Topics in mathematical general relativity

Stefanos Aretakis

University of Toronto

IPAM, Los Angeles
September 21, 2021

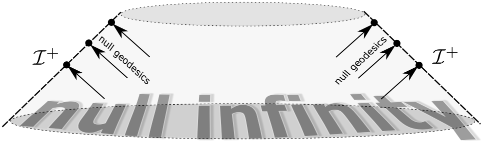
Outline

- Problems: Cosmic censorship conjectures
- Methods: Double null foliations
- Progress:
  - Formation of trapped surfaces,
  - Cauchy horizons,
  - Naked singularities

Null infinity

In order to study isolated systems in the universe we need to investigate the radiation that is emitted from these systems and reaches far away observers (such as ourselves). We therefore need a notion that models the region where radiation scatters. This gives rise to a concept known as future null infinity, an ideal incoming null hypersurface "at infinity", traditionally denoted by $\mathcal{I}^+$. We have the following heuristic definition:

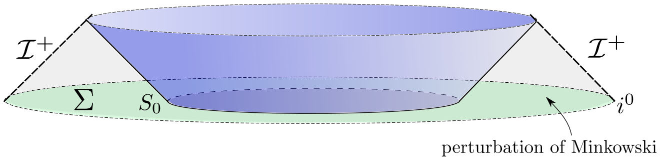
- *future null infinity* $\mathcal{I}^+$ *consists of all limit points of future-directed null geodesics which reach arbitrarily large spatial distances.*



Similar construction can be considered for the past. We define *future timelike infinity* i^+ as the limit point of future-directed timelike geodesics and *past time-like infinity* i^- as the limit point of past-directed timelike geodesics. The above naive definitions can be made precise in terms of conformal tranformations or null foliations and optical functions.

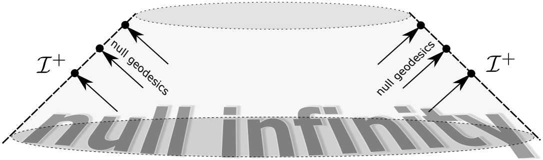
Asymptotical flatness and null infinity

We are mostly interested in studying isolated systems (such as solar systems, black holes and galaxies) in the universe. Hence, we can assume that far away from these systems the spacetime approaches the flat Minkowski spacetime. Such a condition can a priori only be imposed on the initial Cauchy hypersurface Σ. Hence, let us assume that the data on Σ are asymptotically flat, i.e. approach Minkowskean data at infinity. Then, there exists a sphere S_0 in Σ such that the data on the exterior of S_0 in Σ is a small perturbation of the flat Minkowski data. Then it follows by the stability of Minkowski theorem, proved by Christodoulou–Klainerman and Klainerman–Nicolo, that one can attach a piece of future null infinity at the Cauchy development $D^+(\Sigma)$ of Σ:



Completeness of null infinity

Roughly speaking, completeness of null infinity implies that observers on null infinity can receive radiation (for example from isolated systems) for infinite proper time. Thinking of $\mathcal{I}$ as an null hypersurface, its completeness simply corresponds to future and past completeness of its null generators. Another way to think of the completeness of $\mathcal{I}$, in a *limiting* sense, without referring to $\mathcal{I}$ as a concrete entity, is due to Christodoulou.

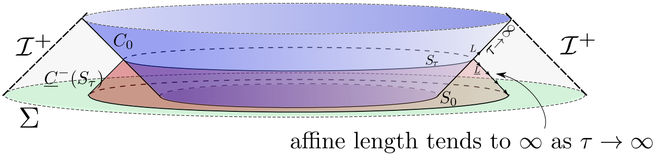


Past completeness of null infinity

Let C_0 be the outgoing null geodesic congruence normal to S_0. Let L be a geodesic null vector field along C_0 with affine parameter τ. We assume that we can take $\tau \to \infty$ along C_0. For each section S_τ, given by the level sets of τ on C_0, we consider the (conjugate) incoming null normal geodesic congruence $\underline{C}^-(S_\tau)$. Let $\underline{L}$ be the past-directed null geodesic vector field on $\underline{C}^-(S_\tau)$ normalized at S_τ such that

$$g(L, \underline{L}) = +1 \quad \text{at} \quad S_\tau.$$

The past-directed null generators of $\underline{C}^-(S_\tau)$ are generated by the vector field $\underline{L}$. Future null infinity is said to be past complete if the affine time it takes the null generators of $\underline{C}^-(S_\tau)$ to intersect Σ, starting from S_τ, tends to infinity as $\tau \to \infty$.

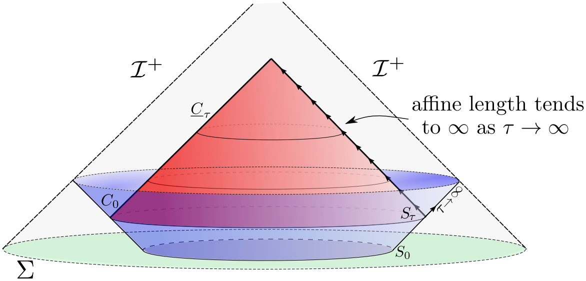


The works on the stability of Minkowski showed, in particular, that

- *Future null infinity of asymptotically flat spacetimes is past complete.*

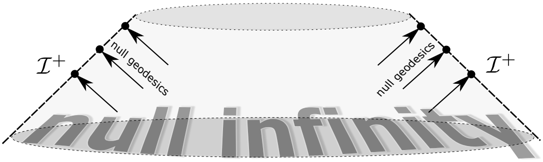
Future completeness of null infinity

Future completeness of future null infinity is defined in a similar way: Consider the null geodesic vector field $\underline{L}$ along $\underline{C}_\tau$ as above. The future-directed null generators of $\underline{C}_\tau$ (spanning the red cone in the figure below) are generated by $-L$. Future null infinity is said to be future complete if the affine time of the future-directed null generators of $\underline{C}_\tau$ starting from S_τ tends to infinity as $\tau \to \infty$ while remaining in the Cauchy development of the interior of S_τ.



Weak cosmic censorship

Unlike past completeness, it is not known if "generic" asymptotically flat spacetimes admit a future complete null infinity. In fact, this is one of the most outstanding open problems in general relativity and is known as the *weak cosmic censorship conjecture*.



- WCC is related to global existence for the Einstein equations.

Black holes

Black holes are regions which cannot communicate with far-away observers (to whom they appear "black"). Since far-away observers are heuristically modeled by future null infinity, we say that black holes cannot "communicate" with future null infinity $\mathcal{I}^+$. In other words

- *The black hole region $\mathcal{BH}$ in a spacetime $\mathcal{M}$ is the complement of the past $\mathcal{J}^-(\mathcal{I}^+)$ of future null infinity. Symbolically,*

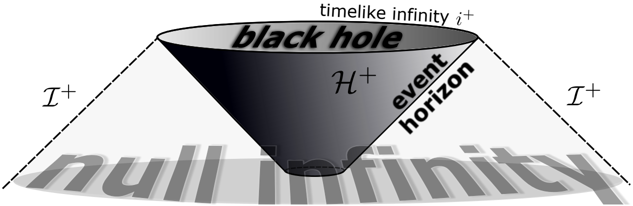
$$\mathcal{BH} = \mathcal{M} - \mathcal{J}^-(\mathcal{I}^+).$$

We have the following important definitions

- The exterior of the black hole region is known as the *domain of outer communications*.
- The boundary of the black hole region is known as the *future event horizon* (an outgoing null hypersurface).

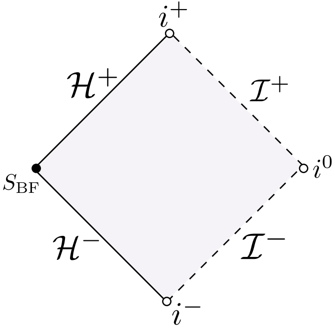
Weak cosmic censorship and black holes

There is a close connection of black holes and the weak cosmic censorship. Indeed, black hole spacetimes with a complete future null infinity have the property that even though observers on $\mathcal{I}$ *live forever* (in view of the future completeness of $\mathcal{I}$) they never receive radiation from the black hole region.
We will make use of the following heuristic (informal) picture to represent the black hole region, the domain of outer communications and future null infinity.



Weak cosmic censorship and black holes

Another representation of (black hole) spacetimes is given by the so-called *Penrose diagrams*. They represent the domain on $\mathbb{R}^2$ of a pair of conjugate optical functions forming a double null coordinate system (which we will discuss in detail later). Informally, the Penrose diagram of the domain of outer communications can be obtained by restricting to an angular slice of the above heuristic diagram as follows



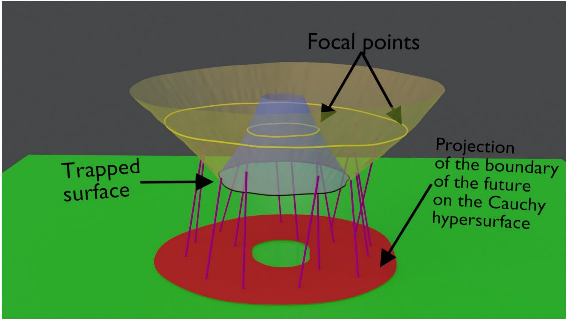
Singularities vs weak cosmic censorship

Does the existence of singularities in a spacetime imply the failure of completeness of its null infinity?

The answer is no, provided the singularity is inside the black hole region. Penrose's incompleteness theorem provides a general setting in which such a situation is possible.

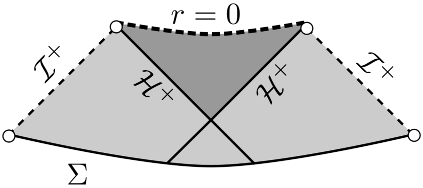
Penrose's incompleteness theorem: Let $(\mathcal{M}, g)$ be a globally hyperbolic vacuum spacetime with a non-compact Cauchy hypersurface $\mathcal{H}$ such that $\mathcal{M}$ contains a trapped surface S. Then $\mathcal{M}$ is future null geodesically incomplete.

The proof does not characterize the origin of the incompleteness, however.



Schwarzschild singularities

The Schwarzschild spacetimes provide examples of spacetimes which contain singularities and still admit a complete null infinity.



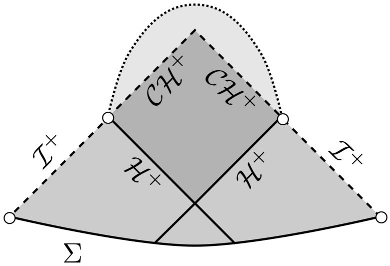
The Schwarzschild manifolds are inextendible as a Lorentzian manifold with C^2 metric (the Kretschmann scalar blows up).

From the PDE perspective, the failure of the metric to be C^2 is in itself insufficient to justify interpreting spacetime as having ended at an essential singularity. Indeed, Klainerman, Rodnianski, and Szeftel showed that the Einstein equations are well-posed for general initial data with curvature only in L^2. Moreover, Rodnianski–Luk proved a general well posedness theorem allowing data with δ-function singularities in curvature on null hypersurfaces.

Sbierski, however, proved that Schwarzschild is indeed C^0 inextendible. Physically speaking, observers crossing the event horizon not only measure infinite curvature but are in fact torn apart by infinite tidal deformations as they approach the singularity $r = 0$.

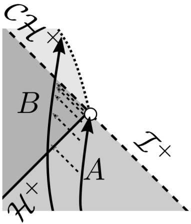
Cauchy horizon and breakdown of determinism

The Kerr family provides examples of spacetimes which are smoothly extendible as is shown below.



The boundary of the above extension is a bifurcate null hypersurface known as a Cauchy horizon. These extensions are non-unique! In other words, Kerr in fact exhibits a manifestation of breakdown of determinism in that predictability fails without any local observer directly measuring that the classical regime has been exited.

Penrose's blueshift instability: The wordline of observer A avoids the black hole and has infinite proper length. Observer B enters the black hole and arrives at the Cauchy horizon in finite proper time. A signal sent by A at constant frequency will be infinitely shifted to the blue when received by B as B approaches his finite crossing time.



The above geometric optics effect is indeed manifested as an instability in the behaviour of the scalar wave equation. Luk–Sbierski and Dafermos–Shlapentokh-Rothman proved that the solution has infinite (non-degenerate) energy on any spacelike hypersurfaces intersecting the Cauchy horizon transversally.

Strong cosmic censorship; various versions

C^2-**version:** For asymptotically flat vacuum initial data, the **maximal Cauchy development** is inextendible as a Lorentzian manifold with C^2 (continuous) metric.

C^0-**version:** For asymptotically flat vacuum initial data, the **maximal Cauchy development** is inextendible as a Lorentzian manifold with C^0 (continuous) metric.

H^1-**version:** For asymptotically flat vacuum initial data, the **maximal Cauchy development** is inextendible as a Lorentzian manifold with Christoffel symbols locally square integrable.

The latter version can be extrapolated from the linear wave equation under the identification $g \sim \psi$ and $\Gamma \sim \partial\psi$.

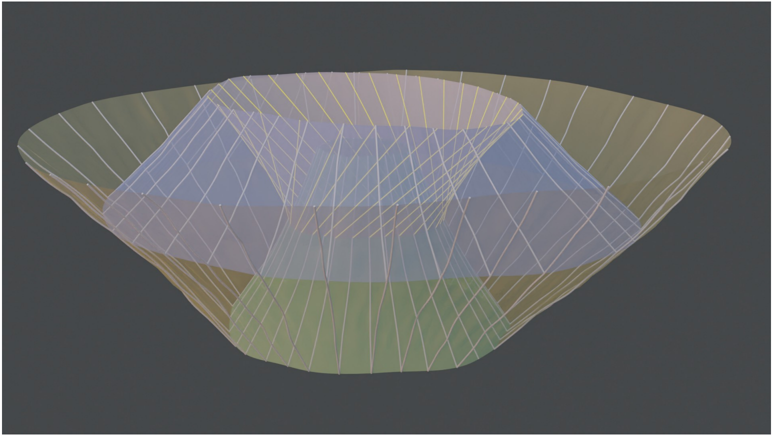
The double null foliation

Before we proceed with presenting more results, we introduce one of the most important methods, namely **the double null foliation**.

The double null foliation

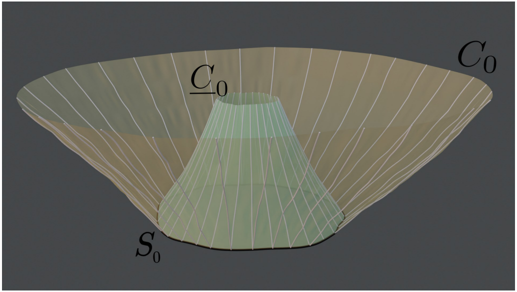
We define the double null foliation simply as the level sets of a pair of two optical functions (u, v):

$$\mathcal{D}_2 = \left\langle C_{u_0} = \{u = u_0\}, \quad \underline{C}_{v_0} = \{v = v_0\}\right\rangle$$



Initial hypersurfaces

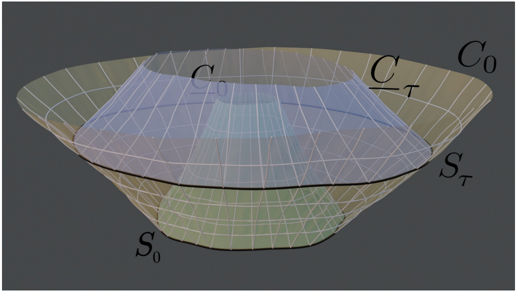
$$\boxed{S_0 \;\leftrightarrow\; \{u=0\}\cap\{v=0\}, \quad C_0 \leftrightarrow \{u=0\}, \quad \underline{C}_0 \leftrightarrow \{v=0\}.}$$



Incoming hypersurfaces

We have the following correspondence for the affinely parametrized geodesic vector fields along incoming null hypersurfaces $\underline{C}$.

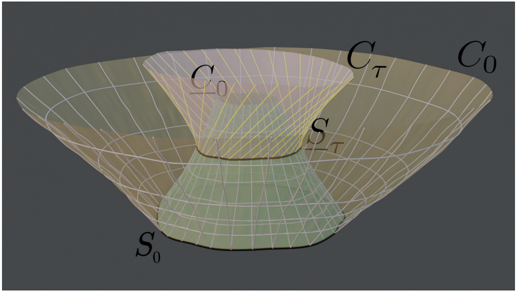
$$\boxed{\nabla v|_{\underline{C}} \leftrightarrow -\underline{L}_{geod}|_{\underline{C}}}$$



Outgoing hypersurfaces

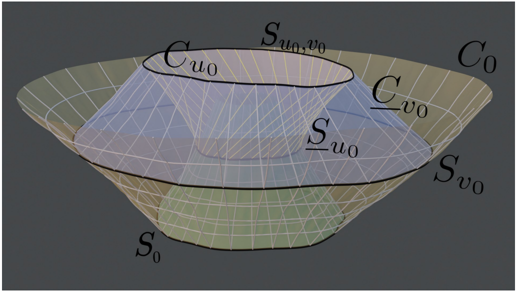
We have the following correspondence for the affinely parametrized geodesic vector fields along outgoing null hypersurfaces C.

$$\boxed{\nabla u|_C \leftrightarrow -L_{geod}|_C}$$



Vector fields

$$\boxed{g\left(\nabla u, \nabla v\right) = -\Omega^{-2}}$$



If we define the C-tangential vector field

$$L = -\Omega^2 \cdot \nabla u$$

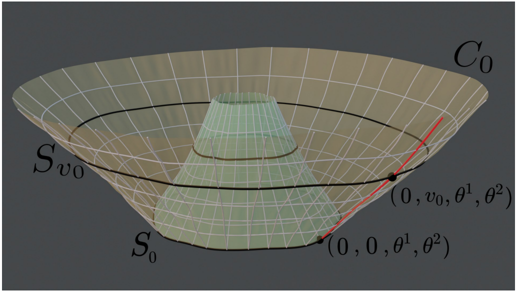
then

$$g\left(L, \nabla v\right) = 1$$

which implies that $Lv = 1$. Similarly we define $\underline{L}$ to obtain the equivariant pair $(L, \underline{L})$.

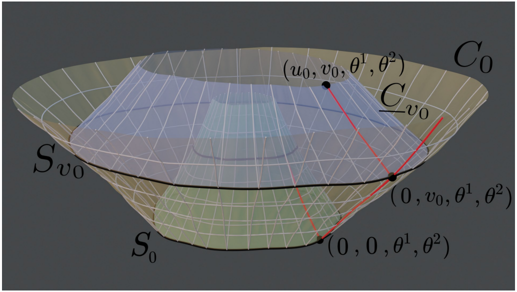
Angular coordinates along C_0

Propagation of angular coordinates along C_0



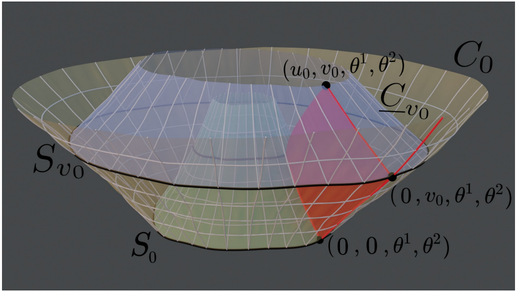
Angular coordinates

Propagation of angular coordinates everywhere



Constant angular coordinates

Surface with constant angular coordinates

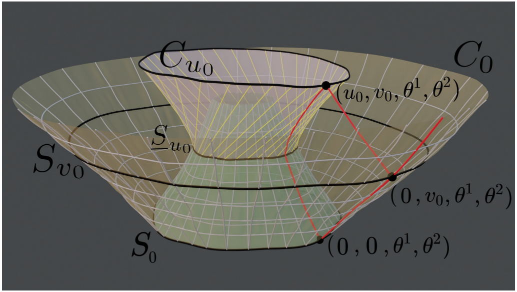


Double null coordinates

We can now introduce the double null coordinates $(u, v, \theta^1, \theta^2)$ and the corresponding coordinate vector fields ∂_v, ∂_u. Note that

$$\partial_v = L + b^i \partial_{\theta^i}$$

for some vector field $b = b^i \partial_{\theta^i}$ which is related to the *torsion* of the double null foliation.



The metric g with respect to the canonical coordinates is given by

$$g = -2\Omega^2 dudv + (b^i b^j \not{g}_{ij})dvdv - 2(b^i \not{g}_{ij})d\theta^j dv + \not{g}_{ij}\, d\theta^i d\theta^j,$$

where $\not{g}$ denotes the induced metric on the 2-surfaces $S_{u,v} = C_u \cap \underline{C}_v$.

Gauge freedom

A double null foliation $\mathcal{D}$ can be completely determined by the following

$$\mathcal{D} = \left\langle S_0,\, L|_{S_0},\, \Omega|_{C_0},\, \Omega|_{\underline{C}_0} \right\rangle .$$

Connection coefficients

We define the normalized pair $e_3 = -\Omega \cdot \nabla v$, $e_4 = -\Omega \cdot \nabla u$ (such that $g(e_3, e_4) = -1$). Let also e_1, e_2 be a frame of the sections S. The connections coefficients $\chi, \underline{\chi}, \eta, \underline{\eta}, \omega, \underline{\omega}, \zeta$ are defined as follows:

$$\chi_{AB} = g(\nabla_A e_4, e_B), \qquad \underline{\chi}_{AB} = g(\nabla_A e_3, e_B),$$
$$\eta_A = g(\nabla_3 e_4, e_A), \qquad \underline{\eta}_A = g(\nabla_4 e_3, e_A),$$
$$\omega = -g(\nabla_4 e_4, e_3), \qquad \underline{\omega} = -g(\nabla_3 e_3, e_4),$$
$$\zeta_A = g(\nabla_A e_4, e_3)$$

where $A, B \in \{1, 2\}$ and $\nabla_\mu = \nabla_{e_\mu}$. The connection coefficients Γ can be recovered by the following relations:

$$\nabla_A e_B = \not\nabla_A e_B + \chi_{AB} e_3 + \underline{\chi}_{AB} e_4,$$
$$\nabla_3 e_A = \not\nabla_3 e_A + \eta_A e_3, \quad \nabla_4 e_A = \not\nabla_4 e_A + \underline{\eta}_A e_4,$$
$$\nabla_A e_3 = \underline{\chi}_A{}^{\sharp B} e_B + \zeta_A e_3, \quad \nabla_A e_4 = \chi_A{}^{\sharp B} e_B - \zeta_A e_4,$$
$$\nabla_3 e_4 = \eta^{\sharp A} e_A - \underline{\omega} e_4, \quad \nabla_4 e_3 = \underline{\eta}^{\sharp A} e_A - \omega e_3,$$
$$\nabla_3 e_3 = \underline{\omega} e_3, \quad \nabla_4 e_4 = \omega e_4,$$

Curvature components

The curvature components are defined as follows

$$\alpha_{AB} = R_{A4B4}, \quad \underline{\alpha}_{AB} = R_{A3B3},$$
$$\beta_A = R_{A434}, \quad \underline{\beta}_A = R_{A334},$$
$$\rho = R_{3434}, \quad \sigma = \frac{1}{2}\not\epsilon^{AB} R_{AB34}.$$

The null structure (vacuum) equations are of the following general form

$$\nabla\!\!\!/_4 \Gamma = R + \Gamma \cdot \Gamma + D\!\!\!/ \Gamma,$$
$$\nabla\!\!\!/_3 \Gamma = R + \Gamma \cdot \Gamma + D\!\!\!/ \Gamma,$$

where $D\!\!\!/ \in \{d\!\!\!/, \nabla\!\!\!/, \mathrm{d\!\!/iv}\}$. Examples:
The second variational formulas

$$\nabla\!\!\!/_4 \chi = -\chi \times \chi - \alpha + \omega\chi,$$
$$\nabla\!\!\!/_3 \underline{\chi} = -\underline{\chi} \times \underline{\chi} - \underline{\alpha} + \underline{\omega}\underline{\chi}.$$

The Codazzi equations

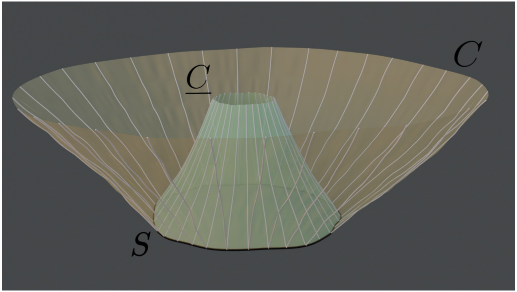
$$\mathrm{d\!\!/iv}\chi - d\!\!\!/ tr\chi + \chi^\sharp \cdot \zeta - (tr\chi) \cdot \zeta = -\beta,$$
$$\mathrm{d\!\!/iv}\underline{\chi} - d\!\!\!/ tr\underline{\chi} - \underline{\chi}^\sharp \cdot \zeta + (tr\underline{\chi}) \cdot \zeta = \underline{\beta}.$$

The characteristic initial value problem

The **Riemannian initial data set** for the Cauchy problem of the vacuum equations consists of the triplet $(\mathcal{H}_0, \overline{g}, k)$, where $\mathcal{H}_0$ is a three-dimensional Riemannian manifold, $\overline{g}$ is the metric on $\mathcal{H}_0$ and k is a symmetric (0,2) tensor field on $\mathcal{H}_0$ and such that $\overline{g}, k$ satisfy the constraint equations:

$$\begin{aligned} \overline{div} k - \overline{d} k =& 0, \\ \overline{R_{sc}} + (trk)^2 - |k|^2 =& 0. \end{aligned}$$

In the characteristic setting, the initial Riemannian (spacelike) Cauchy hypersurface $\mathcal{H}_0$ is replaced by two degenerate (null) hypersurfaces $C \cup \underline{C}$ intersecting at a two-dimensional surface S.



The characteristic initial value problem

Let us assume that $\not{g}$ is a given degenerate metric on $C \cup \underline{C}$. the first variational formula gives us

$$\chi = \frac{1}{2}\not{\mathcal{L}}_L \not{g}\,,$$

but then the Raychaudhuri equation for $\Omega = 1$ (affine foliation)

$$L(tr\chi) = -|\chi|^2$$

This shows that one cannot arbitrarily prescribe a degenerate metric $\not{g}$ on $C \cup \underline{C}$. Instead of prescribing the full metric $\not{g}$ on $C \cup \underline{C}$ and introducing constraint equations, we can freely prescribe the conformal class of the metric:

$$\mathrm{Conf}(\not{g}\,) = \{A\not{g}\,, A \in C^\infty(C \cup \underline{C}), A > 0\}\,.$$

Conformal properties of the double null foliation

Consider two conformal metrics $\not{g}$, $\tilde{\not{g}}$ such that $\tilde{g} = A\not{g}$. Then

$$\chi = \frac{1}{2}\not{\mathcal{L}}_L \not{g}, \quad \tilde{\chi} = \frac{1}{2}\not{\mathcal{L}}_L \tilde{g}.$$

We have

$$\tilde{\chi} = \frac{1}{2}(LA)\not{g} + A\chi.$$

Therefore, for the traceless parts $\hat{\tilde{\chi}}, \hat{\chi}$ we have

$$\hat{\tilde{\chi}} = A\hat{\chi}$$

and hence

$$|\hat{\tilde{\chi}}|^2_{\tilde{g}} = \tilde{g}^{AB}\tilde{g}^{CD}\hat{\tilde{\chi}}_{AC}\hat{\tilde{\chi}}_{BD} = A^{-2}\not{g}^{AB}\not{g}^{CD}A^2\hat{\chi}_{AB}\hat{\chi}_{CD} = |\hat{\chi}|^2_{\not{g}}.$$

Hence, the size $|\hat{\tilde{\chi}}|^2_{\tilde{g}}$ of the shear is conformally invariant! We denote $e = |\hat{\tilde{\chi}}|^2_{\tilde{g}}$. By the Raychaudhuri equation we obtain

$$L(tr\chi) = -\frac{1}{2}(tr\chi)^2 - e.$$

Knowing $tr\chi$ at S_0, the above determines $tr\chi$ on C. This then determines A and hence $\not{g}$ and hence the full χ. Similarly we can compute all the remaining geometric quantities on C and $\underline{C}$.

The characteristic initial value problem

The characteristic initial data set for the Einstein equations consists of a pair of three-dimensional hypersurfaces intersecting at a two-dimensional surface along with the (free) specification of the conformal class Conf($\slashed{g}$) of the degenerate metric $\slashed{g}$ on $C \cup \underline{C}$ as well as the full metric $\slashed{g}$, the expansions $tr\chi$, $tr\underline{\chi}$ and the torsion η on S.

Local well-posedness

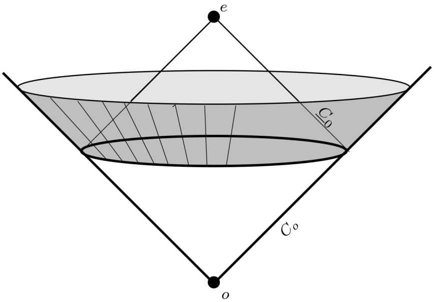
Rendall has shown that for smooth characteristic initial data there exists a unique solution to the Einstein equations in a neighborhood of the surface S. Luk extended the above result to appropriate neighborhoods of the initial hypersurfaces $C, \underline{C}$.

Christodoulou's threorem on trapped surface formation

Assumption 1: Consider characteristic initial data such that on C_0

$$\boxed{\int_0^{\delta} r^2 \cdot e \quad dv > 1}$$

where δ is sufficiently small, along each null generator of C_0.



Christodoulou's threorem on trapped surface formation

Assumption 2: The remaining geometric quantities we have: The connection coefficients satisfy

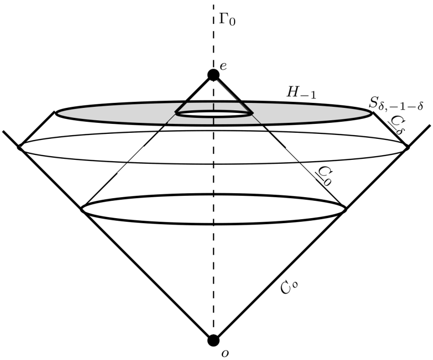
$$
\begin{aligned}
|\Omega| &\leq O(1), \\
|\Omega tr\chi - 2/|u|| &\leq O(|u|^{-2}), \\
|\Omega tr\underline{\chi} + 2/|u|| &\leq O(\delta |u|^{-2}), \\
|\hat{\chi}| &\leq O(\delta^{-1/2}|u|^{-1}), \\
|\underline{\hat{\chi}}| &\leq O(\delta^{1/2}|u|^{-2}), \\
|\eta| &\leq O(\delta^{-1/2}|u|^{-2}), \\
|\underline{\omega}| &\leq O(\delta |u|^{-3}),
\end{aligned}
$$

And curvature components are such that

$$
\begin{aligned}
|\alpha| &\leq O(\delta^{-3/2}|u|^{-1}), \\
|\beta| &\leq O(\delta^{1/2}|u|^{-2}), \\
|\rho| &\leq O(|u|^{-3}), \\
|\sigma| &\leq O(|u|^{-3}), \\
|\underline{\beta}| &\leq O(\delta |u|^{-4}), \\
|\underline{\alpha}| &\leq O(\delta^{3/2}|u|^{-9/2}),
\end{aligned}
$$

Christodoulou's threorem on trapped surface formation

Conclusion: The maximal vacuum development of the data contains a region on which the double null foliation is constructed, bounded in the future by the spacelike hypersurface H_{-1} and the incoming null hypersurface $\underline{C}_{\delta}$. Moreover, it contains a trapped surface.



Remarks

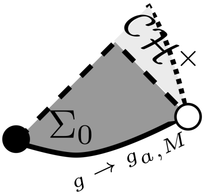
1. The whole setting can be pushed to $u \to -\infty$ so C_0 coincides with past null infinity. In that case, the meaning of the e integral is the flux of incoming radiation per unit solid angle.
2. Trapped surface follows by integrating the Raychaudhuri equation

$$L(tr\chi) = -\frac{1}{2}(tr\chi)^2 - |\hat{\chi}|^2$$

   along C^* and using that, even though $tr\chi \sim 2/r$ on $\underline{C}_0$, we also have $|\hat{\chi}|^2 \sim r^{-2}$.
3. Klainerman–Luk–Rodnianski established a criterion for anisotropic formation of trapped surfaces. In their case the trapped surface is not a section of the double null foliation.

Stong cosmic censorship near the Kerr family

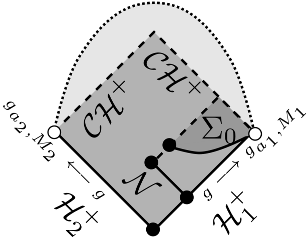
(Dafermos–Luk) Consider general vacuum initial data corresponding to the expected induced geometry of a dynamical black hole settling down to Kerr (with parameters $0 < |a| < M$) on a suitable spacelike hypersurface Σ_0 in the black hole interior. Then the maximal future development spacetime is globally covered by a double null foliation and has a non-trivial Cauchy horizon $\mathcal{CH}^+$ across which the metric is continuously extendible. The future boundary of the maximal development consists of a null piece.



- The above suggests that the C^0 formulation of SCC is wrong.

Global version of the Dafermos–Luk theorem

Consider vacuum initial data on a bifurcate null hypersurface $\mathcal{H}_1^+ \cup \mathcal{H}_2^+$, such that both hypersurfaces are future complete, and globally close to, and asymptote to Kerr metrics with nearby parameters $0 < |a_1| < M_1$, and $0 < |a_2| < M_2$, respectively. Then the maximal future development can be covered by a double null foliation and moreover can be extended as a C^0 metric across a bifurcate Cauchy horizon $\mathcal{CH}^+$.



A spherically symmetric model

Christodoulou obtained a version of the C^0-version of the SCC for the Einstein-scalar field system

$$Ric_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi(\partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \psi \partial_\alpha \psi), \qquad \Box_g \psi = 0.$$

It turns out that Kerr-like Cauchy horizons arising from i^+ cannot occur under any initial conditions. For the same system, Christodoulou proved weak cosmic censorship. He also showed that the genericity assumption is indeed necessary via explicit examples of "naked singularities".

On the other hand, Dafermos and Dafermos–Rodnianski have proved that the black hole interiors of perturbations of the Reissner–Nordström spacetimes for the spherically symmetric Einstein–Maxwell-scalar field system is similar to that for the Kerr case.

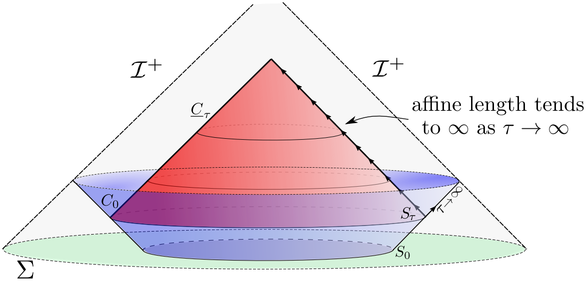
Recent work by Rodnianski–Shlapentokh-Rothman on the existence of vacuum spacetimes with naked singularities.

Stability problems in general relativity

- Stability of Minkowski
  - Methods
  - Memory effect
- Stability of Schwarzschild
- Analysis of propagation of linear wave on black holes
  - Asymptotics on sub-extremal black holes
  - Asymptotics on extremal black holes
    - Signature of extremality

Stability of Minkowski

Christodoulou–Klainerman 1993: The Minkowski spacetime is globally stable under sufficiently small perturbations of its initial data.



Perturbations of Minkowski

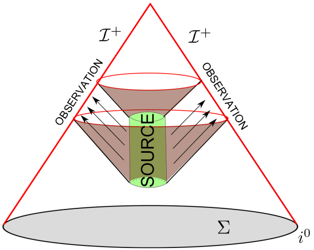
- are geodesically complete with positive ADM mass
- admit a (future) complete null infinity
- contain no black hole regions

Remarks

- The supercriticality of the Einstein-vacuum equations results in having to show not just orbital stability but the full asymptotic stability of Minkowski space. The quantitative decay rates back towards the Minkowski metric should be sufficiently strong so as to ensure that non-linear terms can be understood as error terms which can be integrated in time.
- The modern way to do this is via the vector field method by Klainerman, where energy estimates are applied directly to the nonlinear equations commuted with weighted commutation vector fields.
- Decay towards $\mathcal{I}^+$: Solutions to the wave equation (in 3+1 dimensions) decay along the outgoing null cones only as r^{-1} . This slow decay means that the quadratic nature of the non-linearitiesis insufficient by itself to ensure non-linear stability: one must identify special structure in the nonlinear terms. Such a special structure is called the *null condition* where bad terms can only be coupled with good terms in the non-linearities.
- The geometric structure equations associated to a foliation by maximal hypersurfaces and outgoing null cones capture an analogue of the null condition. Moreover the null structure equations relative to the double null foliation also capture this condition.
- The null condition ensures good behaviour **towards** $\mathcal{I}^+$. The global stability result, however, requires global decay for example **along** $\mathcal{I}^+$. This is possible via teleological normalisations.

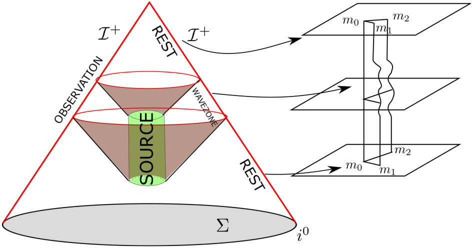
Gravitational wave measurements at $\mathcal{I}^+$

Propagation of gravitational waves and experiments at null infinity

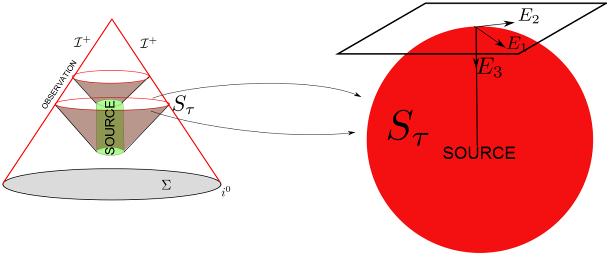


Gravitational wave measurements at $\mathcal{I}^+$

The three mass experiment at null infinity.

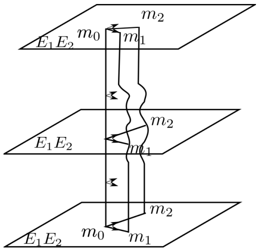


The planes $\langle E_1(t), E_2(t)\rangle$ are tangential to the sections of null infinity along a fixed null generator. In this case, $E_3(t)$ points towards the direction of the source:



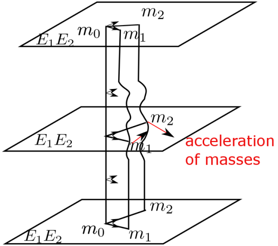
Gravitational wave measurements at $\mathcal{I}^+$

First asymptotic law of gravitational waves: The three masses remain on the orthogonal plane $\langle E_1, E_2 \rangle$.



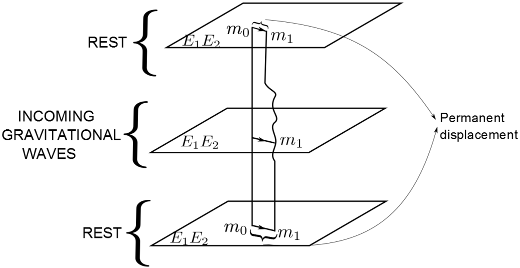
Gravitational wave measurements at $\mathcal{I}^+$

Second asymptotic law of gravitational waves: The accelerations of m_1 and m_2 relative to m_0 are orthogonal.



Memory effect

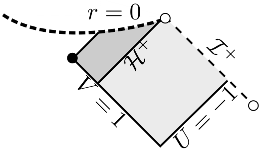
The masses $m_{(i)}$ suffer a permanent displacement $\Delta x_{(i)}$ given by asymptotic quantities of the gravitational field i.e. the asymptotics of the Riemann tensor. This permanent displacement is known as *Christodoulou's memory effect*.



Co-dimension 3 stability of Schwarzschild

(Dafermos, Holzegel, Rodnianski, Taylor): For vacuum initial data sets—with no symmetry assumed—sufficiently close to appropriate Schwarzschild initial data that moreover lie in a codimension-3 submanifold of the moduli space of vacuum initial data, the resulting maximal Cauchy development

- can be covered by appropriate (teleologically normalised) global double null foliations,
- possesses a complete future null infinity $\mathcal{I}^+$ whose past is bounded to the future by a regular, future complete event horizon $\mathcal{H}^+$,
- remains globally close to Schwarzschild in its exterior ,
- asymptotes back to a member of the Schwarzschild family as a suitable notion of time goes to infinity.



Outside the codimension-3 submanifold, one expects solutions to necessarily asymptote to a Kerr solution, since the dimension of linearised Kerr solutions fixing the mass is equal to 3 in our parametrisation.

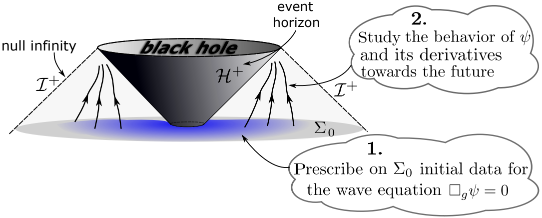
Propagation of linear waves

Scalar perturbations

- Scalar fields: Investigate the evolution of solutions to the wave equation

$$\Box_g \psi = 0$$

on Reissner–Nordström or Kerr backgrounds.

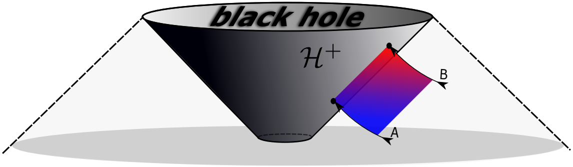


- Motivation: In harmonic gauge $\Box_g x^\mu = 0$ the vacuum equations take the form

$$\Box_g g_{\mu\nu} = N_{\mu\nu}(g, \partial g).$$

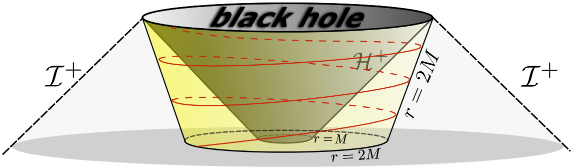
Features of black holes spacetimes

The redshift effect at the horizon



Features of black holes spacetimes

The trapping effect at the photon sphere.



What about black holes?

- Contributors: Dafermos, Rodnianski, Andersson, Tataru, Moschidis, Blue, Holzegel, Sbierski, Shlapentokh-Rothman, Dyatlov, Häfner, Bony, Smulevici, Klainerman, Ionescu, Tohaneanu, Sterbenz, Soffer, Schlue, Luk, Oh, Finster, Kamran, Smoller, Yau, Donninger, Schlag, Vasy, Hintz, Metcalfe, Wald, Franzen, Teixeira da Costa, ...
- Decay for all $|a| < M$ (Dafermos–Rodnianski–Shlapentokh-Rothman)

Kerr asymptotics

(Angelopoulos, A., Gajic): If ψ is a solution to the wave equation on a sub-extremal Kerr space-time with smooth compactly supported initial data then

Asymptotics in the exterior region		
$\psi\|_{\mathcal{H}}$	$\psi\|_{r=R}$	$r\psi\|_{\mathcal{I}}$
$-8I_0^{(1)}[\psi]\cdot\tau^{-3}$	$-8I_0^{(1)}[\psi]\cdot\tau^{-3}$	$-2I_0^{(1)}[\psi]\cdot\tau^{-2}$

- Spherical mean wrt BL spheres.
- Explicit expressions of all constants.

R-N asymptotics

(Angelopoulos, A., Gajic): If ψ is a solution to the wave equation on a sub-extremal R-N space-time with smooth compactly supported initial data then

Asymptotics in the exterior region		
$\psi\vert_{\mathcal{H}}$	$\psi\vert_{r=R}$	$r\psi\vert_{\mathcal{I}}$
$-8I_0^{(1)}[\psi]\cdot\tau^{-3}$	$-8I_0^{(1)}[\psi]\cdot\tau^{-3}$	$-2I_0^{(1)}[\psi]\cdot\tau^{-2}$

- The charge does not seem to affect the asymptotics. Then what about the extremal case?

$I_0^{(1)}[\psi]$ in terms of conservation laws

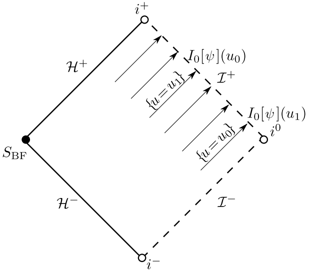
It turns out that the function

$$I_0[\psi](u) = \lim_{r\to\infty} v^2 \partial_v (r\psi_0)$$

is constant, that is independent of u. This yields a **conservation law** along $\mathcal{I}^+$. The associated constant

$$I_0[\psi] := I_0[\psi](u) \tag{1}$$

is called the Newman–Penrose constant of ψ.



Now,

$$\boxed{I_0^{(1)}[\psi] = I_0[T^{-1}\psi]}$$

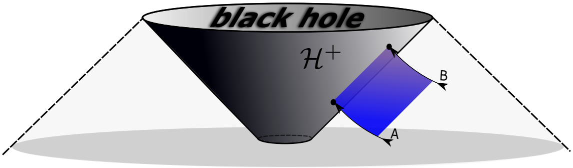
Extremal R-N asymptotics

(Angelopoulos, A., Gajic): If ψ is a solution to the wave equation on a extremal R-N space-time with smooth compactly supported initial data then

Asymptotics in the exterior region		
$\psi\|_{\mathcal{H}}$	$\psi\|_{r=R}$	$r\psi\|_{\mathcal{I}}$
$2M^{-1}H[\psi]\cdot\tau^{-1}$	$\frac{4M}{R-M}H[\psi]\cdot\tau^{-2}$	$\left(4MH[\psi]-2I_0^{(1)}[\psi]\right)\cdot\tau^{-2}$

- Horizon asymptotics significantly slower.
- What about the constant $H[\psi]$?

Degeneracy of the redshift effect at extremal horizons



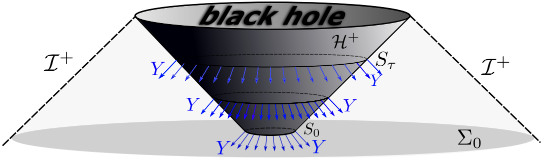
...due to the vanishing of the surface gravity.

Proposition (A.)

If ψ satisfies the wave equation on extremal Reissner–Nordström then the integral

$$H[\psi] = -\int_{S_\tau} \left(Y\psi + \frac{1}{2M}\psi \right) dvol$$

is independent of τ. Here Y is transversal to the horizon.



A signature of extremality at null infinity

Let ψ be a scalar perturbation of Reissner–Nordström (RN) (with mass M, charge e) supported initially near the event horizon.
Let's define:

$$\boxed{s[\psi] := \frac{1}{4M} \lim_{\tau \to \infty} \tau^2 \cdot (r\psi)|_{\mathcal{I}^+} + \frac{1}{8\pi} \int_{\mathcal{I}^+ \cap \{u \geq 0\}} r\psi|_{\mathcal{I}^+}}$$

For all scalar perturbations on sub-extremal RN we have

$$\boxed{\text{If } |e| < M \text{ then } s[\psi] = 0}$$

Moreover,

$$\boxed{\text{If } s[\psi] \neq 0 \text{ then } |e| = M \text{ (ERN) and } s[\psi] = H[\psi]}$$

- **Extremal black holes admit classical externally measurable hair.**
- **The horizon hair $H[\psi]$ could potentially serve as an observational signature.**
- **For extremal black holes information "leaks" from the event horizon to null infinity.**

Thank you!