Workshop "Mathematical and Computational Challenges in the Era of GW astronomy" | 14 Sept 2021

Introduction to Numerical Relativity

- Textbooks: • M. Alcubierre "Intro to 3+1 NR" 2008
  - T. Baumgarte & S. Shapiro: "Numerical Relativity" 2010
    "NR: Starting from scratch" 2021
  - M. Shibata: "Numerical Relativity" 2015

[also online lecture notes, eg, Gourgoulhon gr-qc/0703035; LRRs]

- What is numerical relativity?
  - solving EEs (or extensions) in 3+1 dims typically using HPC to model the nonlinear regime of gravity

- Science cases

GW source modelling:

$v/c \ll 1$, "$\Phi < 1$" post Newtonian

$v/c \lesssim 1$ NR nonlinear

- ringdown
- perturbation theory
- NR

Ingredients

1) theoretical model

TODAY EEs in vacuum in 4D asympt. flat

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0$$

2) Spacetime decomposition
   - → space, time-dep. explicit ←
   - → kinematic evol.

3) 3+1 decomposition of field eqs.
→ dynamics (hyperbolic PDEs)
→ constraint (elliptic PDEs)

FYI xTensor tutorial

4) Initial data
$(g_{\mu\nu}, \partial_t g_{\mu\nu})|_{t=0}$
→ phys. system under consideration (e.g. single or binary BHs)
→ solve constraints (4 coupled, elliptic PDEs)

5) Evol. eqs → well-posed initial value problems

6) Gauge choices, treatment of BH

7) Observables
→ wave extraction (Newman-Penrose form., metric perturbations)
→ apparent horizon

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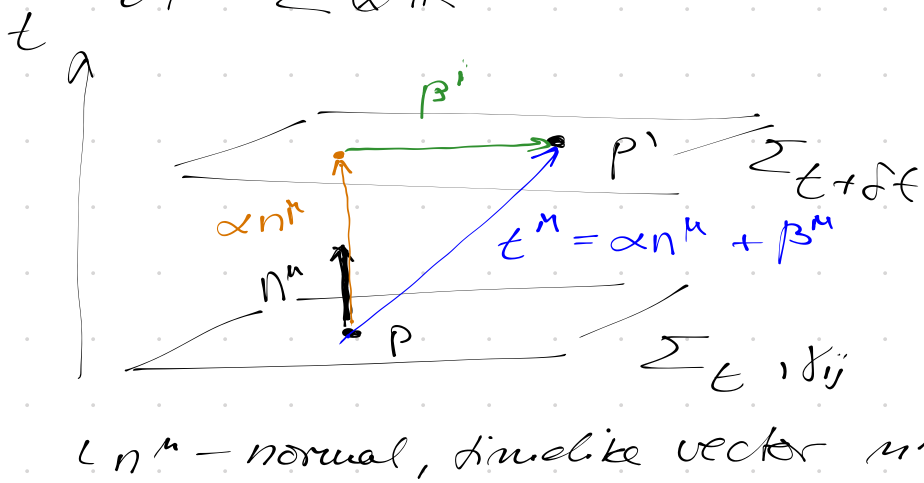
2) 3+1 of spacetime

∟ foliate 4D manifold $(\mathcal{M}, g)$ into spacelike hypersurfaces (Σ, γ) parametrized by $t \in \mathbb{R}$

$\mathcal{M} = \Sigma \otimes \mathbb{R}$

$\mu, \nu, \ldots = 0, \ldots, 3$
$i, j, \ldots = 1, 2, 3$
x^μ - coords



∟ n^μ – normal, timelike vector $n^\mu n_\mu = -1$
∟ on Σ: induced, spatial metric γ_{ij}
measure proper distances $dl^2 = \gamma_{ij}\, dx^i dx^j$

└ lapse α: proper time between hypersurfaces measured by observer moving along n^μ

└ shift β^i: relative velocity between normal observer and lines of const. spat. coords.

by construction $\beta^\mu n_\mu = 0$

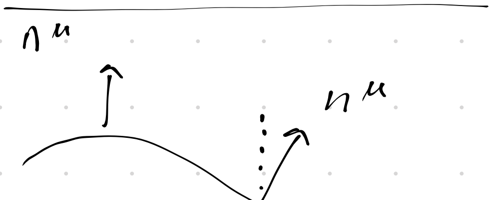
└ metric $g_{\mu\nu}$ in terms of $(\gamma_{ij}, \alpha, \beta^i)$

$$
\begin{aligned}
ds^2 &= g_{\mu\nu}\, dx^\mu \cancel{\otimes} dx^\nu \\
&= -\left(\alpha^2 - \beta^i \beta_i\right) dt^2 \\
&\quad + 2\gamma_{ij}\beta^i\, dt\, dx^j + \gamma_{ij}\, dx^i dx^j
\end{aligned}
$$

└ note: $\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$

- γ defines a projection op.
  $$\gamma^\mu{}_\nu = \delta^\mu{}_\nu + n^\mu n_\nu$$
- a vector $V^\mu = \mathcal{V} n^\mu + v^\mu$
  $$\mathcal{V} = -n_\nu V^\nu \qquad v^i = \gamma^i{}_\nu V^\nu$$

└ extrinsic ~~cu~~curvature $K_{\mu\nu}$



$$K_{\mu\nu} = \underbrace{-}_{\text{sign convention}} \gamma^\kappa{}_\mu \nabla_\kappa n_\nu$$

one can show:
$$
\begin{aligned}
K_{\mu\nu} &= -\frac{1}{2\alpha}\mathcal{L}_n \gamma_{\mu\nu} \\
&= -\frac{1}{2\alpha}\left(\partial_t - \mathcal{L}_\beta\right)\gamma_{\mu\nu}
\end{aligned}
$$

$$\longrightarrow \boxed{\left(\partial_t - \mathcal{L}_\beta\right)\gamma_{ij} = -2\alpha K_{ij}}$$ kinematic eq.

3) 3+1 split of field equations

↳ dynamic

(i) Gauss-Codazzi relations) → ${}^{(4)}R_{\mu\nu\rho\sigma} \leftrightarrow {}^{(3)}R_{\mu\nu\rho\sigma}$

$\nabla_\mu \qquad D_\mu = (\gamma D)_\mu$

on Monday 20 Sept.
xtensor for decomposition
- download notebook
- install xtensor (xact.es)

(ii) projection of energy-momentum tensor $T_{\mu\nu}$:

- $\rho = T_{\mu\nu} n^\mu n^\nu$ (energy density)
- $j_\mu = - T_{\rho\nu} \gamma_\mu{}^\rho n^\nu$ (energy-mom. flux)
- $S_{\mu\nu} = T_{\rho\sigma} \gamma^\rho{}_\mu \gamma^\sigma{}_\nu$ (spatial stress tensor)

(iii) decomposition of EEs:

- Hamiltonian constraint $\mathcal{H} = 2\, G_{\mu\nu} n^\mu n^\nu$

$$= 0$$
$$= {}^{(3)}R - K_{ij}K^{ij} + K^2$$

- Momentum constraint

$$\mathcal{M}_i = - \gamma_i{}^\mu n^\nu G_{\mu\nu} = D_j K_{ij} - D_i K = 0$$

Evolution Eqs:

$$\gamma_\mu^\rho \gamma_\nu^\sigma G_{\rho\sigma} = 0 \implies \mathcal{L}_n K_{ij}$$

dynamical $\partial_t K_{ij} = \mathcal{L}_\beta K_{ij} - D_i D_j \alpha + \alpha \left[{}^{(3)}R_{ij} + K K_{ij} - 2 K_{ik} K^k{}_j \right]$

kinematic: $\partial_t \gamma_{ij} = \mathcal{L}_\beta \gamma_{ij} - 2\alpha K_{ij}$

note: structure:

$$\underbrace{\partial_t u}_{\text{vector of evol vars}} \simeq F\left[\partial_i \partial_j u, \partial_i u, u\right]$$

in practise: adopt Method of Lines

→ rhs $(F(\partial_i \partial_j u, \partial_i u, u))$

└ choose a type discretization for spatial [finite differences, spectral method]

→ lhs: $\partial_t u$

└ choose time integrator (e.g. Runge-Kutta 4^{th} order)

└ note: adopt a free evolution scheme, ie,

- solve constraints only for initial data
- monitor constraints during evolution (check that they remain satisfied)
- use BI to show that constraints remain satisfied if • satisfied @ $t=0$
  • system evolves according to evol. system.

Well-posed initial value formulation

Def: A system of PDEs

$$\begin{cases} \partial_t f = A^p \partial_p f + B f \\ f(t=0) = g \end{cases}$$

is said to be a well(-posed) IVP if there exists a unique solution that depends continously on smooth ID in particular, a system is well-posed, if ($\exists$ k = const, a = const. s.t. for all f)

we have

$$\|f(t,\cdot)\| \le k e^{a\cdot t} \|f(t=0,\cdot)\|$$

Lay person: compare to wave-eqn

V1: 4D EES

$$R_{\mu\nu} = 0$$
$$= \eta^{\kappa\lambda}\partial_\kappa\partial_\lambda g_{\mu\nu}$$
$$+ \cancel{\eta^{\kappa\lambda}\partial_\mu\partial_\nu g_{\kappa\lambda}}$$
$$+ \cancel{\eta\,\partial\partial g}$$
$$+ \text{l.o.t.}$$

scalar wave eqn

$$\Box\phi = \eta^{\mu\nu}\partial_\mu\partial_\nu\phi = 0$$

harmonic gauge

$$\Box x^\mu = 0$$

V2: 3+1 EES

$$\partial_t \gamma_{ij} \simeq -K_{ij}$$
$$\partial_t K_{ij} \simeq -\partial_i\partial_j\alpha + \alpha\,{}^{(3)}R_{ij} + \text{l.o.t}$$
$$\simeq -\partial_i\partial_j\alpha + \alpha\left[\gamma^{lm}\partial_l\partial_m\gamma_{ij} + \gamma^{lm}\partial_i\partial_j\gamma_{lm} + \dots\right] + \text{l.o.t.}$$

cause for illposedness

Scalar wave eq

$$\Box\phi = 0$$

introduce $\Pi = -\partial_t\phi$

$$\partial_t\phi = -\Pi$$
$$\partial_t\Pi = -\Delta\phi = -\eta^{ij}\partial_i\partial_j\phi$$

"cure": • Baumgarte - Shapiro (1999) - Shibata - Nakamura (1995) (BSSN) form.

vars: $W = \gamma^{-1/6}$ ($\cdot 24$)

$$\tilde\gamma_{ij} = W^2\gamma_{ij} \quad (\text{s.t. } \tilde\gamma = 1)$$
$$K = \gamma^{ij}K_{ij}, \quad \tilde A_{ij} = W^2 A_{ij} = W^2\left(K_{ij} - \tfrac{1}{3}\gamma_{ij}K\right)$$
$$\tilde\Gamma^i = \tilde\gamma^{kl}\tilde\Gamma^i_{kl} = -\partial_j\tilde\gamma^{ij}$$

Gauge choices, i.e. (α, β^i)

note: simple is not always best
(e.g. $\alpha = 1$, $\beta^i = 0$ => reach BH sing. in finite time!

Wishlist: • avoid reaching singularity
• evol eqs + gauge cond. form a well-posed PDE system
• easy to implement
spec., avoid elliptic eqs (comput. expensive)
use evol. eqs.

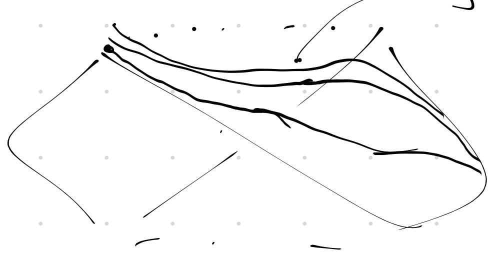
common choices: puncture gauge

1+log slicing

$$\boxed{\partial_t \alpha = -2\alpha (K - K_0)}$$

↳ $\alpha \to 0$ near singularity
↳ "singularity avoidance"
spat. hypersurfaces cannot be arbitrarily close to singularity

$r_s = \frac{3}{2} M$



$\tilde{\Gamma}$-driver shift

$$\partial_t \beta^i = \beta_\Gamma \tilde{\Gamma}^i - \eta_\beta \beta^i$$

advection

$\simeq \partial_j \tilde{\gamma}^{ij}$

Initial data

Goal: prescribe $(\gamma_{ij}, K_{ij})\big|_{t=0}$

- count: specify 12 ind. components
- solve constraints: specify <u>4</u> ind. comp.
- "free" choice for remaining γ:
  - motivated by physical system
  - simplify

- conformal decomposition:

$$\gamma_{ij} = \psi^4 \hat{\gamma}_{ij}$$
$$K_{ij} = \psi^{-2} \hat{A}_{ij} + \frac{1}{3}\psi^4 \hat{\gamma}_{ij} K$$

- Example: single BH:
  - time-symmetry: $K_{ij} = 0$

    $\Rightarrow \mathcal{M}_i = 0$ ✓
  - conformal flatness, ie, $\hat{\gamma}_{ij} = \eta_{ij}$

    $\hat{\Delta} = \hat{\Delta}_{fl}$
  - asymp. flatness, ie, $\boxed{\lim_{r\to\infty} \psi = 1}$

$$\Rightarrow \boxed{\mathcal{H} = \hat{\Delta}_{fl} \psi}$$

- simplest non-trivial solution:

$$\psi = 1 + \frac{k}{R}$$

identify $k = \frac{M}{2} \longrightarrow \psi = 1 + \frac{M}{2R}$

$\rightarrow ds^2 = -\alpha^2 dt^2 + \left(1 + \frac{M}{2R}\right)^4 \eta_{ij}\, dx^i dx^j$

Schwarzschild in isotropic coordinates

- <u>note</u>: superpose for headon (zero momenta)
  - seed solution for BBH, together with Bowen-York extrinsic cur