Introduction to Numerical Relativity

- Textbooks: H. Alcubierre, "Intro to 3+1 NR", 2008
  T. Baumgarte & S. Shapiro, "Numerical Relativity", 2010
  "NR: Starting from scratch", 2021
  [Also online lecture notes, e.g., Bourguignon gr-qc/0703035, LRRs]
- What is numerical relativity?
  - Solving EEs (or extensions) in 3+1 dimensions typically using HPC
  - To model the nonlinear regime of gravity

- Science cases

- GW source modelling:
  \[ \rho \]
  \[ 1 \leq \rho \leq 1 \]
  Post-Newtonian
  \[ \rho < 1 \]

Ingredients

1) Theoretical model

   - TODAY EEs in vacuum in 4D
   - Asymptotic flat
   \[ g_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = 0 \]

2) Spacetime decomposition
   - Space, time - dep. explicit
   - Kinematic evol.
3) \(3+1\) decomposition of field eqs.
   \[\rightarrow\] dynamics (hyperbolic PDEs)
   \(\rightarrow\) constraint (elliptic PDEs)

4) Initial data
   \((9_{\mu
\nu}, \delta 9_{\mu
\nu})\) \(t=0\)
   \(\rightarrow\) physical system under consideration
   \(\rightarrow\) solve constraints
   \(\rightarrow\) initial value problem

5) Evol. eqs. \(\rightarrow\) well-posed initial value problems

6) Gauge choices, treatment of BH

7) Observables
   \(\rightarrow\) wave extraction (Newman-Penrose formalism)
   \(\rightarrow\) apparent horizon

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2) \(3+1\) of spacetime

Prolate 4D manifold \((M, g)\) into
space-like hypersurfaces \((\Sigma, \nu)\) parametrized
by \(t \in \mathbb{R}\)
\[M = \Sigma \times \mathbb{R}\]

\(\nu^\mu\) - normal, timelike vector \(\nu^\mu \nu_\mu = -1\)
\(\nu^\mu\) - on \(\Sigma\): induced, spatial metric \(\nu_{ij}\)
measure proper distances \(ds^2 = \eta_{ij} \nu^i \nu^j\)
Lapse $\alpha$: proper time between hypersurfaces measured by observer moving along $n^\mu$.

Shift $\beta^i$: relative velocity between normal observer and lines of constant coords. by construction $n^\mu n_\mu = 0$.

L-metric $g_{\mu\nu}$ in terms of $(\delta_{ij}, \alpha, \beta^i)$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$= -\left(\alpha^2 - \beta^i\beta_i\right)dt^2 + 2 \beta^i dt dx^i + f_{ij} dx^i dx^j$$

**Note:** $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$

- $\gamma^\mu_\nu$ defines a projection op.
  $$\gamma^\mu_\nu = \delta^\mu_\nu + n^\mu n_\nu$$
- a vector $V^\mu = \omega^\mu n^\nu + \nu^\mu$
  $$\omega = -n_\nu V^\nu$$
  $$\nu^i = \gamma^i_\nu V^\nu$$

**Extrinsic curvature** $K_{\mu\nu}$

$$K_{\mu\nu} = -\frac{1}{2} \left( \partial_\mu \ln f_{\nu\nu} \right)$$

one can show: $K_{\mu\nu} = -\frac{1}{2\alpha} (\partial_\mu - \frac{\nabla_\mu}{\alpha}) f_{\nu\nu}$

$$\rightarrow (\partial_\mu - \frac{\nabla_\mu}{\alpha}) f_{ij} = -2\alpha K_{ij}$$

\[\text{kinematic eq.}\]
3 + 1 Split of Field Equations

\( \rightarrow \) Dynamic

(i) Gauss-Codazzi relations

\( (4) \) \( \nabla \rho \rightarrow (3) \nabla \rho \)

\( \delta_\mu \nabla_\mu \rho = (f \Phi)_\mu \)

(ii) Projection of Energy-Momentum Tenser

\( T_{\mu \nu} = \rho - \sigma = T_{\mu \nu} n^\mu n^\nu \) (Energy density)

\( j_\mu = -T_{\nu \mu} n^\nu \) (Energy-Momentum flux)

\( \mathcal{S}_\mu = T_{\mu \nu} n^\nu (\text{Spatial Stress Tensor}) \)

(iii) Decomposition of EEs:

- Hamiltonian constraint

\( H = 2 \, g_{\mu \nu} n^\mu n^\nu = 0 \)

\( = (3) \, R - R_{ij} K^{ij} + K^2 \)

- Momentum constraint

\( \mathcal{H}_i = -j_\mu n^\mu g_{\nu \mu} = D_J K_{ij} - D_i K^i_j = 0 \)

Evolution Eqs:

\( \delta_{\mu \nu} g_{\rho \sigma} = 0 \Rightarrow \nabla_\mu K_{ij} \)

\( \delta_\mu \nabla_\rho g_{\sigma \rho} = 0 \Rightarrow \nabla_\mu K_{ij} \)

Dynamic

\( \partial_\tau K_{ij} = \nabla_\mu K_{ij} - D_i D_j \alpha \)

\( + \alpha [R_{ij} + k K_{ij} - 2 K_{ik} K_{kj}] \)

Kinematic:

\( \partial_\tau \mathcal{L} = \nabla_\mu \mathcal{L} - 2 \alpha K_{ij} \)
Note: structure:
\[ \frac{\partial}{\partial t} u = F \left[ \partial, \partial_t u, \partial_{xx} u \right] \]

\text{vector of evol vars}

in practice: adopt Method of lines

\[ \text{rhs: } F \left( \partial, \partial_t u, \partial_{xx} u \right) \]

\text{choose a type discretization for spatial [finite differences, spectral method]}

\[ \text{lhs: } \partial_t u \]

\text{choose time integrator (e.g. Runge-Kutta 4th order)}

Note: adopt a free evolution scheme, i.e.

- solve constraints only for initial data
- monitor constraints during evolution (check that they remain satisfied)
- use BI to show that constraints remain satisfied if satisfied at \( t = 0 \)

\text{system evolves according to evol. system.}

Well-posed minimal value formulation

\text{Def: A system of PDEs}

\[ \begin{cases} \frac{\partial}{\partial t} f = A \partial_{xx} f + B f \\ f(t=0) = g \end{cases} \]

is said to be well-posed IVP if there exists a unique solution that depends continuously on smooth ID
in particular, a system is well-posed if
$\frac{\partial}{\partial x_i} \phi \text{ and } \frac{\partial}{\partial x_j} \phi \text{ are bounded}
$
we have
$||f(x, t)|| \leq k \ e^{-\alpha t} ||f(t=0)||$

lay person: compare to wave eqn

V1: 4D

$\mathbf{R}_{\mu\nu} = 0$
$-\gamma \kappa_{\mu\nu} g_{\mu\nu}$
$+ \gamma \kappa_{\mu\nu} g_{\mu\nu}$
$+ \gamma \kappa_{\mu\nu} g_{\mu\nu}$
$+ \ldots$

scalar wave eqn
$\Box \phi = 0$

harmonic gauge
$\phi + \nabla^2 \phi = 0$

V2: 3+1 EES

$\partial^2 \chi_{ij} = -K_{ij}$
$\partial^2 K_{ij} = -\partial_{(i} \partial_{j)} \chi$
$+ \partial X_{ij} + \ldots$

Scalar wave eqn
$\Box \phi = 0$
introduce $\Pi = -\partial^2 \phi$

cause for illposedness

"Cure": Baumgarte - Shapiro (1999)

Casus: $W = \frac{1}{2} f_{ij}^2 (s, k, f = 1)$
$K_{ij} = f_{ij}^2 f_{ij}^2$
$X_{ij} = k_{ij}^2 A_{ij} - W^2 (k_{ij}^2 f_{ij}^2)$
Gauge choices, \( \psi, (\alpha, \beta) \)

Note: simple is not always best.
(p. g. \( \alpha = 1 \), \( \beta = 0 \) = reach BL sing.

Wishlist:
- Avoid reading singularity
- Evol. eqs + gauge cond. form a well-posed PDE system
- Easy to implement
  - Spec., avoid elliptic eqs (comp: exp. ask)
  - Use evol. eqs

Common choices: puncture gauge

1 + log slicing

\[ \dot{\alpha} = -2\alpha (K - K_0) \]

\( \lambda \to 0 \) near singularity

"Singularity avoidance" spatial hypersurfaces cannot be arbitrarily close to singularity

\[ \mathcal{I} = \frac{3}{2} M \]
Main data

Goal: prescribe \((K_{ij}) | t = 0\)

1. Count: specify 12 components

2. Solve constraints: specify 4 independ.

3. "Free" choice for 8:
   - motivated by physical system
   - simplify

- Conformal decomposition:
  \[
  \begin{align*}
  f_{ij} &= \eta^{4} f_{ij} \\
  K_{ij} &= \eta^{-2} \xi_{ij} + \frac{1}{3} \eta^{4} f_{ij} \eta
  \end{align*}
  \]

- Example: Single BH:
  - time-symmetry: \(\xi_{ij} = 0\)
    \[\eta_{i} = 0\]
  - conformal flatness, i.e. \(\eta_{ij} = \eta_{ij}\)

- Asymptotic flatness, i.e. \(\lim_{r \to \infty} \eta_{ij} = 0\)
  \[\Rightarrow |\mathcal{H} = \Delta_{r} \eta_{ij}\]

- Simplest non-trivial solution:
  \(\eta = 1 + \frac{R}{r}\)

Identify \(R = \frac{M}{2} \Rightarrow \eta = 1 + \frac{M}{2r}\)

\[ds^2 = -\left(1 + \frac{M}{2r}\right)^2 dt^2 + \left(1 + \frac{M}{2r}\right)^4 d\xi \cdot d\xi\]

Schwarzschild in isotropic coordinates

Note: suppose for heads (zero moment).

Seed solution for BBH, together with \(\eta_{ij}\) expressions.