions

Underappreciated (?) issues

- *'linearization stability'* : soln to linearized eqns <-> solution to full problem in linearized regime
  - e.g. take Navier Stokes → all modes decay (*no turbulence!*)

- *'regime of applicability'* : higher operators / corrections stay small.
  - But, we have both 'stability of Minkowski' and *singularity theorems* in GR already. Why take an EFT approach if we know the latter happens?

‘Over’ appreciated issues

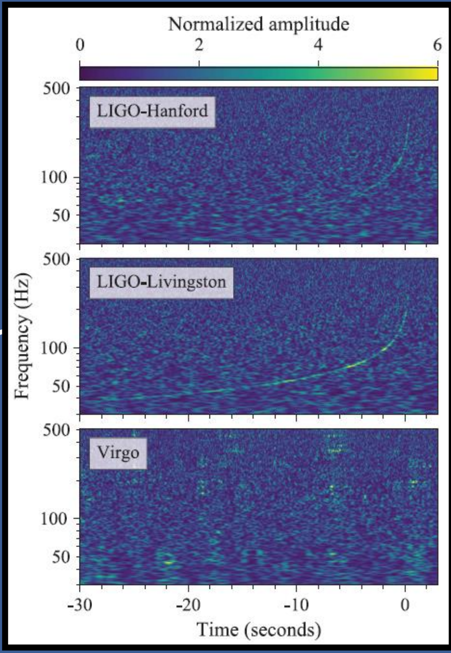
- 2^{nd} order equations → lack of Ostrogradski’s ghost, we’re good….
- Reduction of order strategy → treat corrections perturbatively (as in S-matrix calculations), we’re good….

Modified Coupling Between Geometry and Matter

- Prototypical example, scalar tensor theories

$$L \propto R(g_{ab}) + L_\theta(\theta, g_{ab}) + L_\varphi(\varphi, f(\varphi) g_{ab})$$

  - Einstein gravity with a scalar field θ, but other matter ϕ experiences "physical" geometry through a scalar field rescaled metric



- These are arguably the only class of modified gravity theories shown to be *<u>interesting and viable</u>*; however
  - Relevant for NS due to "scalarization" process, for BH one needs a suitable(?) potential for departures from GR.
  - with Einstein-Maxwell-Dilaton, need black holes with a significant charge [which would be unnatural]

What's in the EOMs ?

Most general 2nd order theory (with 1 extra d.o.f)

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(\Sigma_{i=1}^{5}\mathcal{L}_i) \tag{1}$$

where,

$$\mathcal{L}_1 = R + X - V(\phi)\,, \tag{2}$$
$$\mathcal{L}_2 = \mathcal{G}_2(\phi, X)\,, \tag{3}$$
$$\mathcal{L}_3 = \mathcal{G}_3(\phi, X)\Box\phi\,, \tag{4}$$
$$,\mathcal{L}_4 = \mathcal{G}_4(\phi, X)R + \partial_X\mathcal{G}_4(\phi, X)\delta^{ac}_{bd}\nabla_a\nabla^b\phi\nabla_c\nabla^d\phi\,, \tag{5}$$
$$\mathcal{L}_5 = \mathcal{G}_5(\phi, X)G_{ab}\nabla^a\nabla^b\phi - \frac{1}{6}\partial_X\mathcal{G}_5(\phi, X)\delta^{ace}_{bdf}\nabla_a\nabla^b\phi\nabla_c\nabla^d\phi\nabla_e\nabla^g\phi\,. \tag{6}$$

with $X = -1/2\nabla_a\phi\nabla^a\phi$, G_{ab} the Einstein tensor, $\mathcal{G}_i$ are functions of the scalars $\{\phi, X\}$, V is a potential and $\delta^{b_1..b_n}_{a_1..a_n}$ is the generalised Kronecker delta symbol.

Linearized study: Papallo-Real's, in a special frame, better be that $G_4=G_5=0$ or a generic problem *will be ill-posed*

Counter argument: analysis relies in a particular gauge, could this be regarded as too restrictive?

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[(1+\mathcal{G}_4(\phi))R + X - V(\phi) + \mathcal{G}_2(\phi,X)\right] \; ,$$

- The 'look' of the theory can be simplified by a conformal transformation…

form $\tilde{g}_{ab} = \Omega^2 g_{ab}$ with $\Omega = \sqrt{1+\mathcal{G}_4(\phi)})$ allows one to rewrite the above action as,

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-\tilde{g}}\left\{\tilde{R} + \frac{1}{(1+\mathcal{G}_4(\phi))^2}\left[\left(3[\mathcal{G}_4'(\phi)]^2 + 1 + \mathcal{G}_4(\phi)\right)\tilde{X} - V(\phi) + \mathcal{G}_2\left(\phi,(1+\mathcal{G}_4(\phi))\tilde{X}\right)\right]\right\} \; , \tag{8}$$

where $\tilde{X} = -1/2\tilde{\nabla}_c\phi\,\tilde{\nabla}^c\phi$. From this action, the equation of motions are

$$\tilde{G}_{ab} = \left[\frac{3[\mathcal{G}_4'(\phi)]^2 + 1 + \mathcal{G}_4(\phi)}{2(1+\mathcal{G}_4(\phi))^2}\tilde{X} + \frac{-V(\phi)+\mathcal{G}_2(\phi,X)}{2(1+\mathcal{G}_4(\phi))^2}\right]\tilde{g}_{ab} + \left[\frac{3[\mathcal{G}_4'(\phi)]^2}{2(1+\mathcal{G}_4(\phi))^2} + \frac{1+\partial_X\mathcal{G}_2(\phi,X)}{2(1+\mathcal{G}_4(\phi))}\right]\tilde{\nabla}_a\phi\tilde{\nabla}_b\phi \; , \tag{9}$$

$$\begin{aligned}\tilde{\Box}\phi = {} & \frac{1}{3[\mathcal{G}_4'(\phi)]^2 + (1+\mathcal{G}_4(\phi))(1+\partial_X\mathcal{G}_2(\phi,X))}\left\{V'(\phi) - \partial_\phi\mathcal{G}_2(\phi,X) - 2\,\mathcal{G}_4'(\phi)\frac{V(\phi)-\mathcal{G}_2(\phi,X)}{1+\mathcal{G}_4(\phi)}\right. \\ & + \left[2(1+\mathcal{G}_4(\phi))\partial^2_{\phi X}\mathcal{G}_2(\phi,X) + \mathcal{G}_4'(\phi)\left(6\mathcal{G}_4''(\phi) - 1 - 3\partial_X\mathcal{G}_2(\phi,X) - 6\frac{[\mathcal{G}_4'(\phi)]^2}{1+\mathcal{G}_4(\phi)}\right)\right]\tilde{X} \\ & \left. + 2\mathcal{G}_4'(\phi)(1+\mathcal{G}_4(\phi))\partial^2_{XX}\mathcal{G}_2(\phi,X)\tilde{X}^2 + (1+\mathcal{G}_4(\phi))^2\partial^2_{XX}\mathcal{G}_2(\phi,X)\tilde{\nabla}^a\phi\tilde{\nabla}^b\phi\tilde{\nabla}_a\tilde{\nabla}_b\phi\right\} .\end{aligned} \tag{10}$$

[Bernard-Luna-LL]

- Or rather…

$$\left[\tilde{g}^{ab} - \frac{(1+\mathcal{G}_4(\phi))^2 \partial^2_{XX}\mathcal{G}_2(\phi,X)}{3[\mathcal{G}_4'(\phi)]^2 + (1+\mathcal{G}_4(\phi))(1+\partial_X \mathcal{G}_2(\phi,X))} \tilde{\nabla}^a\phi\tilde{\nabla}^b\phi\right]\tilde{\nabla}_a\tilde{\nabla}_b\phi$$
$$= \frac{1}{3[\mathcal{G}_4'(\phi)]^2 + (1+\mathcal{G}_4(\phi))(1+\partial_X \mathcal{G}_2(\phi,X))}\left\{V'(\phi) - \partial_\phi \mathcal{G}_2(\phi,X) - 2\,\mathcal{G}_4'(\phi)\frac{V(\phi)-\mathcal{G}_2(\phi,X)}{1+\mathcal{G}_4(\phi)}\right.$$
$$+\left[2(1+\mathcal{G}_4(\phi))\partial^2_{\phi X}\mathcal{G}_2(\phi,X) + \mathcal{G}_4'(\phi)\left(6\mathcal{G}_4''(\phi) - 1 - 3\partial_X\mathcal{G}_2(\phi,X) - 6\frac{[\mathcal{G}_4'(\phi)]^2}{1+\mathcal{G}_4(\phi)}\right)\right]\tilde{X}$$
$$\left.+2\mathcal{G}_4'(\phi)(1+\mathcal{G}_4(\phi))\partial^2_{XX}\mathcal{G}_2(\phi,X)\tilde{X}^2\right\}.$$

- Thus, the field propagates according to a different metric that depends on its gradients
  - For ‘weak data’ the eqn satisfies the ‘null condition’ (Klainerman) which together with Straus’ condition in V → global solutions
  - If the $v_a = X\,\phi_{,\alpha}$ is ‘twist free’, a field redefinition takes it to a ‘standard’ wave equation wrt to conformal metric

Beyond these cases, can we expect good behavior?

- If shocks arise → uniqueness is lost -> thus well posedness
- The character of the equation might change!
- Issues are *independent* of the gauge adopted for EEs

example case: $\mathcal{G}_2(X) = -gX^2$

Monitor in a spherical scenario eigenvalues of effective metric (λ), propagation speed of scalar field (V)

Regardless of sign of "g" : 'weak' data → smooth solutions; stronger → equation changes character! (no longer hyperbolic).

Lessons drawn from weak coupling can't be pushed to stronger ones, but what weak is, depends also on the ID considered

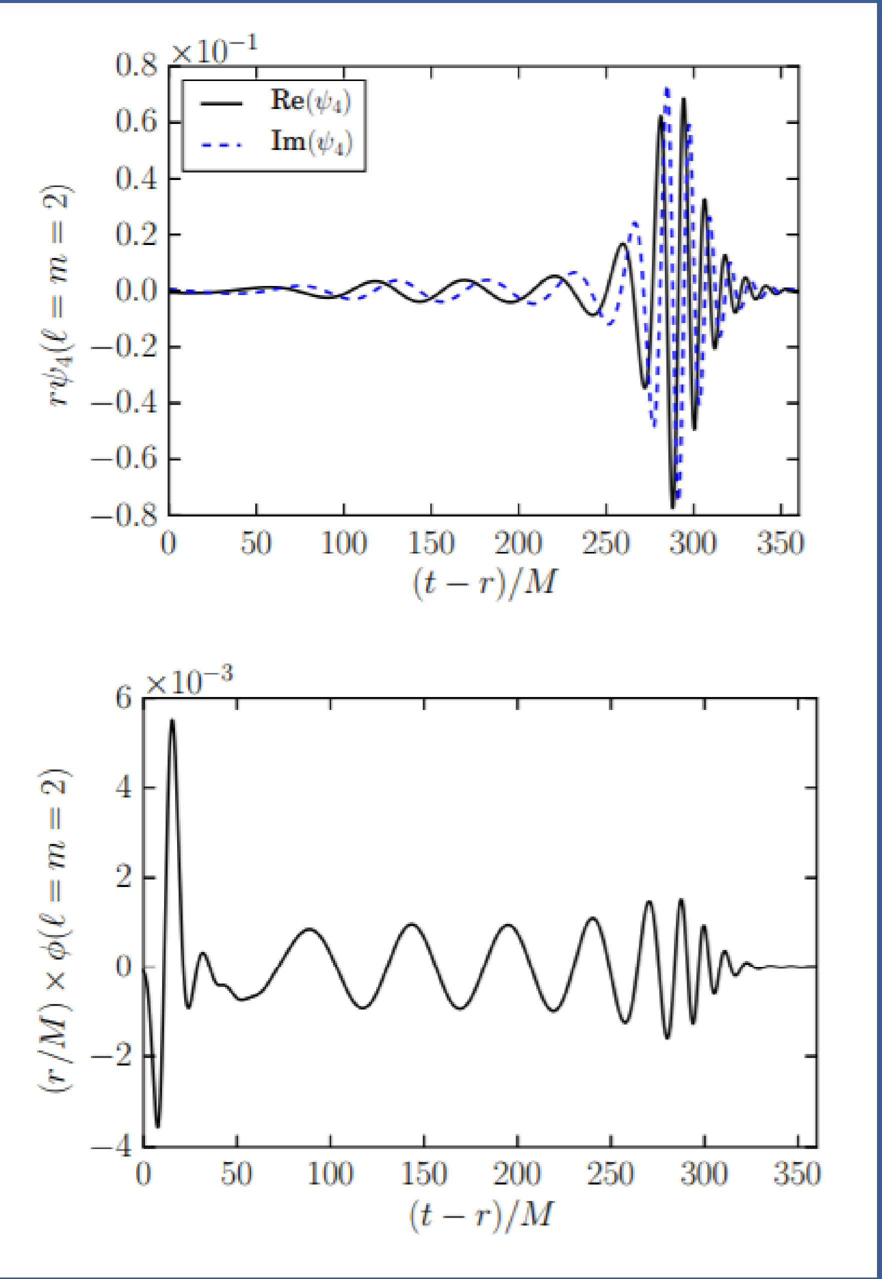
This is but one example of phenomenology that might arise… and we are 'only' dealing with 2nd order equations still.

[Also: Rippley-Pretorius & Figueras-Franca]

- With a suitable gauge [Kovacs-Reall], and excising the BH interior East-Ripley'20 evolve fully non-linearly EDGB
- (see also Wittek's talk)

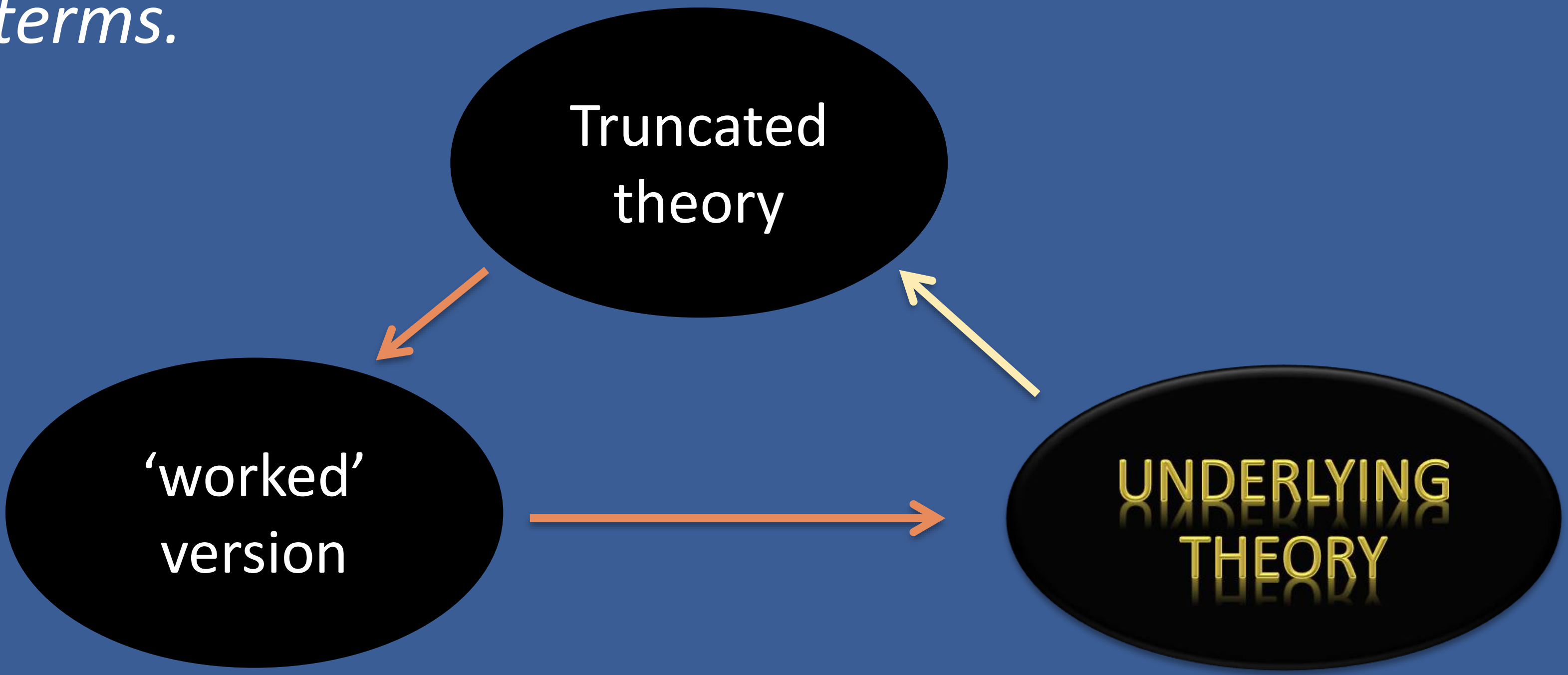
- Slightly earlier merger + scalar radiation. However for λ/m^2 > '0.23' -> hyperbolicity is lost outside the horizon

- (see also Figueras-Franca '20 for gravitational collapse)



So, that's at 2nd order.....beyond?

- Effective Field Theories: degrees of freedom at some short scales are integrated out. Their effects enter through suitable expansions wrt a given scale.
  - This generically introduces *higher order derivative terms.*



Example!

- Burgess-Williams '14

$$\frac{S}{v^2} = -\int d^4x \left[\frac{1}{2}\partial_\mu\rho\,\partial^\mu\rho + \frac{1}{2}(1+\rho)^2\partial_\mu\theta\,\partial^\mu\theta + V(\rho)\right]$$

where

$$V(\rho) = \frac{M^2}{2}\left(\rho^2 + \rho^3 + \frac{1}{4}\rho^4\right).$$

Varying this action gives the classical equations of motion:

$$\Box\rho - (1+\rho)(\partial\theta)^2 - V'(\rho) = 0$$
$$\partial_\mu\left[(1+\rho)^2\partial^\mu\theta\right] = 0\,,$$

- Integrating out ϱ $\rightarrow$ $-\int d^4x \left\{\frac{1}{2}(\partial\theta)^2 - \frac{1}{2M^2}(\partial\theta)^4 + \frac{2}{M^4}(\theta_{\mu\nu}\theta^{\mu\rho})\,(\partial_\rho\theta\,\partial^\nu\theta)\right\}$

- and EOM

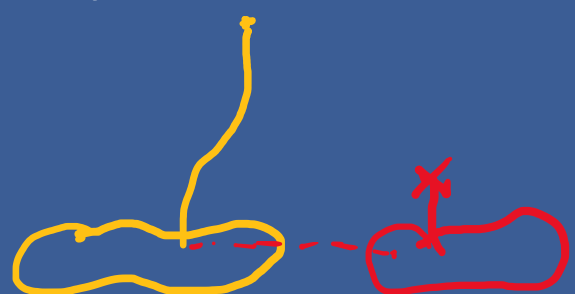
$$\begin{aligned} 0 \;=\; & -\Box\phi \\ & + \frac{1}{M^2}\left(4\phi^{,l}\phi^{,m}\phi_{,lm} + 2\phi_{,l}\phi^{,l}\Box\phi\right) \\ & + \frac{1}{M^4}\left(4\phi^{,l}\phi^{,m}(\Box\phi)_{,lm} + 4\phi^{,lm}\phi_{,lm}\Box\phi + 8\phi_{,m}\phi_{,lq}\phi^{,mlq} + 4\phi_{,q}(\Box\phi)\partial^q\Box\phi + 4\phi_{,q}\phi^{,mq}\partial_m\Box\phi\right) \end{aligned}$$

What’s the task?

- Deal with eqns with a structure as:

$$R_{ab} = \boldsymbol{\Lambda}\{\partial g\, \partial^2 g + \partial^2 g\, \partial^2 g + \partial^{>2} g + \ldots\}$$

- predict the behavior of relevant systems and observable’s dependence on $\boldsymbol{\Lambda}$
- In the non-linear regime → numerics
  - Any potential problem will likely be triggered by numerical noise
  - Whatever method, should lead to a consistent discretization and the ability to check faithfulness of solution

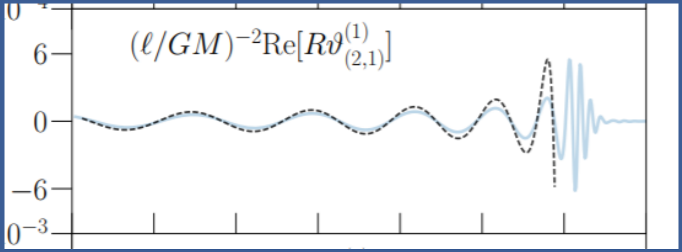


Strategies?

- ‘Iteration/perturbation’: corrections are small

$$B(g^*) = 0 \ \rightarrow B(g) = S(g^*)$$

  - Rinse and repeat: but during what time frame?
  - Perturbative hierarchy [e.g. Okounkova+ ‘17]



  - justified? can one guarantee the fidelity of the solution obtained?

- ‘Modification’: high frequencies are spurious

$$B(g) = F\ ; F_t = -\lambda(F - S(g))$$

  - Modify system of equations to ‘fix’ problems [Cayuso+ ‘17]
  - Introduces a new timescale λ, can one guarantee the fidelity of the solution obtained?

- *Option 1:*

Take Navier Stokes, solution of:

$$\partial_t \hat{v} + \hat{v}\partial\hat{v} = 0 \ (\sim \eta \partial^2 v)$$

displays turbulence for all wavelengths. What recovers the laminar regime? [short wavelength!]

$$\partial_t v + \hat{v}\partial v + v\partial\hat{v} = \eta\partial^2 v$$

Introduces a secular dependence, and if resummation is possible, it would reveal the laminar/turbulent regimes.

Option 2

- Modify the system of eqns, in an ad-hoc manner to control higher gradients and prevent wild runaway to the UV
- E.g. Israel-Stewart formultion of viscous relativistic hydrodynamics: $T = T^{pf}$ + gradient terms
  - Define $\Pi =$ (shear/bulk)$_{ab}$ + Grad(shear/bulk..)$_{ab}$ as new and independent variable
  - Force an eqn on Π such that $\Pi \sim$ (shear/bulk)$_{ab}$ to leading order always

    $\tau\, \Pi_{,t} = - \Pi +$ (shear/bulk)$_{ab}$ .... [Geroch, details shouldn't matter]

So, mathematically can be 'in check'. How about physically? Without a complete problem it is hard to tell

[large M cases]

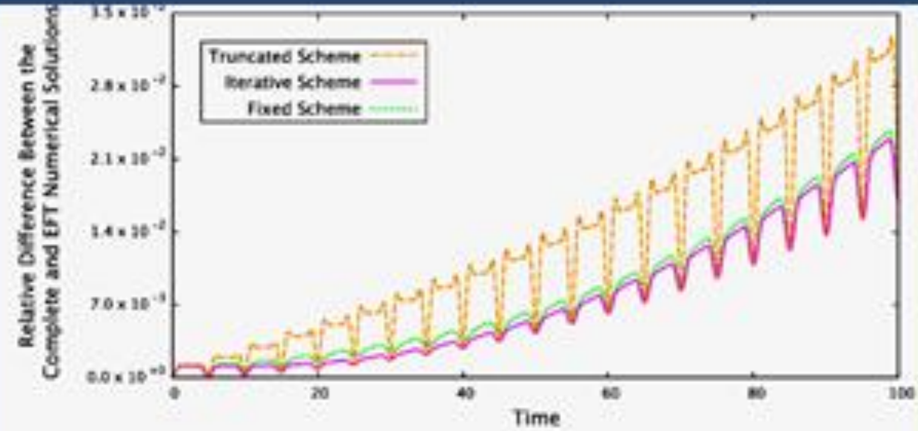


Figure 2: Plot of the relative difference between the UV-complete and EFT solutions for three different numerical techniques: the truncated, iterative and fixed schemes, using $M = 0.1$.

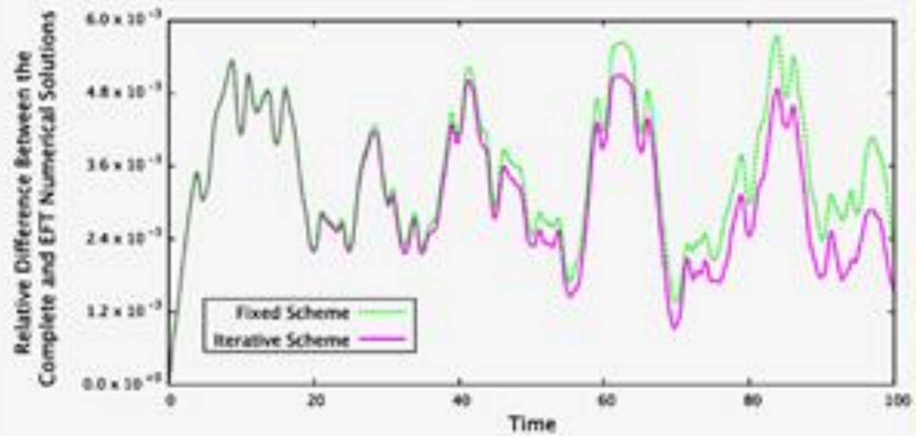


Figure 3: Plot of the relative difference between the UV-complete and EFT solutions for two different numerical techniques: the iterative and fixed schemes, using $M = 1$.

Small m case

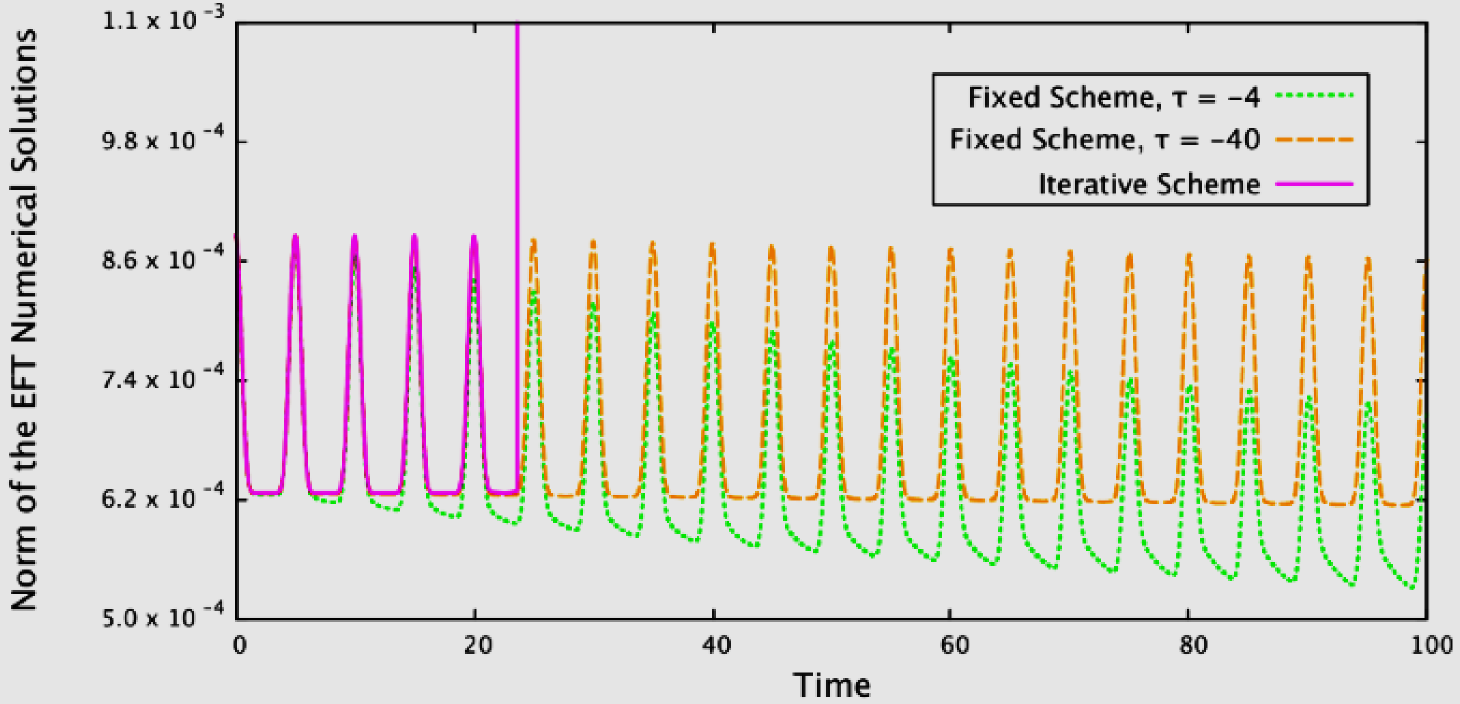


Figure 4: Plot of the L_2-norm of the EFT solutions for two numerical techniques: the iterative and fixed schemes, using $M = 0.01$.

‘Gravity application’

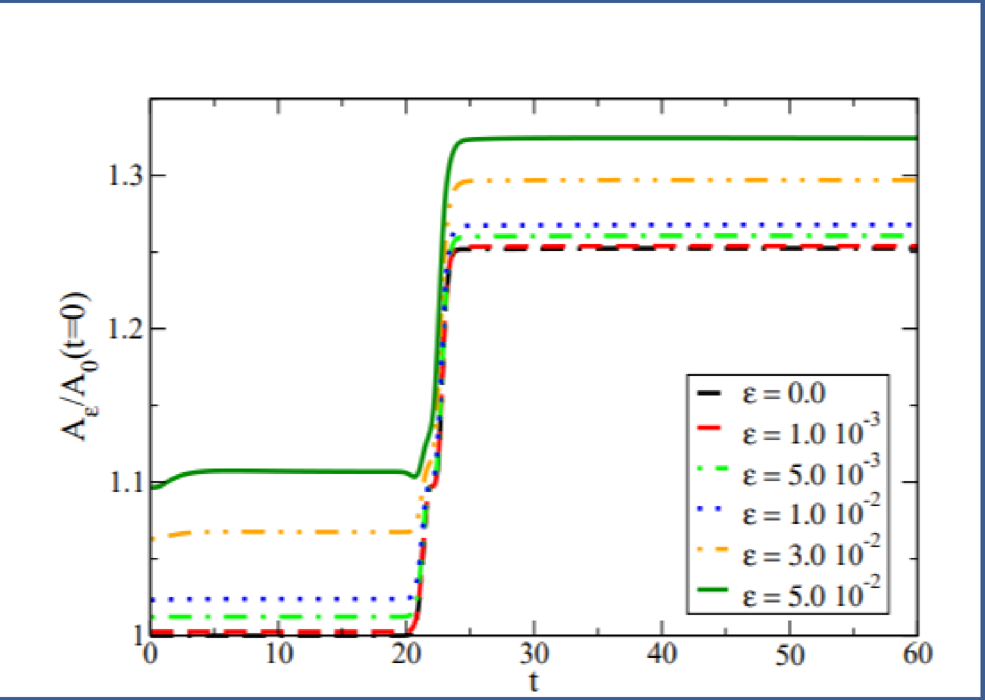
$$S_{eff} = \int d^4x\sqrt{-g}2M_{pl}^2\left(R - \frac{\mathcal{C}^2}{\Lambda^6} - \frac{\widetilde{\mathcal{C}}^2}{\widetilde{\Lambda}^6} - \frac{\mathcal{C}\widetilde{\mathcal{C}}}{\Lambda_-^6} + \dots\right), \tag{1}$$

where $\mathcal{C} \equiv R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ and $\widetilde{\mathcal{C}} \equiv R_{\alpha\beta\gamma\delta}\widetilde{R}^{\alpha\beta\gamma\delta}$, with $\widetilde{R}^{\alpha\beta\gamma\delta} = \epsilon^{\alpha\beta}{}_{\mu\nu}R^{\mu\nu\gamma\delta}$, and the $+\dots$ correspond to sub-leading contributions.

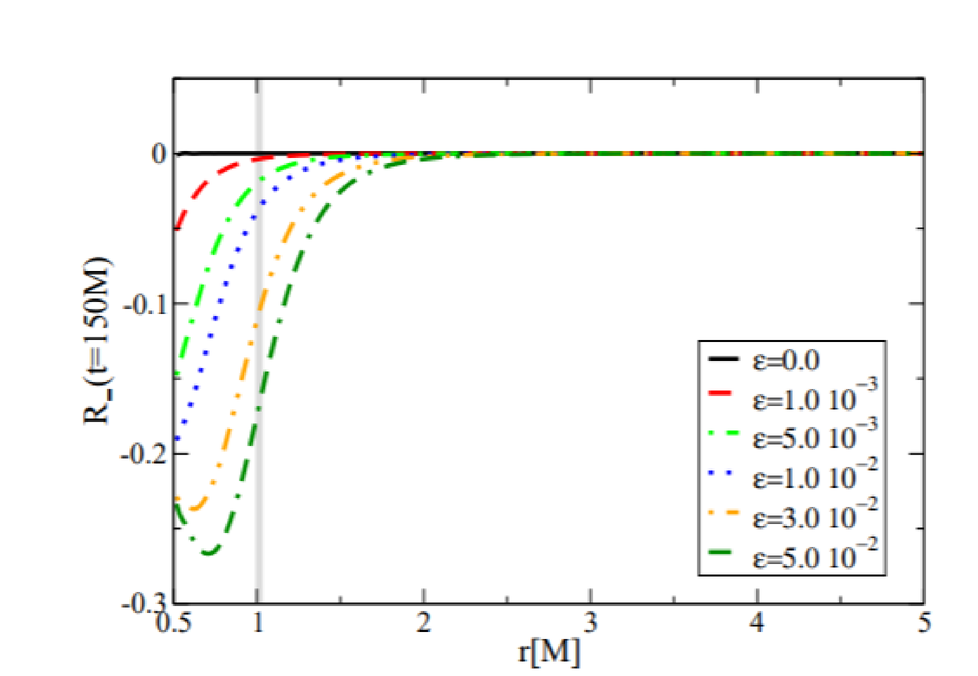
$$\begin{aligned} G_{\mu\nu} = 8\pi T_{\mu\nu} + \frac{1}{\Lambda^6}\Big(&- 8\,\mathcal{C}\,R_\mu{}^\alpha R_{\nu\alpha} + 8\,\mathcal{C}\,R^{\alpha\beta}R_{\mu\alpha\nu\beta} \\ &+ 4\,\mathcal{C}\,R_\mu{}^{\alpha\beta\gamma}R_{\nu\alpha\beta\gamma} - \frac{1}{2}g_{\mu\nu}\mathcal{C}^2 - 4\mathcal{C}\nabla_\mu\nabla_\nu R \\ &- 32R^{\beta\gamma\sigma\delta}\nabla_{(\mu}R_{\nu)}^{\alpha}\nabla_\alpha R_{\beta\gamma\sigma\delta} + 8\mathcal{C}\nabla_\alpha\nabla^\alpha R_{\mu\nu} \\ &+ 32R^{\beta\gamma\sigma\delta}\nabla_\alpha R_{\beta\gamma\sigma\delta}\nabla^\alpha R_{\mu\nu} + 8R_\mu{}^\alpha{}_\nu{}^\beta\nabla_\beta\nabla_\alpha\mathcal{C}\Big), \end{aligned} \tag{2}$$

$$\nabla^\mu T_{\mu\nu} = 0. \tag{3}$$

[Cayuso,LL ‘21]

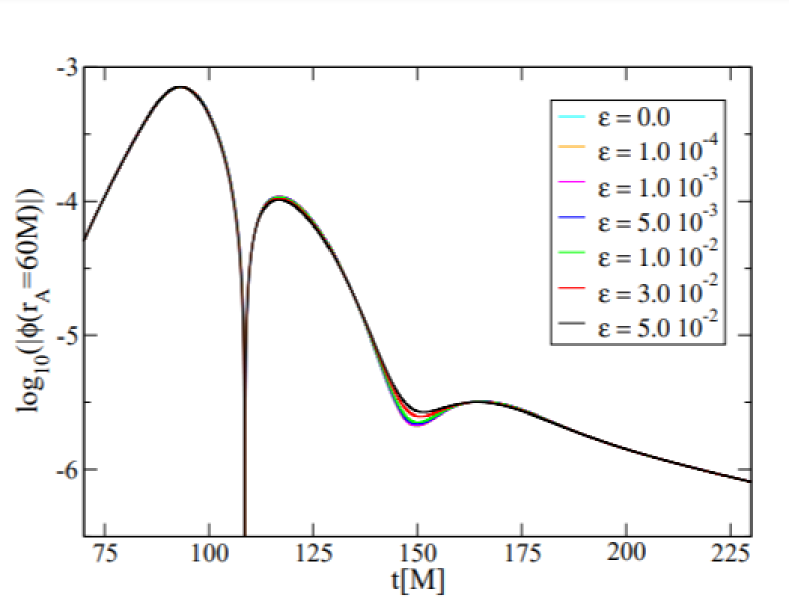


- Horizon growth



- Null convergence condition violation

Quasi normal modes of SF



$$\omega^R = \omega^R_{GR}(1 - 0.54\epsilon + 0.77\epsilon^2),$$
$$\omega^I = \omega^I_{GR}(1 + 0.45\epsilon - 1.33\epsilon^2).$$

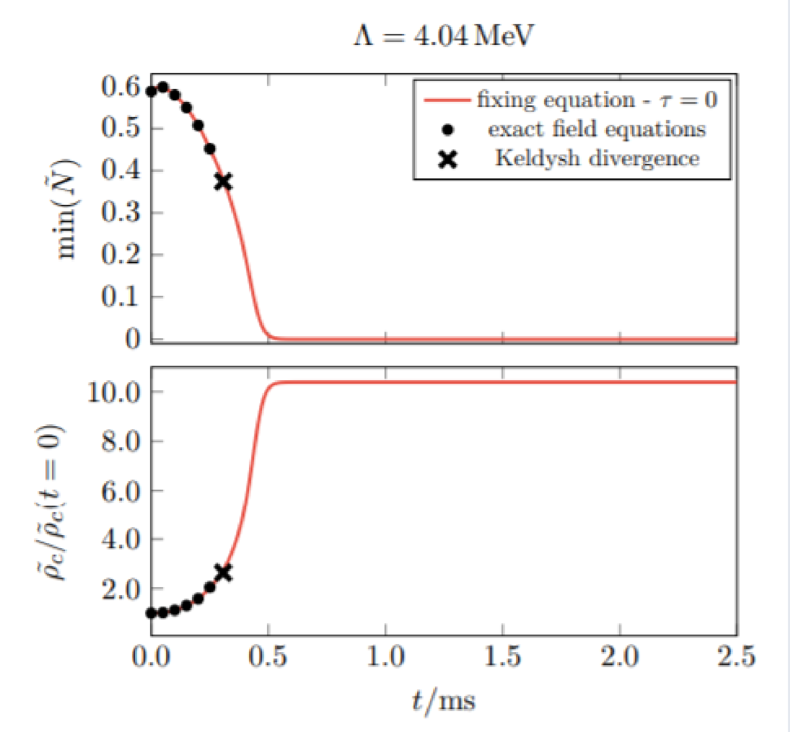
Other examples

- Gravitational collapse within Horndenski-class
- $L \sim R + K\left((\partial\varphi)^2\right) + matter$

- Star oscillations, scalar waves, but collapse issues

- Introduced aux field, and 'fixed' the eqns →

- Agreeing with result with a different gauge that avoids problems



[Bezares+, '21]

some more....

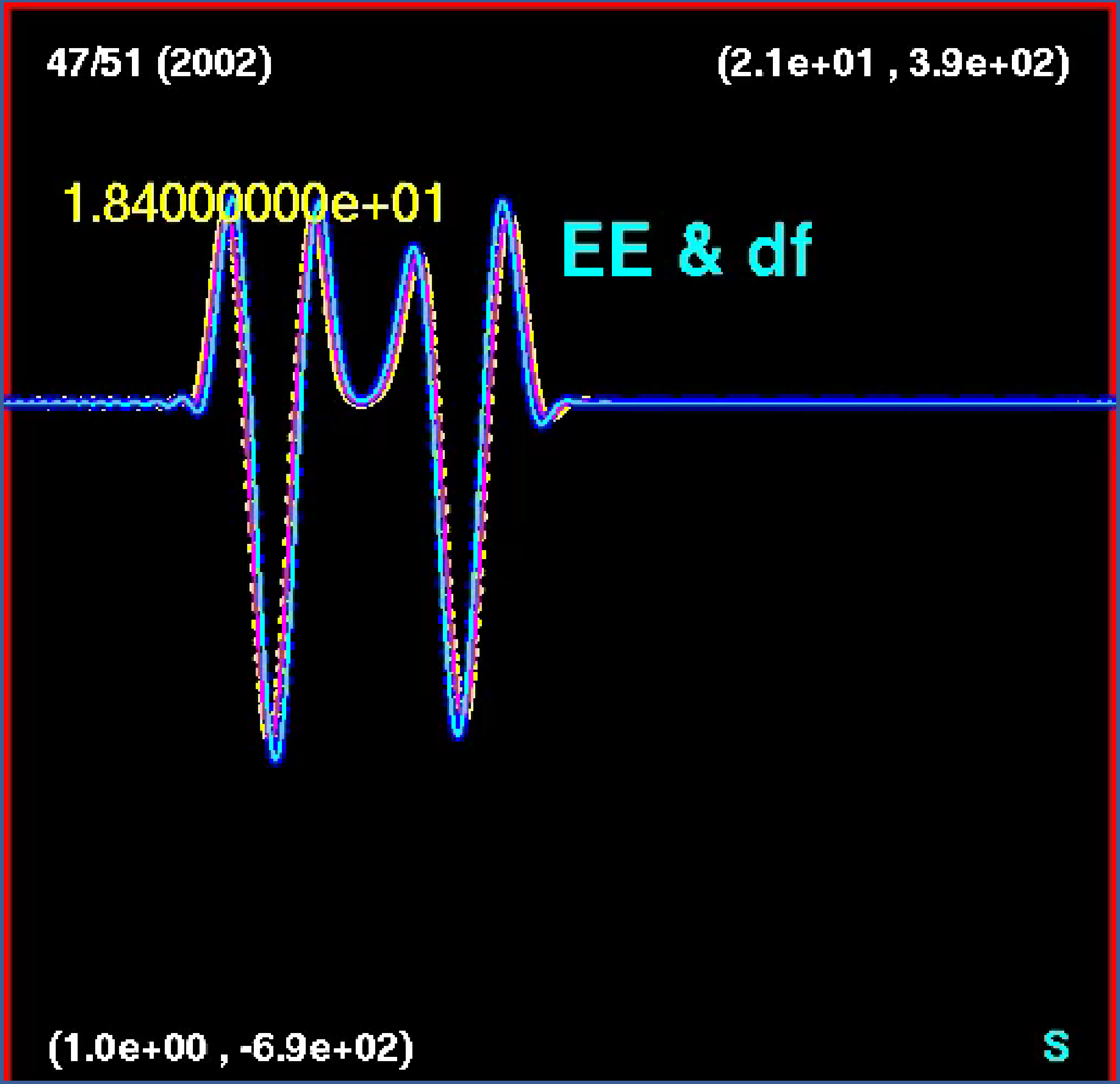
$$\phi_{,tt} = \phi_{,xx} + \lambda \partial_x^4 \phi \longrightarrow \begin{cases} \phi_{,tt} = \phi_{,xx} + \lambda E_{,xx} \\ E_{,tt} = E_{,xx} - \tau \partial_t E \\ \qquad + \sigma(-E + \phi_{,xx}) \end{cases}$$

— o —

$$\phi_{,t} = \lambda \phi_{,xx} \longrightarrow \begin{cases} \phi_{,t} = \lambda \Pi_{,x} \\ \partial_t \Pi = -\sigma(\Pi - \phi_{,x}) \end{cases}$$

— o —

$$\begin{aligned} \phi_{,t} = -6\phi\partial_x\phi - \partial_x^3\phi \\ + \varepsilon \partial^2\phi \end{aligned} \longrightarrow \begin{cases} \phi_{,t} = -6\phi\partial_x\phi - \partial_x^3\phi \\ \qquad + \varepsilon K \\ K_{,t} = -\tau(K - \phi_{,xx}) \end{cases}$$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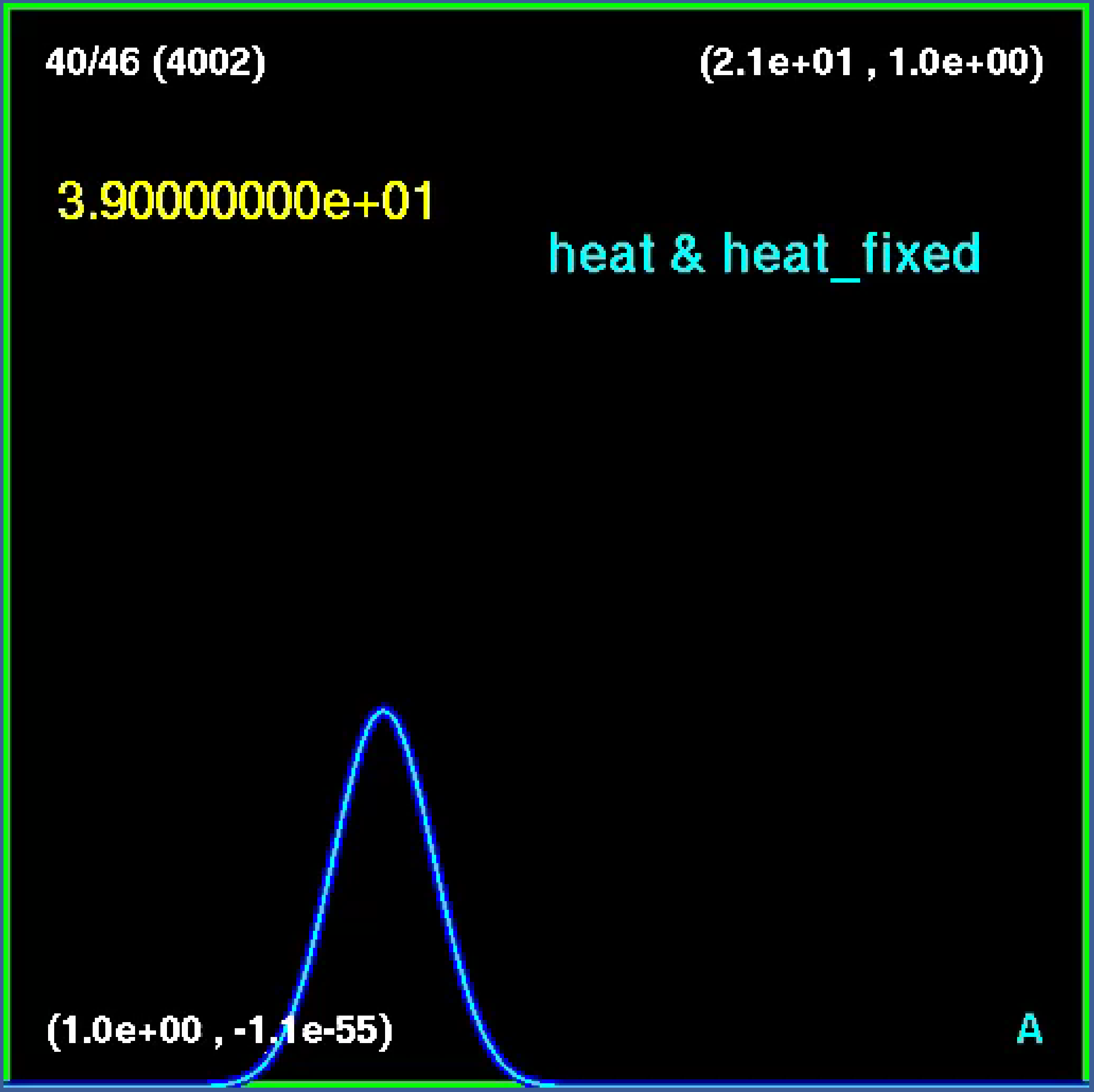
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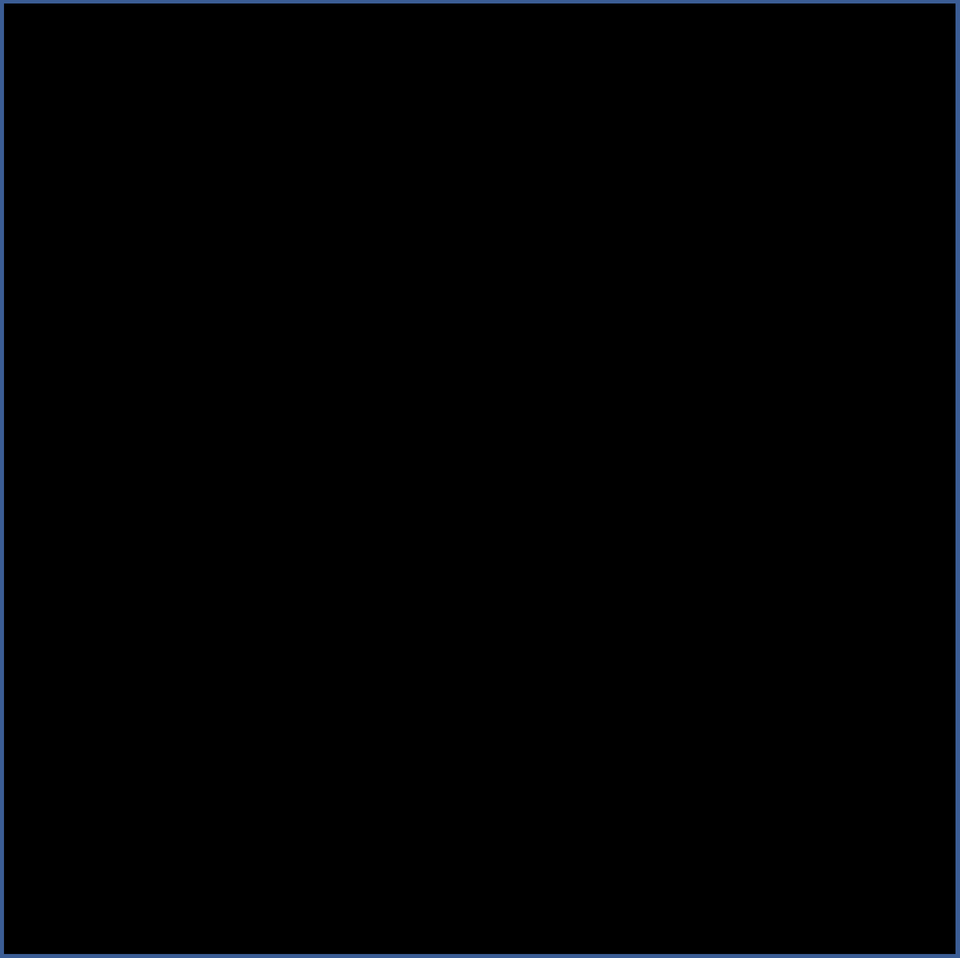
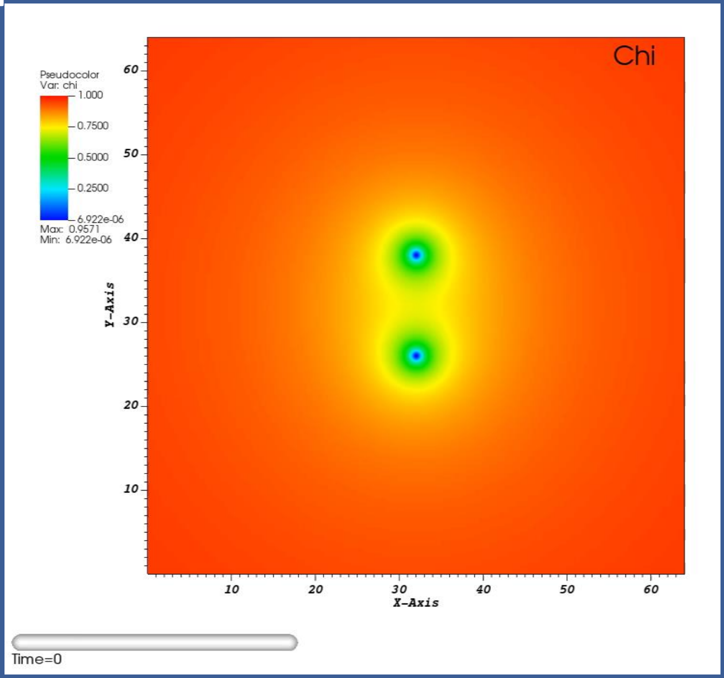


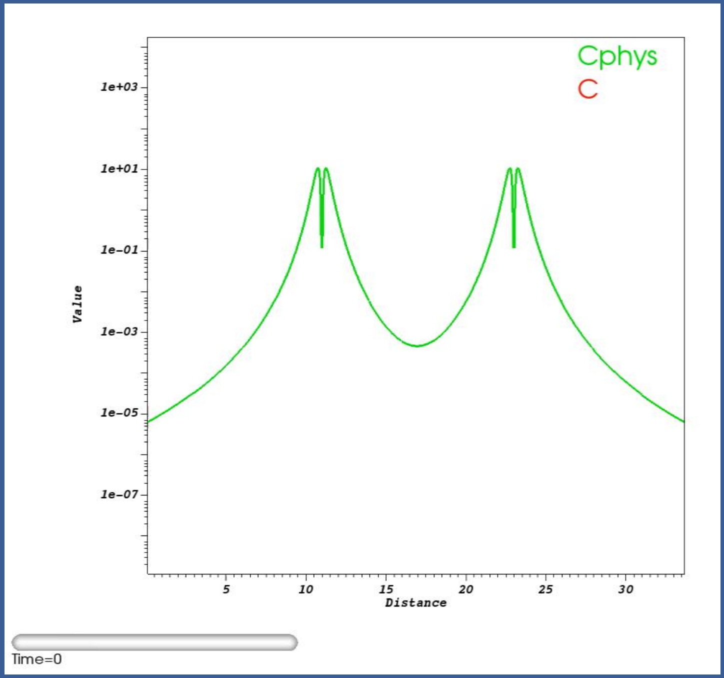
- See also [Galvez Ghersi-Stein ‘21] use of Renormalization Group flow to address secular effects

EFT, higher derivs

- $L \sim R + e\,(R_{abcd}\,R^{abcd})^2$
- EOM -> $R_{ab} = l\left(4W\,R_a^{pcd}R_{bpcd} - \frac{3}{2}W^2 g_{ab} + R_{ab}^{cd}\nabla_c\nabla_d W\right)$

- Again, introduce a new variable that has 2nd derivatives of the metric tensor (and convenient to reexpress the rhs) and 'fix' the eqn.





[Franca,Figueras,Cayuso,LL]

Drive carefully!

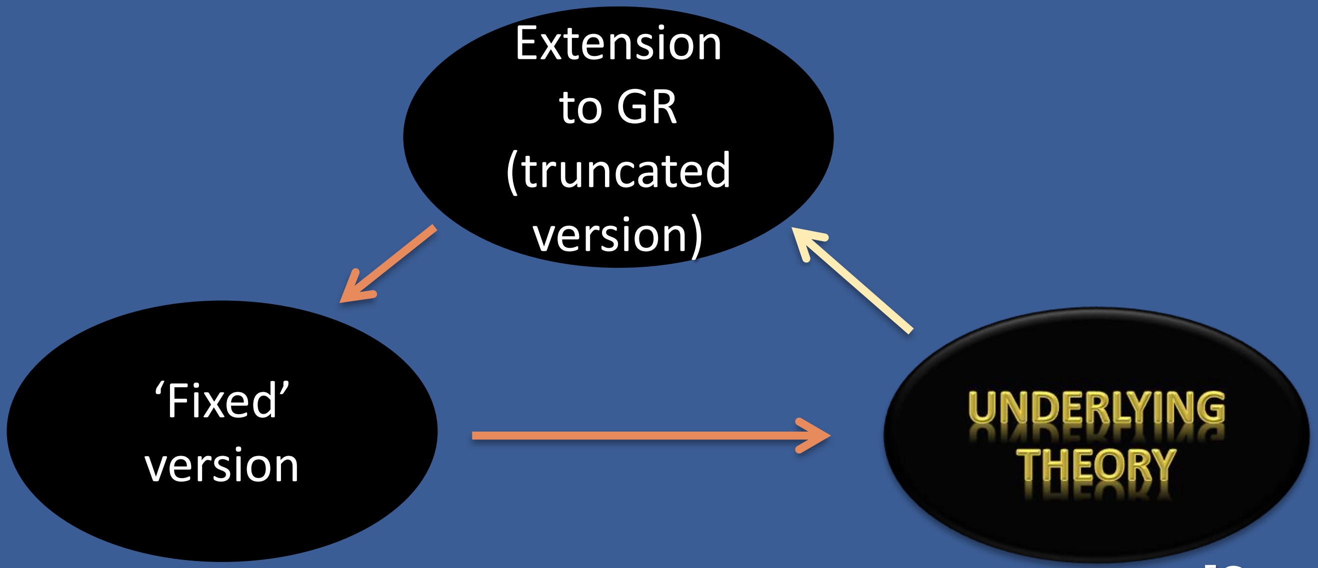
- You’d be playing God!
- Dependency with extra scale --lack there off!-- required to support validity of solution

- If so, dispersive properties of the basic system (GR) wins over corrections for long wavelengths (ie. in the regime of applicability that led to the new system)

- If not, perhaps (?) explore partially the system a la Quark-Gluon plasma in accelerators

- *Option 1: delicate (& uncertain?). Option 2: fine but can it be justified? One could argue yes in 3+1 dimensions but not above*
  - (drawing from LIGO/VIRGO, fluid-gravity correspondence and specific '2nd order' BH perturbation calculations) → all methods are intrinsically relying on this!

There seems to be a way to avoid 'not going to non-linear-land' with (many) GR alternatives and face upcoming data



[Cayuso,Ortiz,LL '17]

Further freedom!

$$\theta \to \theta + \frac{2}{M^4} \left(\partial_\mu \partial_\nu \theta\right) \left(\partial^\mu \theta\right) \left(\partial^\nu \theta\right) . \tag{36}$$

We can then rewrite the EFT action given by eq. (28) as

$$\frac{S}{v^2} \simeq -\int_{\mathbb{R}^4} \mathrm{d}^4 x \left\{ \frac{1}{2} \partial_\nu \theta(x) \cdot \partial^\nu \theta(x) - \frac{1}{2M^2} [\partial_\nu \theta(x) \cdot \partial^\nu \theta(x)]^2 \right.$$
$$\left. + \frac{2}{M^4} \left[\partial_\mu \partial_\nu \theta(x) \cdot \partial^\sigma \partial^\mu \theta(x) \cdot \partial_\sigma \theta(x) \cdot \partial^\nu \theta(x) + \partial_\sigma \theta(x) \cdot \partial^\sigma \left[\partial_\mu \partial_\nu \theta(x) \cdot \partial^\mu \theta(x) \cdot \partial^\nu \theta(x) \right] \right] \right\} . \tag{37}$$

By integrating the last term by parts, we obtain

$$\frac{S}{v^2} \simeq -\int_{\mathbb{R}^4} \mathrm{d}^4 x \left\{ \frac{1}{2} \partial_\nu \theta(x) \cdot \partial^\nu \theta(x) - \frac{1}{2M^2} [\partial_\nu \theta(x) \cdot \partial^\nu \theta(x)]^2 \right.$$
$$\left. + \frac{2}{M^4} \partial_\mu \partial_\nu \theta(x) \cdot \left[\partial^\sigma \partial^\mu \theta(x) \cdot \partial_\sigma \theta(x) \cdot \partial^\nu \theta(x) - \Box \theta(x) \cdot \partial^\mu \theta(x) \cdot \partial^\nu \theta(x) \right] \right\} . \tag{38}$$

↓ [Solomon-Trodden '18]

$$- \Box \theta + \frac{2}{M^2} \left[\left(\partial_\nu \theta\right) \left(\partial^\nu \theta\right) \Box \theta + 2 \left(\partial_\mu \partial_\nu \theta\right) \left(\partial^\nu \theta\right) \left(\partial^\mu \theta\right) \right]$$
$$+ \frac{2}{M^4} \left[3 \Box \theta \left(\partial_\mu \partial_\nu \theta\right) \left(\partial^\mu \partial^\nu \theta\right) - (\Box \theta)^3 - 2 \left(\partial_\mu \partial_\nu \theta\right) \left(\partial_\sigma \partial^\mu \theta\right) \left(\partial^\sigma \partial^\nu \theta\right) \right] = 0 .$$

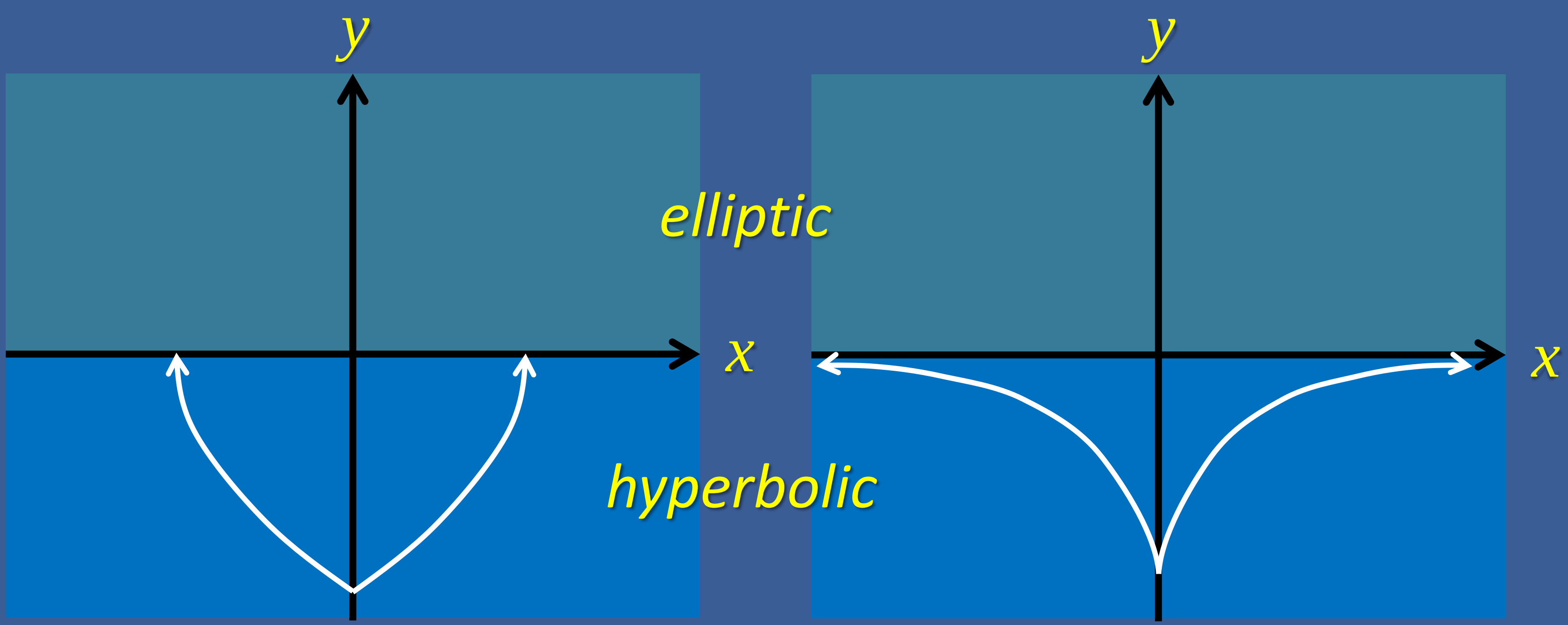
Two "Canonical" Mixed Type Equations

- The Tricomi Equation

$$\partial_y^2 u(x,y) + y\,\partial_x^2 u(x,y) = 0$$

- The Keldysh Equation

$$\partial_y^2 u(x,y) + \frac{1}{y}\,\partial_x^2 u(x,y) = 0$$



- Characteristics approach the sonic line (y=0) orthogonally, and corresponding speeds go to zero there
- Characteristics approach the sonic line tangentially, and speeds go to infinity there

Existence of regular solutions for the Tricomi Equation

$$\partial_y^2 u(x,y) + y\,\partial_x^2 u(x,y) = 0$$

- Morawetz considered a boundary consisting of three segments, two intersecting characteristic curves A and B in the hyperbolic region, connected to a smooth curve C in the elliptic region

- To get a unique, regular solution in the interior, free Dirichlet data be specified on C and *one* of the characteristic surfaces A (*or* B) but *not* the other