High Post-Minkowskian Orders for Binary Systems

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IPAM Workshop
Computational Challenges in Gravitational Wave Astronomy

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ZB, C. Cheung, R. Roiban, C.H. Shen, M. Solon, M. Zeng, arXiv:1901.04424 and in preparation

Outline

1. Can particle physics help with core issues for gravitational wave astronomy?

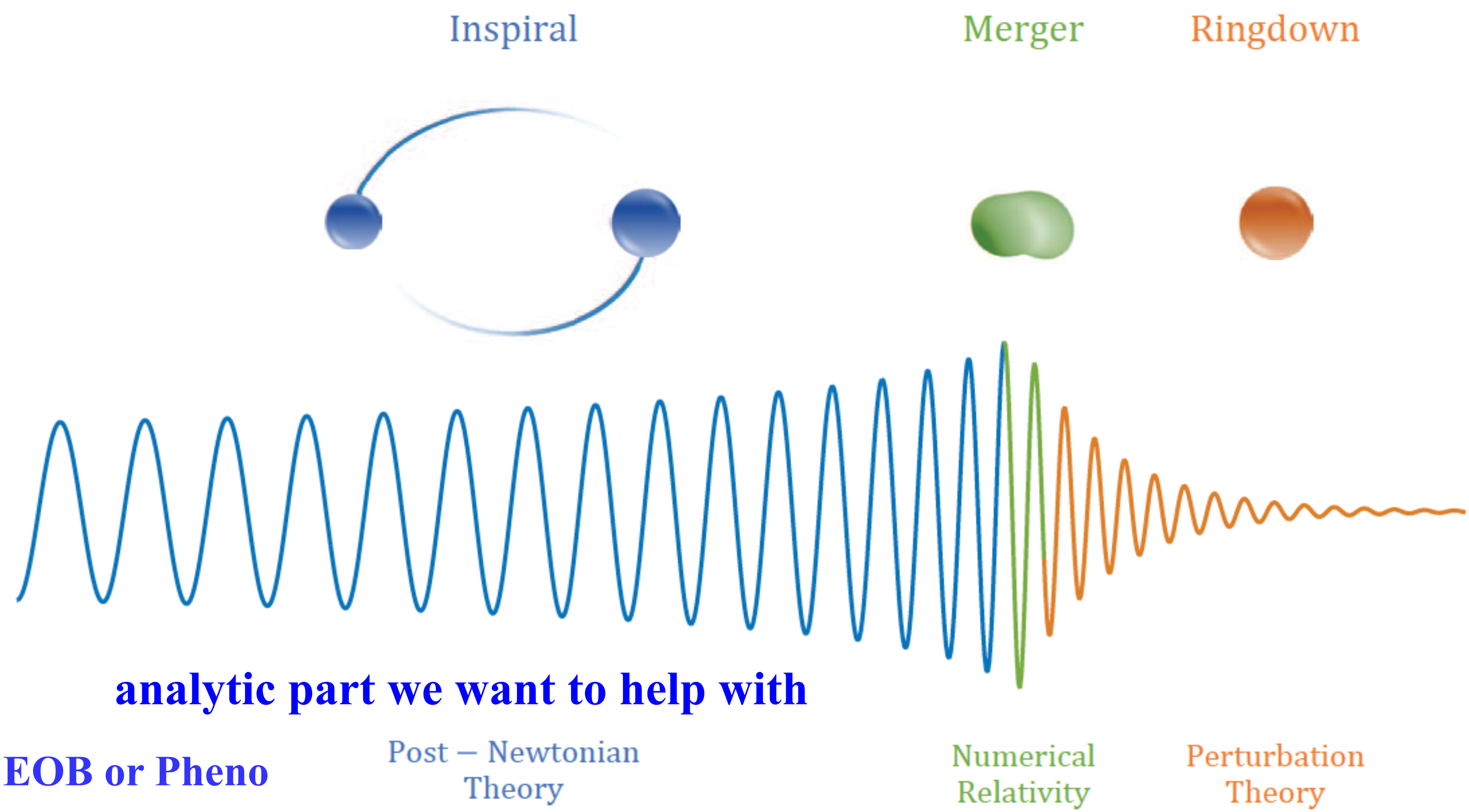
2. Nontriviality of gravity calculations. Particle theorists understand perturbative gravity!

3. Modern idea from particle theory:
   — Effective field theory (EFT).
   — Generalized unitarity.
   — Double copy and duality between color and kinematics.

4. Can we calculate something new and nontrivial of direct interest for LIGO/Virgo? Yes! 3rd post-Minkowskian 2 body Hamiltonian.

5. Prospects for future.

Goal: Improve on post-Newtonian Theory



analytic part we want to help with

PN + EOB or Pheno

Small errors accumulate. Need for high precision.

From Antelis and Moreno, arXiv:1610.03567

Post Newtonian Approximation

For orbital mechanics:

Expand in *G* and v^2

$$v^2 \sim \frac{GM}{r} \ll 1$$

virial theorem

In center of mass frame:

$$m = m_A + m_B, \quad \nu = \mu/M,$$

$$\mu = m_A m_B/m, \quad P_R = P \cdot \hat{R}$$

$$\frac{H}{\mu} = \frac{P^2}{2} - \frac{Gm}{R} \quad \leftarrow \textbf{Newton}$$

$$+ \frac{1}{c^2}\left\{ -\frac{P^4}{8} + \frac{3\,\nu\,P^4}{8} + \frac{Gm}{R}\left(-\frac{{P_R}^2\,\nu}{2} - \frac{3\,P^2}{2} - \frac{\nu\,P^2}{2} \right) + \frac{G^2 m^2}{2R^2} \right\}$$

$$+ \ldots$$

1PN: Einstein, Infeld, Hoffmann

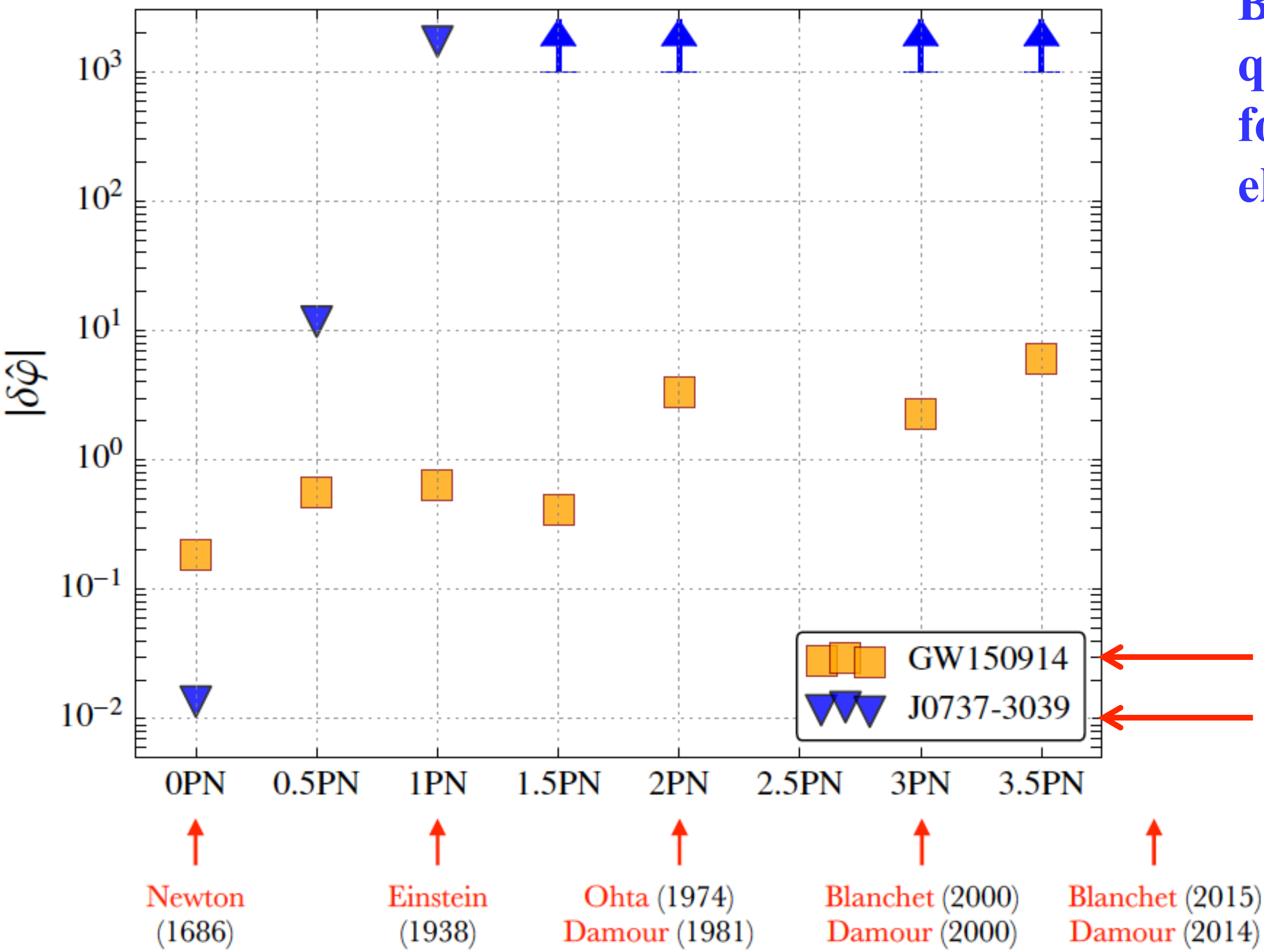
Hamiltonian known to 4PN order.

2PN: **Ohta, Okamura, Kimura and Hiida.**
3PN: **Damour, Jaranowski and Schaefer; L. Blanchet and G. Faye.**
4PN: **Damour, Jaranowski and Schaefer (2014)**

Importance of higher orders for LIGO

LIGO/Virgo Collaboration arXiv:1602.03841

Binary pulsar confirms quadrupole radiation formula and not much else.



LIGO

Binary pulsar

LIGO/Virgo sensitive to high PN orders.

PN versus PM expansion for conservative two-body dynamics

From Buonanno Amplitudes 2018

$$\mathcal{L} = -Mc^2 + \underbrace{\frac{\mu v^2}{2} + \frac{GM\mu}{r}}_{\downarrow} + \frac{1}{c^2}\left[\ldots\right] + \frac{1}{c^4}\left[\ldots\right] + \ldots$$

$$E(v) = -\frac{\mu}{2}v^2 + \cdots$$

non-spinning compact objects

		0PN	1PN	2PN	3PN	4PN	5PN	...
0PM:	1	v^2	v^4	v^6	v^8	v^{10}	v^{12}	...
1PM:		$1/r$	v^2/r	v^4/r	v^6/r	v^8/r	v^{10}/r	...
2PM:			$1/r^2$	v^2/r^2	v^4/r^2	v^6/r^2	v^8/r^2	...
3PM:				$1/r^3$	v^2/r^3	v^4/r^3	v^6/r^3	...
4PM:					$1/r^4$	v^2/r^4	v^4/r^4	...
...						...	...	

(credit: Justin Vines)

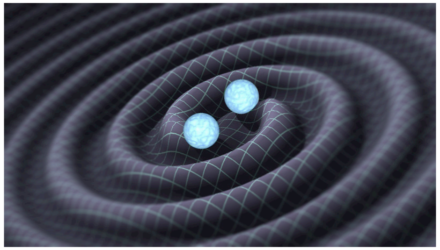
$$1 \to Mc^2, \qquad v^2 \to \frac{v^2}{c^2}, \qquad \frac{1}{r} \to \frac{GM}{rc^2}.$$

current known PN results — current known PM results — overlap between PN & PM results — unknown

- **PM results** (Westfahl 79, Westfahl & Goller 80, Portilla 79-80, Bel et al. 81, Ledvinka et al. 10, Damour 16-17, Guevara 17, Vines 17, Bini & Damour 17-18, Vines in prep)

Which problem to solve?

Some problems for (analytic) theorists:



1. Spin.
2. Finite size effects.
3. Radiation.
4. High orders in perturbation theory.

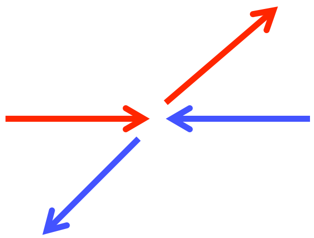
Which one to solve?

- Needs to be extremely difficult using standard methods.
- Needs to be of importance to LIGO.
- Needs to be in a form that LIGO can use.

3rd post-Minkowskian order 2-body Hamiltonian

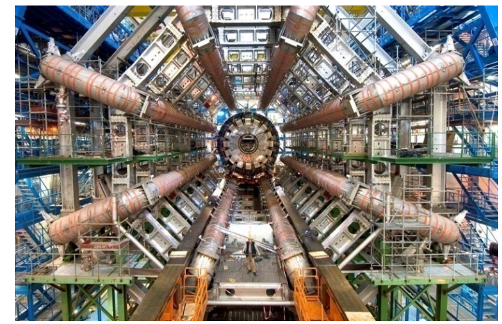
Quantum Field Theory and Scattering Amplitudes

Scattering amplitudes give us quantum mechanical description of events at particle colliders.

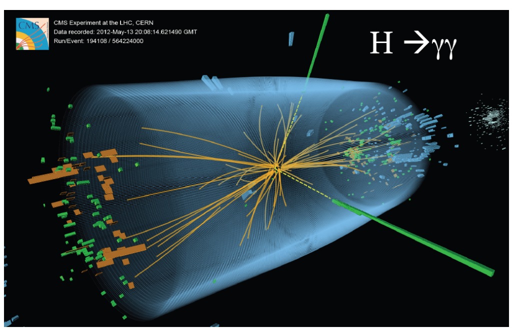


particle scattering

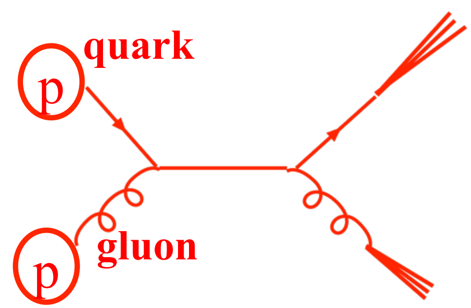
Large Hadron Collider



ATLAS Detector



Higgs boson event

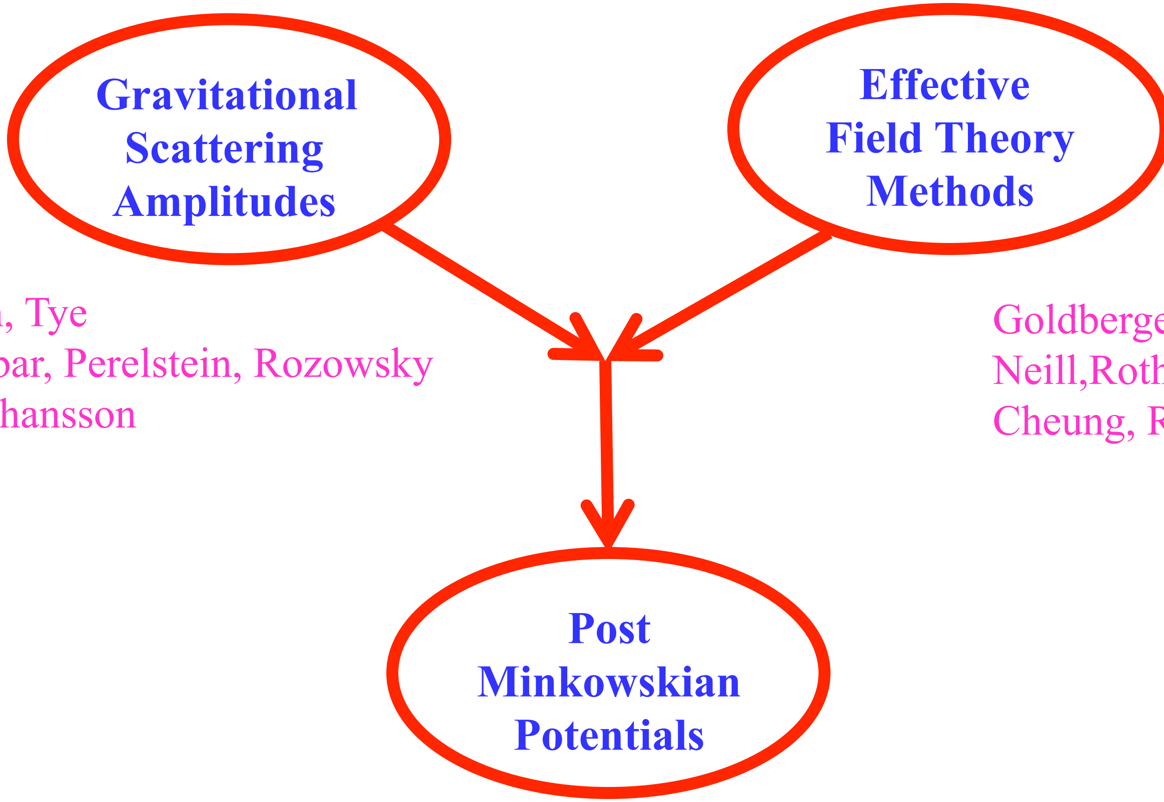


QCD Feynman diagram

Does not seem to have much to do with gravitational waves

Our Approach

ZB, Cheung, Roiban, Shen, Solon, Zeng



Kawai, Lewellen, Tye
ZB, Dixon, Dunbar, Perelstein, Rozowsky
ZB, Carrasco, Johansson

Goldberger, Rothstein
Neill,Rothstein
Cheung, Rothstein, Solon

Inefficient: Start with quantum theory and take $\hbar \to 0$

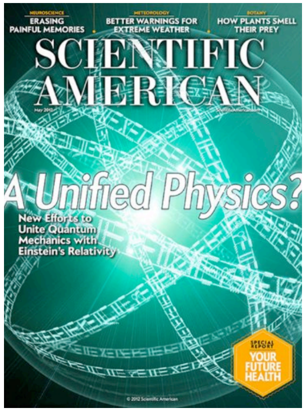
Efficient: Almost magical simplifications for gravity amplitudes.
EFT methods efficiently target pieces we want.

Will show efficiency wins

Scattering Amplitudes Revolution

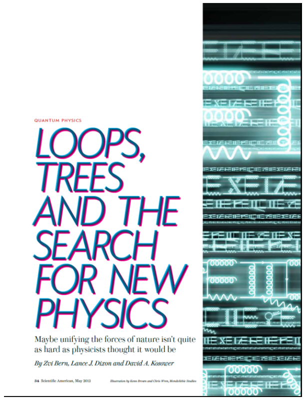
Z. Bern, L. Dixon, D. Kosower,
May 2012 *Scientific American*

Over the years we have developed tools for calculating scattering amplitudes that are "impossibly complicated"



Don't use Lagrangians or Feynman diagrams.

1. **Generalized unitarity. Complicated amplitudes assembled from simpler ones. Loops from trees.**
2. **Double-copy relations. Gravity amplitudes built from much simpler gauge-theory ones.**



Many more advances, involving also beautiful mathematics.

Parke and Taylor; ZB, Dixon, Dunbar, Kosower; ZB, Dixon, Dunbar, Kosower; ZB, Dixon, Dunbar, Perelstein, Rozowsky; Witten; Britto, Cachazo, Feng, Witten; ZB, Carrasco, Johansson; Bourjaily, Cachazo, Goncharov, Postnikov, Trnka + 1000s more.

Harness advances to extract GR corrections to Newtonian potential.

Gravity vs Gauge Theory

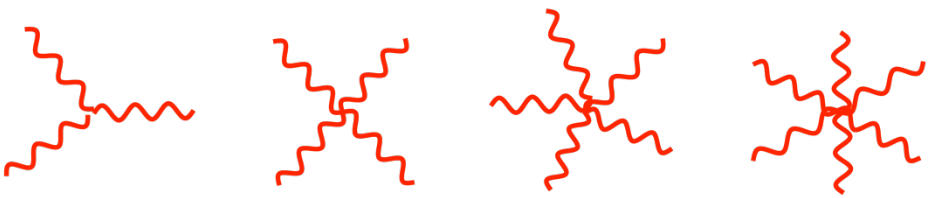
Consider the Einstein gravity Lagrangian

curvature — Flat-space metric — graviton field — metric

$$L_{\text{gravity}} = \frac{2}{\kappa^2}\sqrt{-g}\,R \qquad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$$

$$\kappa^2 = 32\pi G_{\text{Newton}}$$

Infinite number of complicated interactions

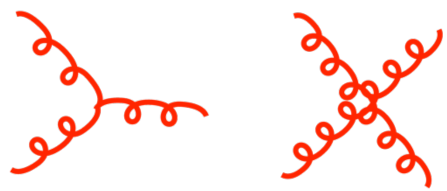


$+ \cdots$

terrible mess

Compare to gauge-theory Lagrangian on which QCD is based

$$L_{\text{YM}} = \frac{1}{g^2} F^2$$



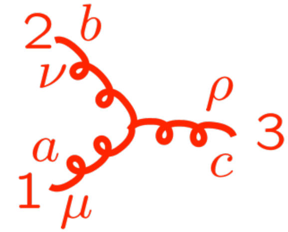
Only three and four point interactions

Gravity seems so much more complicated than gauge theory.

Theories do not look related!

Three Vertices

Standard Feynman diagram approach.



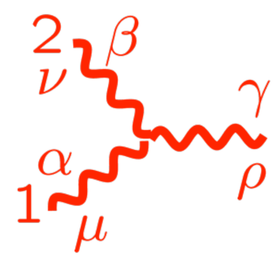
Three-gluon vertex:

$$V_{3\,\mu\nu\sigma}^{abc} = -gf^{abc}(\eta_{\mu\nu}(k_1-k_2)_\rho+\eta_{\nu\rho}(k_1-k_2)_\mu+\eta_{\rho\mu}(k_1-k_2)_\nu)$$

Three-graviton vertex:

$$k_i^2 = E_i^2 - \vec{k}_i^2 \neq 0$$

$$\begin{aligned}
&G_{3\mu\alpha,\nu\beta,\sigma\gamma}(k_1,k_2,k_3) = \\
&\quad \mathrm{sym}[-\tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}) - \tfrac{1}{2}P_6(k_{1\nu}k_{1\beta}\eta_{\mu\alpha}\eta_{\sigma\gamma}) + \tfrac{1}{2}P_3(k_1\cdot k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma}) \\
&\qquad + P_6(k_1\cdot k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}) + 2P_3(k_{1\nu}k_{1\gamma}\eta_{\mu\alpha}\eta_{\beta\sigma}) - P_3(k_{1\beta}k_{2\mu}\eta_{\alpha\nu}\eta_{\sigma\gamma}) \\
&\qquad + P_3(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + P_6(k_{1\sigma}k_{1\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}) + 2P_6(k_{1\nu}k_{2\gamma}\eta_{\beta\mu}\eta_{\alpha\sigma}) \\
&\qquad + 2P_3(k_{1\nu}k_{2\mu}\eta_{\beta\sigma}\eta_{\gamma\alpha}) - 2P_3(k_1\cdot k_2\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu})]
\end{aligned}$$

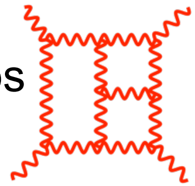


About 100 terms in three vertex

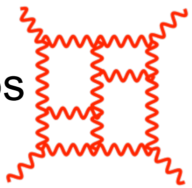
Naïve conclusion: Gravity is a nasty mess.

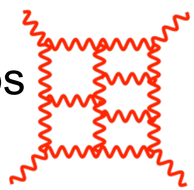
Feynman Diagrams for Gravity

Spectacularly poor scaling in GR

3 loops  $\sim 10^{20}$ TERMS

No surprise it has never been calculated via Feynman diagrams.

4 loops  $\sim 10^{26}$ TERMS

5 loops  $\sim 10^{31}$ TERMS

More terms than atoms in your brain!

- Such calculations seemed utterly hopeless!
- Seemed destined for dustbin of undecidable questions.

Modern methods make such calculations routine, but challenging.

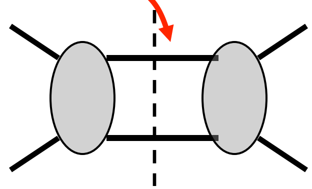
Generalized Unitarity Method

No Feynman rules; no need for virtual particles. Keep Ein

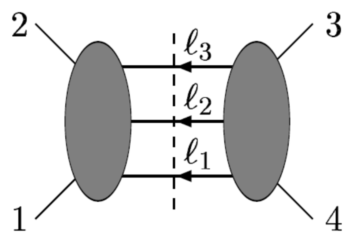
$$E^2 = \vec{p}^{\,2} + m^2$$ ← **on-shell**

ZB, Dixon, Dunbar and Kosower (1994)

Two-particle cut:

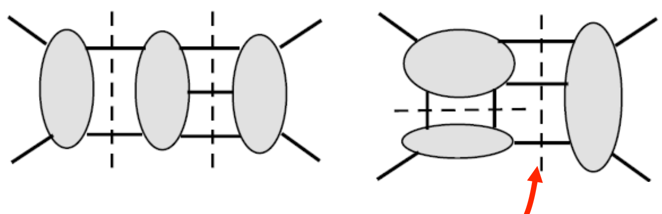


Three-particle cut:



- **Systematic assembly of complete amplitudes from other amplitudes.**
- **Works for any number of particles or loops.**

Generalized unitarity as a practical tool.



on-shell

ZB, Dixon and Kosower;
ZB, Morgan;
Britto, Cachazo, Feng;
Ossala,Pittau,Papadopoulos;
Ellis, Kunszt, Melnikov;
Forde; Badger
and many others

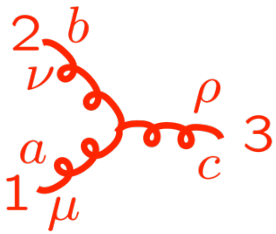
Reproduces Feynman diagrams except intermediate steps of calculation based on gauge invariant quantities.

Simplicity of Gravity Amplitudes

People were looking at gravity amplitudes the wrong way.
On-shell viewpoint much more powerful.

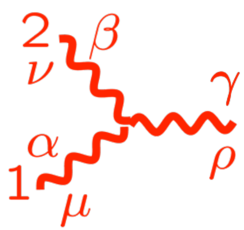
***On-shell* three vertices contains all information:** $k_i^2 = 0$

Yang-Mills gauge theory:



$$-g f^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$

Einstein gravity:



$$i\kappa(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$
$$\times(\eta_{\alpha\beta}(k_1 - k_2)_\gamma + \text{cyclic})$$

"square" of Yang-Mills vertex.

Very simple!

Gravity Amplitudes

KLT (1985)

Kawai-Lewellen-Tye string relations in low energy limit:

gravity — gauge theory color ordered

$$M_4^{\text{tree}}(1,2,3,4) = -is_{12} A_4^{\text{tree}}(1,2,3,4)\, A_4^{\text{tree}}(1,2,4,3)\,,$$

$$M_5^{\text{tree}}(1,2,3,4,5) = is_{12}s_{34} A_5^{\text{tree}}(1,2,3,4,5)\, A_5^{\text{tree}}(2,1,4,3,5) + is_{13}s_{24} A_5^{\text{tree}}(1,3,2,4,5)\, A_5^{\text{tree}}(3,1,4,2,5)$$

Pattern gives explicit all-leg form.

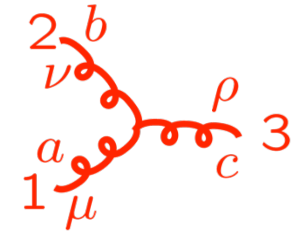
1. **Gravity is derivable from gauge theory. Standard Lagrangian methods offers no hint why this is possible.**
2. **It is very generally applicable.**

Duality Between Color and Kinematics

ZB, Carrasco, Johansson (2007)

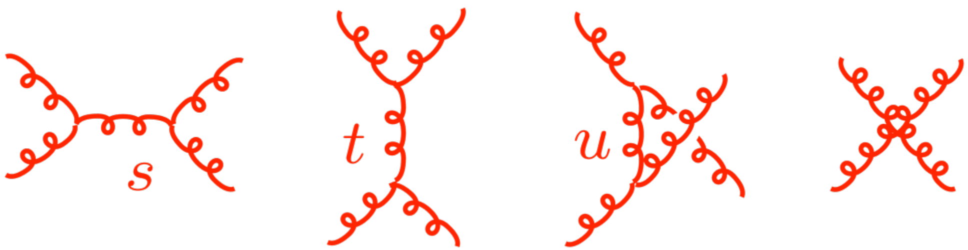
coupling constant → **color factor** ← **momentum dependent kinematic factor**

$$-gf^{abc}(\eta_{\mu\nu}(k_1 - k_2)_\rho + \text{cyclic})$$



Color factors based on a Lie algebra: $[T^a, T^b] = if^{abc}T^c$

Jacobi Identity $f^{a_1 a_2 b} f^{b a_4 a_3} + f^{a_4 a_2 b} f^{b a_3 a_1} + f^{a_4 a_1 b} f^{b a_2 a_3} = 0$



Use 1 = *s/s* = *t/t* = *u/u* to assign 4-point diagram to others.

$s = (k_1 + k_2)^2 \qquad t = (k_1 + k_4)^2$

$u = (k_1 + k_3)^2$

$$\mathcal{A}_4^{\text{tree}} = g^2 \left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u} \right)$$

Color factors satisfy Jacobi identity: $c_u = c_s - c_t$

Numerator factors satisfy similar identity: $n_u = n_s - n_t$

Proven at tree level

ZB, Carrasco, Johansson; Kiermaier; Bjerrum-Bohr, Damgaard, Sondergaard, Vanhove; Cachazo, etc

Duality Between Color and Kinematics

Consider five-point tree amplitude:

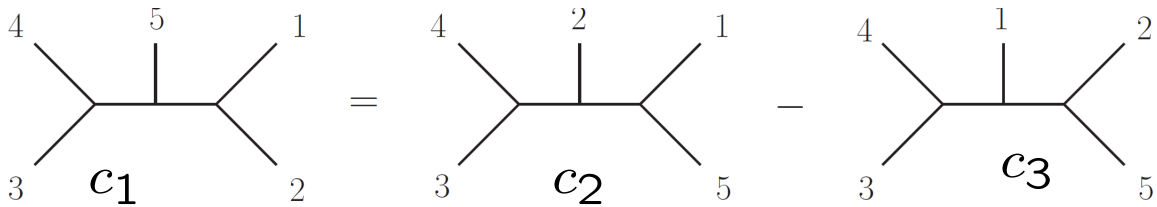
ZB, Carrasco, Johansson (BCJ)

$$\mathcal{A}_5^{\text{tree}} = \sum_{i=1}^{15} \frac{c_i n_i}{\prod_{\alpha_i} p^2_{\alpha_i}}$$

color factor

kinematic numerator factor

Feynman propagators



$$c_1 = f^{a_3 a_4 b} f^{b a_5 c} f^{c a_1 a_2} \qquad c_2 = f^{a_3 a_4 b} f^{b a_2 c} f^{c a_1 a_5} \qquad c_3 = f^{a_3 a_4 b} f^{b a_1 c} f^{c a_2 a_5}$$

$$n_i \sim k_4 \cdot k_5 \, k_2 \cdot \varepsilon_1 \, \varepsilon_2 \cdot \varepsilon_3 \, \varepsilon_4 \cdot \varepsilon_5 + \cdots$$

$$c_1 + c_2 + c_3 = 0 \quad \Leftrightarrow \quad n_1 + n_2 + n_3 = 0$$

Claim: We can always find a rearrangement so color and kinematics satisfy the *same* algebraic constraint equations.

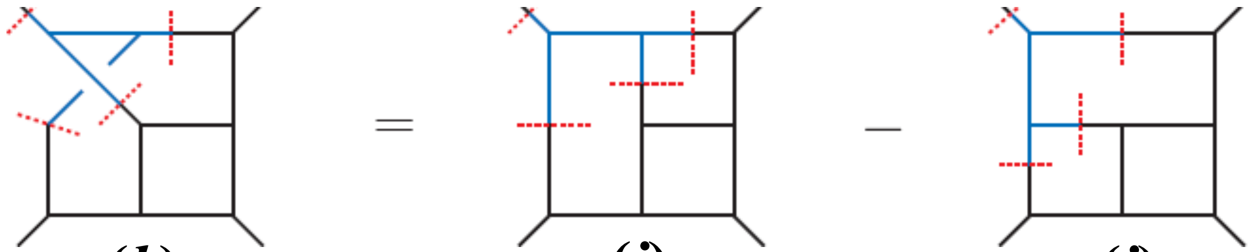
Progress on unraveling relations.

BCJ, Bjerrum-Bohr, Feng, Damgaard, Vanhove, ; Mafra, Stieberger, Schlotterer;
Tye and Zhang; Feng, Huang, Jia; Chen, Du, Feng; Du, Feng, Fu; Naculich, Nastase, Schnitzer
O'Connell and Montiero; Bjerrum-Bohr, Damgaard, O'Connell and Montiero; O'Connell, Montiero, White;
Du, Feng and Teng, Song and Schlotterer, etc.

BCJ

Gravity Loop Integrands from Gauge Theory

Ideas conjectured to generalize to loops:



color factor

$$c_k = c_i - c_j$$

$$n_k = n_i - n_j$$

kinematic numerator

If you have a set of duality satisfying numerators.

To get:

gauge theory → gravity theory

simply take

color factor → kinematic numerator

$$c_k \longrightarrow n_k$$

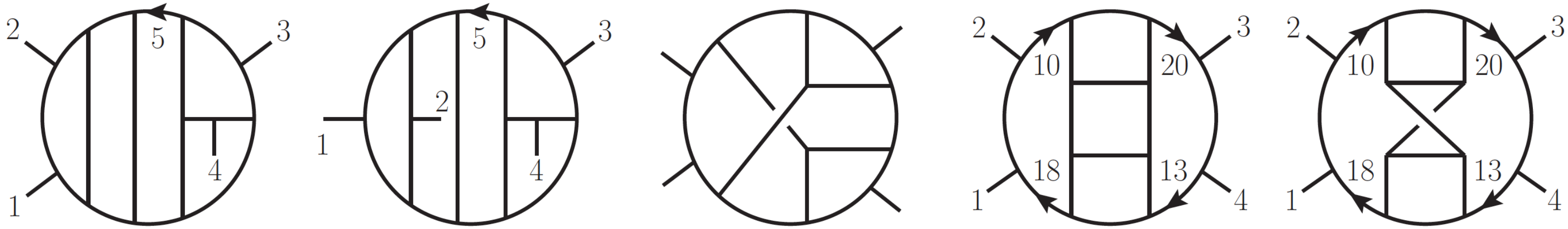
Gravity loop integrands follow from gauge theory!

$\mathcal{N} = 8$ Supergravity at Five loops

ZB, Carrasco, Chen, Edison, Johansson, Roiban, Parra-Martinez, Zeng (2018)

$\mathcal{N} = 8$ supergravity is a theory, studied as a model of unification.

To make a long story short: Even 5 loop calculations are possible



Evaluated leading divergence behavior to help answer fundamental questions on nonrenormalizability of gravity theories:

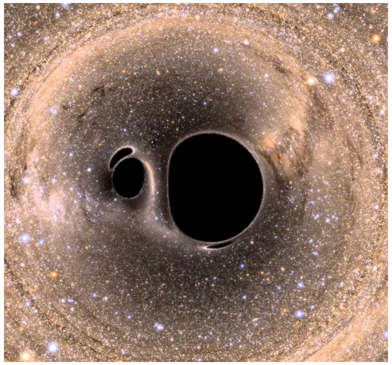
Key point: "Impossible" calculations are pretty standard by now.

Want to harness these advances for LIGO

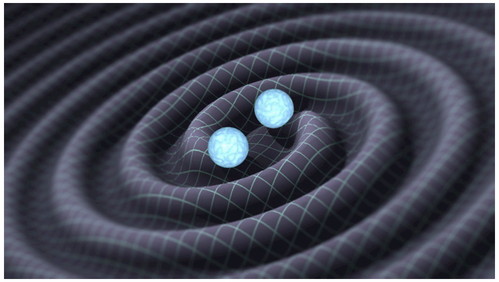
Double Copy for Classical Solutions

Goal is to formulate gravity solutions directly in terms of gauge theory

Variety of special cases:



- **Schwarzschild and Kerr black holes.**
- **Taub-NUT space.**
- **Solutions with cosmological constant.**
- **Radiation from accelerating black hole.**
- **Maximally symmetric space times.**
- **Plane wave background.**
- **Gravitational radiation.**

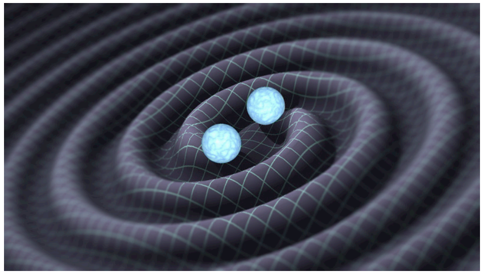


Luna, Monteiro, O'Connell and White;
Luna, Monteiro, Nicholsen, O'Connell and White;
Ridgway and Wise; Carrillo González, Penco, Trodden;
Adamo, Casali, Mason, Nekovar;
Goldberger and Ridgway; Chen;
Luna, Monteiro, Nicholson, Ochirov;
Bjerrum-Bohr, Donoghue, Vanhove;
O'Connell, Westerberg, White; Kosower, Maybee, O'Connell, etc

Still no general understanding.
But plenty of examples.

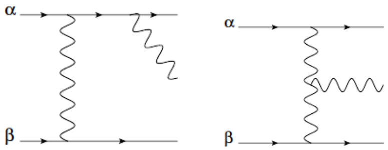
Double Copy and Gravitational Radiation

Can we simplify the types of calculations needed for LIGO?



A small industry has developed to study this.

- **Connection to scattering amplitudes.**
  Bjerrum-Bohr, Donoghue, Holstein, Plante, Pierre Vanhove; Luna, Nicholson, O'Connell, White
  Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon; Damour;
  Plefka, Steinhoff, Wormsbecher



- **Worldline approach for radiation.**
  Goldberger and Ridgway; Goldberger, Li, Prabhu, Thompson; Chester; Shen
- **Removing the dilaton contamination.**
  Luna, Nicholson, O'Connell, White; Johansson, Ochirov; Johansson, Kalin;
  Henrik Johansson, Gregor Kälin, Mogull.

Key Question: Can double copy help us calculate something of direct interest to LIGO/Virgo *beyond* state of the art?

PN versus PM expansion for conservative two-body dynamics

$$\mathcal{L} = -Mc^2 + \underbrace{\frac{\mu v^2}{2} + \frac{GM\mu}{r}} + \frac{1}{c^2}\left[\ldots\right] + \frac{1}{c^4}\left[\ldots\right] + \ldots$$

From Buonanno Amplitudes 2018

$$E(v) = -\frac{\mu}{2}v^2 + \cdots$$

non-spinning compact objects

		0PN	1PN	2PN	3PN	4PN	5PN	$\ldots$
0PM:	1	v^2	v^4	v^6	v^8	v^{10}	v^{12}	$\ldots$
1PM:		$1/r$	v^2/r	v^4/r	v^6/r	v^8/r	v^{10}/r	$\ldots$
2PM:			$1/r^2$	v^2/r^2	v^4/r^2	v^6/r^2	v^8/r^2	$\ldots$
3PM:				$1/r^3$	v^2/r^3	v^4/r^3	v^6/r^3	$\ldots$
4PM:					$1/r^4$	v^2/r^4	v^4/r^4	$\ldots$
$\ldots$						$\ldots$	$\ldots$	

(credit: Justin Vines)

$$1 \to Mc^2, \qquad v^2 \to \frac{v^2}{c^2}, \qquad \frac{1}{r} \to \frac{GM}{rc^2}.$$

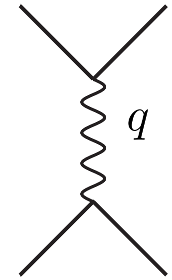
current known PN results — current known PM results — overlap between PN & PM results — unknown

- **PM results** (Westfahl 79, Westfahl & Goller 80, Portilla 79-80, Bel et al. 81, Ledvinka et al. 10, Damour 16-17, Guevara 17, Vines 17, Bini & Damour 17-18, Vines in prep)

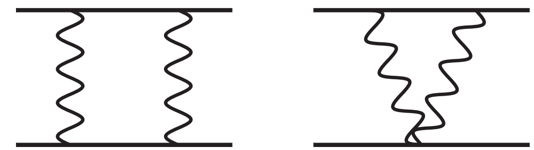
Potentials and Amplitudes

Tree-level: Fourier transform gives classical potential.

$$V(r) = \int \frac{d^3q}{(2\pi)^3} e^{-i\mathbf{q}\cdot\mathbf{r}} A^{\text{tree}}(\mathbf{q})$$



At higher orders things quickly become less obvious:



- **What we learned in grad school on $\hbar$ counting is wrong. Loops with masses can have classical pieces.**
- **Double counting and iteration.**
- **$1/\hbar$ scaling of loop amplitudes.**
- **Non-uniqueness of potential.**
- **Cross terms between $1/\hbar$ and $\hbar$.**

$$e^{iS_{\text{classical}}/\hbar}$$

$1/\hbar^L$ at L loops

Piece of loops are classical: Our task is to extract these pieces.

We harness EFT to clean up confusion

EFT is a Clean Approach

No need to re-invent the wheel.
Build EFT from which we can read off potential.

Goldberger and Rothstein
Neill, Rothstein
Cheung, Rothstein, Solon (2018)

$$L_{\text{kin}} = \int_{\boldsymbol{k}} A^\dagger(-\boldsymbol{k})\left(i\partial_t + \sqrt{\boldsymbol{k}^2 + m_A^2}\right)A(\boldsymbol{k}) + \int_{\boldsymbol{k}} B^\dagger(-\boldsymbol{k})\left(i\partial_t + \sqrt{\boldsymbol{k}^2 + m_B^2}\right)B(\boldsymbol{k})$$

***A*, *B* scalars represents spinless black holes**

$$L_{\text{int}} = -\int_{\boldsymbol{k},\boldsymbol{k}'} V(\boldsymbol{k},\boldsymbol{k}')\, A^\dagger(\boldsymbol{k}')A(\boldsymbol{k})B^\dagger(-\boldsymbol{k}')B(-\boldsymbol{k})$$

Match amplitudes of this theory to the full theory in classical limit to extract a potential.

EFT Matching

full Einstein's theory
(complicated)

↓ Amplitude methods, double copy

tree amplitude

$\hbar \to 0$

↓ generalized unitarity

loop integrand

↓ loop integration

GR loop amplitude

effective theory
(simpler)

↓ build ansatz

potential

↓ Feynman diagrams

loop integrand

↓ loop integration

EFT loop amplitude

GR loop amplitude $=$ EFT loop amplitude (identical physics)

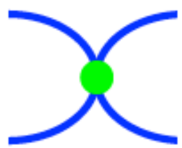
Roundabout but efficiently determines potential

Feynman diagrams for EFT

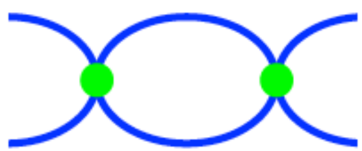
- **EFT scattering amplitudes easy to compute using Feynman diagrams.**
- **No need for advanced methods.**

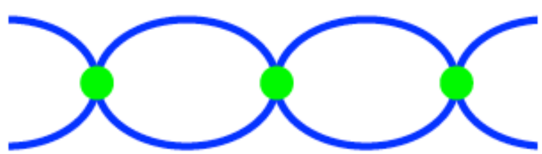
Newton's constant

$$A_{\rm EFT} = \sum_{i=1}^{\infty} G^i A_{\rm EFT}^{(i)}$$

$=$ 

vertices contain all powers of *G*

$+$ 

$+$  $+ \cdots$

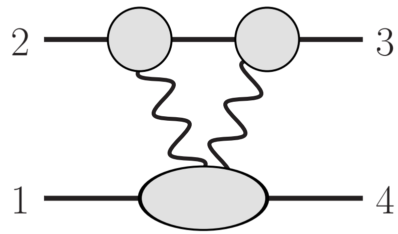
Match to Full Theory

Full Theory: Unitarity + Double Copy

- **Long range force: Two matter lines must be separated by cut propagators.**
- **Classical potential: 1 matter line per loop is cut (on-shell).**

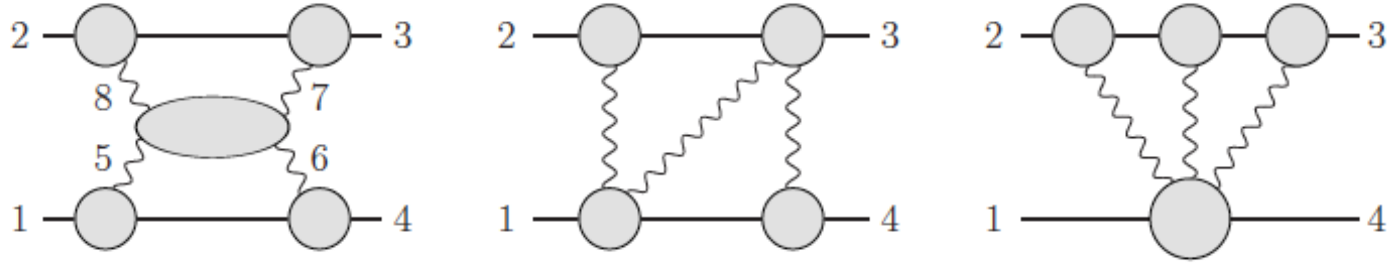
Neill and Rothstein ; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon

Only independent unitarity cut for 2 PM.



exposed lines on-shell (long range).
Classical pieces simple!

Independent generalized unitarity cuts for 3 PM.



What about the dilaton?

Double copy constructions naturally have unwanted states

(gluon) x (gluon) = graviton + dilaton + axion

2 states ↑ (gluon) **2 states** ↑ (gluon) **2 states** ↑ (graviton)

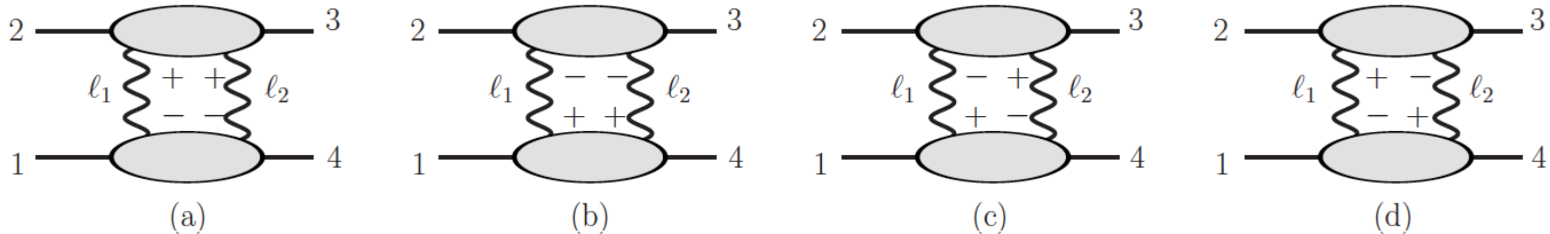
For physical gravitational waves must eliminate dilaton and axion.

Will explain that this is *not* a problem, merely a minor annoyance.

Generalized Unitarity Cuts

Primary means of construction uses BCJ, but KLT should have better scaling at high loops and easier to explain:

2PM



$$C_{\rm GR} = \sum_{h_1,h_2=\pm} M^{\rm tree}(3, \ell_2^{h_2}, -\ell_1^{h_1}, 2) \times M^{\rm tree}(1, \ell_1^{-h_1}, -\ell_2^{-h_2}, 4)$$

$$= -\sum_{h_1,h_2=\pm} s_{23}^2 [A^{\rm tree}(3, \ell_2^{h_2}, -\ell_1^{h_1}, 2) \times A^{\rm tree}(1, \ell_1^{-h_1}, -\ell_2^{-h_2}, 4)]$$

$$\times [A^{\rm tree}(2, \ell_2^{h_2}, -\ell_1^{h_1}, 3) \times A^{\rm tree}(4, \ell_1^{-h_1}, -\ell_2^{-h_2}, 1)]$$

By correlating gluon helicities, removing dilaton is trivial.

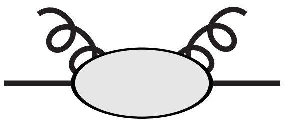
$h^-_{\mu\nu} \to A^-_\mu A^-_\mu$ $\qquad$ $h^+_{\mu\nu} \to A^+_\mu A^+_\mu$ $\qquad$ **Forbid:** $A^+_\mu A^-_\mu$

Problem of computing the generalized cuts in gravity is reduced multiplying and summing gauge-theory tree amplitudes.

Gauge-Theory Building Blocks for 2 PM Gravity

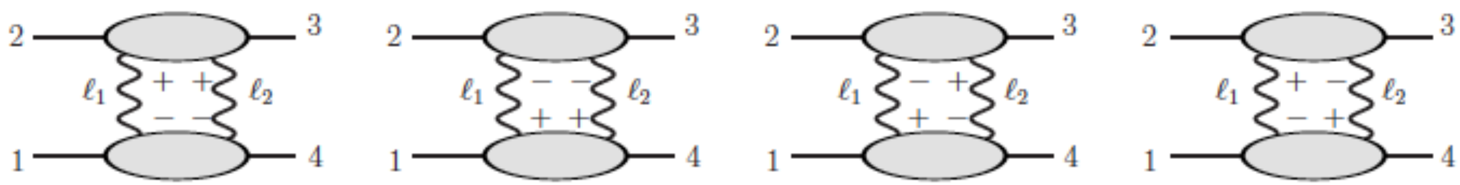
$$A^{\text{tree}}(1^s, 2^+, 3^+, 4^s) = i\frac{m_1^2[23]}{\langle 23\rangle t_{12}}$$

$$A^{\text{tree}}(1^s, 2^+, 3^-, 4^s) = -i\frac{\langle 3|1|2]^2}{\langle 23\rangle[23]t_{12}}$$



color-ordered gauge-theory tree amplitudes

- **This is all you need for 2 PM.**
- **Scaling with number of external legs is brilliant.**



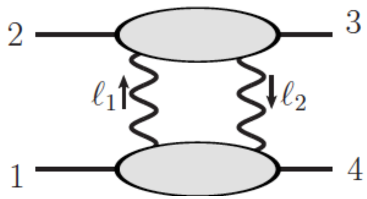
$$C_{\text{YM}} = 2\left(\frac{\mathcal{E}^2 + \mathcal{O}^2}{s_{23}^2} + m_1^2 m_2^2\right)\frac{1}{t_{1\ell_1}\, t_{2\ell_1}}$$

gauge theory

$$\mathcal{E}^2 = \frac{1}{4}\Big[-t_{12}s_{23} + s_{23}t_{1\ell_1} - s_{23}t_{2\ell_1} + 2t_{1\ell_1}t_{2\ell_1}\Big]^2$$

$$\mathcal{O}^2 = \mathcal{E}^2 - (s_{23}m_1^2 + s_{23}t_{1\ell_1} + t_{1\ell_1}^2)(s_{23}m_2^2 - s_{23}t_{2\ell_1} + t_{2\ell_1}^2)$$

One loop gravity warmup



Apply unitarity and KLT relations.
Import gauge-theory results.

$$C_{\rm GR} = 2\left[\frac{1}{t^4}\left(\mathcal{E}^4 + \mathcal{O}^4 + 6\mathcal{E}^2\mathcal{O}^2\right) + m_1^4 m_2^4\right]\left[\frac{1}{t_{1\ell_1}} + \frac{1}{t_{4\ell_1}}\right]\left[\frac{1}{t_{2\ell_1}} + \frac{1}{t_{3\ell_1}}\right]$$

- **Same building blocks as gauge theory!**
- **Double copy is visible even though we have removed dilaton and axion.**

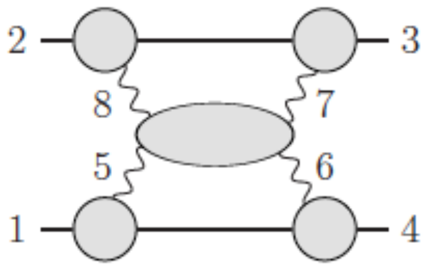
We can extract classical scattering angles or potentials following literature Damour; Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove; Cheung, Rothstein, Solon

This is $2^{\rm nd}$ PM order

$s_{23} = (p_2 + p_3)^2$
$t_{ij} = 2p_i \cdot p_j$

Two Loops for 3 PM

ZB, Cheung, Shen, Roiban, Solon, Zeng



- **Use KLT and sum over helicities**
- **Very similar to one loop**

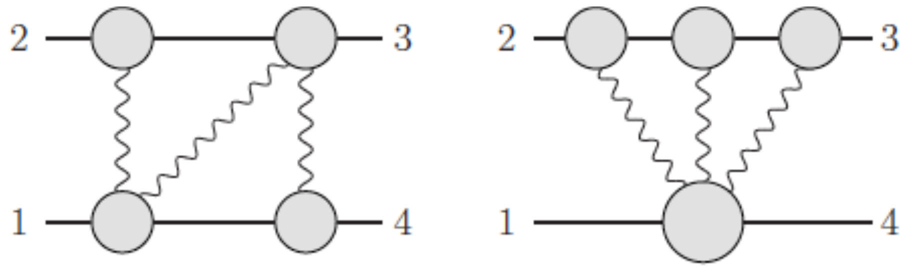
$$C^{\text{H-cut}} = 2i\left[\frac{1}{(p_5 - p_8)^2} + \frac{1}{(p_5 + p_7)^2}\right]$$
$$\times \left[s_{23}^2 m_1^4 m_2^4 + \frac{1}{s_{23}^6}\sum_{i=1,2}\left(\mathcal{E}_i^4 + \mathcal{O}_i^4 + 6\mathcal{O}_i^2\mathcal{E}_i^2\right)\right]$$

$$\mathcal{E}_1^2 = \frac{1}{4}s_{23}^2(t_{18}t_{25} - t_{12}t_{58})^2, \qquad \mathcal{O}_1^2 = \mathcal{E}_1^2 - m_1^2 m_2^2 s_{23}^2 t_{58}^2,$$
$$\mathcal{E}_2^2 = \frac{1}{4}s_{23}^2(t_{17}t_{25} - t_{12}t_{57} - s_{23}(t_{17} + t_{57}))^2,$$
$$\mathcal{O}_2^2 = \mathcal{E}_2^2 - m_1^2 m_2^2 s_{23}^2 t_{57}^2.$$

- **Double copy is visible.**
- **Remarkably simple, given it is two-loop gravity.**

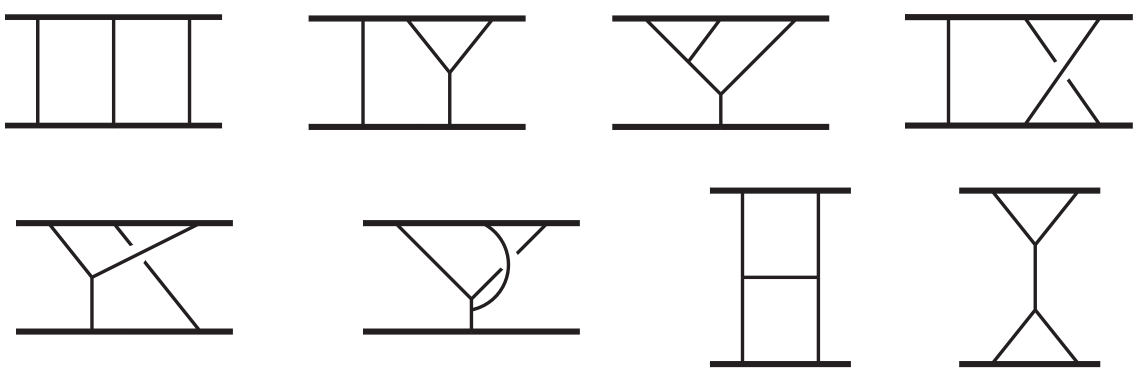
Two loops and 3 PM

Also need contributions from other cuts.

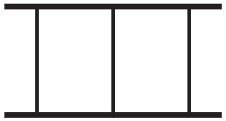


- These unitarity cuts more complicated than previous cut.
- Evaluated using BCJ double copy, with KLT double copy as check.
- To interface easily with EFT approach merged unitarity cuts into diagrams to get integrand.

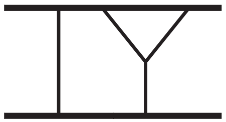
Integrand organized into 8 independent diagrams that can contribute in classical limit:



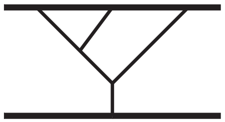
Two-Loop Diagram Numerators



$$(t_{12}^2 - 2m_1^2 m_2^2)^3$$



$$2m_2^3 t_{47}^2 (t_{12}^2 - 2m_1^2 m_2^2)$$



$$2m_2^4(s_{23}^4 + s_{23}^3(2t_{12} + 2t_{15} - t_{47} - 2t_{67}) - 2m_1^2 m_2^2 (s_{23} - t_{67})^2 + (t_{15}t_{56} + (t_{12} - t_{47})t_{67})^2$$
$$+ s_{23}^2(t_{12}^2 + t_{15}^2 + t_{47}^2 - t_{47}t_{56} + t_{12}(4t_{15} - 2t_{47} + t_{56} - 4t_{67}) + t_{15}(-2t_{47} + t_{56} - 2t_{67})$$
$$+ 2t_{47}t_{67} + t_{67}^2) + s_{23}(t_{15}(t_{56}^2 + 2(-2t_{12} + t_{47})t_{67} - t_{56}t_{67})$$
$$+ t_{67}(-2t_{12}^2 + t_{47}(-2t_{47} + t_{56} - t_{67}) + t_{12}(4t_{47} - t_{56} + 2t_{67}))))$$

etc. Remaining 5 diagrams somewhat more complicated but not a big deal.

- **Very simple compared to the usual Feynman diagram explosion.**
- **Higher-loop integrand constructions definitely possible!**

Integration + Extraction of Potential

ZB, Cheung, Shen, Roiban, Solon, Zeng

To integrate follow methods of Cheung, Rothstein and Solon.

- **Integrals reduce via residues to simple 3 dimensional integrals.**
- **Efficiently targets the classical pieces we want.**
- **Incorporates matching to effective field theory.**
- **Good scaling with perturbative order.**

Checks on integrals using standard tools of QCD:

- **Mellin-Barnes integration.** V. Smirnov; Czakon
- **Sector decompostion.** Binoth and Heinrich, A. Smirnov
- **Integration by parts.** K. G. Chetyrkin and F. V. Tkachov, Laporta; A. Smirnov; Maierhöfer, Usovitsch, Uwer
- **Differential equations.** ZB, Dixon, Kosower; Remiddi and Gehrmann
- **Method of regions.** Beneke, V. Smirnov; A. Smirnov.

Amplitude in Classical Limit

ZB, Cheung, Roiban, Shen, Solon, Zeng (2019)

Classical limit. The O(G^3) or 3PM terms are:

rapidity

$$\mathcal{M}_3 = \frac{\pi G^3 \nu^2 m^4 \log \boldsymbol{q}^2}{6\gamma^2 \xi}\left[3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3 - \frac{48\nu\left(3 + 12\sigma^2 - 4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}} - \frac{18\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{(1+\gamma)(1+\sigma)}\right] + \frac{8\pi^3 G^3 \nu^4 m^6}{\gamma^4 \xi}\left[3\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)F_1 - 32 m^2\nu^2\left(1-2\sigma^2\right)^3 F_2\right]$$

$$m = m_A + m_B, \qquad \mu = m_A m_B/m, \qquad \nu = \mu/m, \qquad \gamma = E/m,$$
$$\xi = E_1 E_2/E^2, \qquad E = E_1 + E_2, \qquad \sigma = p_1 \cdot p_2/m_1 m_2,$$

F_1 and F_2 IR divergent iteration terms that don't affect potential.

Two loop gravity.

Simplicity of result is remarkable!

Conservative 3PM Potential

ZB, Cheung, Roiban, Shen, Solon, Zeng

Follow EFT strategy:

The 3PM Hamiltonian:

$$H(\boldsymbol{p},\boldsymbol{r}) = \sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2} + V(\boldsymbol{p},\boldsymbol{r})$$

$$V(\boldsymbol{p},\boldsymbol{r}) = \sum_{i=1}^{\infty} c_i(\boldsymbol{p}^2) \left(\frac{G}{|\boldsymbol{r}|}\right)^i,$$

$$c_1 = \frac{\nu^2 m^2}{\gamma^2 \xi}\left(1-2\sigma^2\right), \qquad c_2 = \frac{\nu^2 m^3}{\gamma^2 \xi}\left[\frac{3}{4}\left(1-5\sigma^2\right) - \frac{4\nu\sigma\left(1-2\sigma^2\right)}{\gamma\xi} - \frac{\nu^2(1-\xi)\left(1-2\sigma^2\right)^2}{2\gamma^3\xi^2}\right],$$

$$\begin{aligned} c_3 = \frac{\nu^2 m^4}{\gamma^2 \xi}\Bigg[&\frac{1}{12}\left(3-6\nu+206\nu\sigma-54\sigma^2+108\nu\sigma^2+4\nu\sigma^3\right) - \frac{4\nu\left(3+12\sigma^2-4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}} \\ &- \frac{3\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{2(1+\gamma)(1+\sigma)} - \frac{3\nu\sigma\left(7-20\sigma^2\right)}{2\gamma\xi} - \frac{\nu^2\left(3+8\gamma-3\xi-15\sigma^2-80\gamma\sigma^2+15\xi\sigma^2\right)\left(1-2\sigma^2\right)}{4\gamma^3\xi^2} \\ &+ \frac{2\nu^3(3-4\xi)\sigma\left(1-2\sigma^2\right)^2}{\gamma^4\xi^3} + \frac{\nu^4(1-2\xi)\left(1-2\sigma^2\right)^3}{2\gamma^6\xi^4}\Bigg], \end{aligned}$$

$$m = m_A + m_B, \qquad \mu = m_A m_B/m, \qquad \nu = \mu/m, \qquad \gamma = E/m,$$

$$\xi = E_1 E_2/E^2, \qquad E = E_1 + E_2, \qquad \sigma = p_1 \cdot p_2/m_1 m_2,$$

Checks

ZB, Cheung, Roiban, Shen, Solon, Zeng

Primary check:

Compare to 4PN Hamiltonian of Damour, Jaranowski, Schäfer

Need canonical transformation:

$$(\boldsymbol{r}, \boldsymbol{p}) \to (\boldsymbol{R}, \boldsymbol{P}) = (A\,\boldsymbol{r} + B\,\boldsymbol{p}, C\,\boldsymbol{p} + D\,\boldsymbol{r})$$

$$A = 1 - \frac{Gm\nu}{2|\boldsymbol{r}|} + \cdots\,, \quad B = \frac{G(1-2/\nu)}{4m|\boldsymbol{r}|}\boldsymbol{p}\cdot\boldsymbol{r} + \cdots$$

$$C = 1 + \frac{Gm\nu}{2|\boldsymbol{r}|} + \cdots\,, \quad D = -\frac{Gm\nu}{2|\boldsymbol{r}|^3}\boldsymbol{p}\cdot\boldsymbol{r} + \cdots\,,$$

For overlap terms of our Hamiltonian equivalent to 4PN Hamitonian.
Explicit canonical transformation found.

4 PN Hamiltonian

Damour, Jaranowski, Schaefer

$$\mathbf{n} = \hat{\mathbf{r}}$$

$$\widehat{H}_{\mathrm{N}}(\mathbf{r},\mathbf{p}) = \frac{\mathbf{p}^2}{2} - \frac{1}{r},$$

$$c^2\,\widehat{H}_{1\mathrm{PN}}(\mathbf{r},\mathbf{p}) = \frac{1}{8}(3\nu-1)(\mathbf{p}^2)^2 - \frac{1}{2}\Big\{(3+\nu)\mathbf{p}^2 + \nu(\mathbf{n}\cdot\mathbf{p})^2\Big\}\frac{1}{r} + \frac{1}{2r^2},$$

$$c^4\,\widehat{H}_{2\mathrm{PN}}(\mathbf{r},\mathbf{p}) = \frac{1}{16}\left(1-5\nu+5\nu^2\right)(\mathbf{p}^2)^3 + \frac{1}{8}\Big\{\left(5-20\nu-3\nu^2\right)(\mathbf{p}^2)^2 - 2\nu^2(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - 3\nu^2(\mathbf{n}\cdot\mathbf{p})^4\Big\}\frac{1}{r}$$
$$+ \frac{1}{2}\Big\{(5+8\nu)\mathbf{p}^2 + 3\nu(\mathbf{n}\cdot\mathbf{p})^2\Big\}\frac{1}{r^2} - \frac{1}{4}(1+3\nu)\frac{1}{r^3},$$

$$c^6\,\widehat{H}_{3\mathrm{PN}}(\mathbf{r},\mathbf{p}) = \frac{1}{128}\left(-5+35\nu-70\nu^2+35\nu^3\right)(\mathbf{p}^2)^4 + \frac{1}{16}\Big\{\left(-7+42\nu-53\nu^2-5\nu^3\right)(\mathbf{p}^2)^3$$
$$+ (2-3\nu)\nu^2(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + 3(1-\nu)\nu^2(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 - 5\nu^3(\mathbf{n}\cdot\mathbf{p})^6\Big\}\frac{1}{r}$$
$$+ \Big\{\frac{1}{16}\left(-27+136\nu+109\nu^2\right)(\mathbf{p}^2)^2 + \frac{1}{16}(17+30\nu)\nu(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \frac{1}{12}(5+43\nu)\nu(\mathbf{n}\cdot\mathbf{p})^4\Big\}\frac{1}{r^2}$$
$$+ \Big\{\Big(-\frac{25}{8} + \Big(\frac{\pi^2}{64} - \frac{335}{48}\Big)\nu - \frac{23\nu^2}{8}\Big)\mathbf{p}^2 + \Big(-\frac{85}{16} - \frac{3\pi^2}{64} - \frac{7\nu}{4}\Big)\nu(\mathbf{n}\cdot\mathbf{p})^2\Big\}\frac{1}{r^3} + \Big\{\frac{1}{8} + \Big(\frac{109}{12} - \frac{21}{32}\pi^2\Big)\nu\Big\}\frac{1}{r^4},$$

G^4

4 PN Hamiltonian

Damour, Jaranowski, Schaefer

$$\mathbf{n} = \hat{\mathbf{r}}$$

$$
\begin{aligned}
c^8\,\widehat{H}^{\rm local}_{\rm 4PN}(\mathbf{r},\mathbf{p}) = {} & \left(\frac{7}{256} - \frac{63}{256}\nu + \frac{189}{256}\nu^2 - \frac{105}{128}\nu^3 + \frac{63}{256}\nu^4\right)(\mathbf{p}^2)^5 \\
& + \left\{\frac{45}{128}(\mathbf{p}^2)^4 - \frac{45}{16}(\mathbf{p}^2)^4\,\nu + \left(\frac{423}{64}(\mathbf{p}^2)^4 - \frac{3}{32}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 - \frac{9}{64}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2\right)\nu^2\right. \\
& + \left(-\frac{1013}{256}(\mathbf{p}^2)^4 + \frac{23}{64}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 + \frac{69}{128}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{64}(\mathbf{n}\cdot\mathbf{p})^6\mathbf{p}^2 + \frac{35}{256}(\mathbf{n}\cdot\mathbf{p})^8\right)\nu^3 \\
& \left. + \left(-\frac{35}{128}(\mathbf{p}^2)^4 - \frac{5}{32}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 - \frac{9}{64}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{32}(\mathbf{n}\cdot\mathbf{p})^6\mathbf{p}^2 - \frac{35}{128}(\mathbf{n}\cdot\mathbf{p})^8\right)\nu^4\right\}\frac{1}{r} \\
& + \left\{\frac{13}{8}(\mathbf{p}^2)^3 + \left(-\frac{791}{64}(\mathbf{p}^2)^3 + \frac{49}{16}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 - \frac{889}{192}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 + \frac{369}{160}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu\right. \\
& + \left(\frac{4857}{256}(\mathbf{p}^2)^3 - \frac{545}{64}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + \frac{9475}{768}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 - \frac{1151}{128}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu^2 \\
& \left. + \left(\frac{2335}{256}(\mathbf{p}^2)^3 + \frac{1135}{256}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 - \frac{1649}{768}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 + \frac{10353}{1280}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu^3\right\}\frac{1}{r^2} \\
& + \left\{\frac{105}{32}(\mathbf{p}^2)^2 + \left(\left(\frac{2749\pi^2}{8192} - \frac{589189}{19200}\right)(\mathbf{p}^2)^2 + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right)(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right)(\mathbf{n}\cdot\mathbf{p})^4\right)\nu\right. \\
& + \left(\left(\frac{18491\pi^2}{16384} - \frac{1189789}{28800}\right)(\mathbf{p}^2)^2 + \left(-\frac{127}{3} - \frac{4035\pi^2}{2048}\right)(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \left(\frac{57563}{1920} - \frac{38655\pi^2}{16384}\right)(\mathbf{n}\cdot\mathbf{p})^4\right)\nu^2 \\
& \left. + \left(-\frac{553}{128}(\mathbf{p}^2)^2 - \frac{225}{64}(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - \frac{381}{128}(\mathbf{n}\cdot\mathbf{p})^4\right)\nu^3\right\}\frac{1}{r^3} \\
& + \left\{\frac{105}{32}\mathbf{p}^2 + \left(\left(\frac{185761}{19200} - \frac{21837\pi^2}{8192}\right)\mathbf{p}^2 + \left(\frac{3401779}{57600} - \frac{28691\pi^2}{24576}\right)(\mathbf{n}\cdot\mathbf{p})^2\right)\nu\right. \\
& \left. + \left(\left(\frac{672811}{19200} - \frac{158177\pi^2}{49152}\right)\mathbf{p}^2 + \left(\frac{110099\pi^2}{49152} - \frac{21827}{3840}\right)(\mathbf{n}\cdot\mathbf{p})^2\right)\nu^2\right\}\frac{1}{r^4} \quad \longleftarrow G^4 \\
& + \left\{-\frac{1}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169199}{2400}\right)\nu + \left(\frac{7403\pi^2}{3072} - \frac{1256}{45}\right)\nu^2\right\}\frac{1}{r^5}. \quad \longleftarrow G^5
\end{aligned}
$$

After canonical transformation we match all but G^4 and G^5 terms

Mess is partly due to their gauge choice.

Ours is all orders in *p* at G^3

Additional Tests

Additional (somewhat redundant) tests:

1. Calculated classical scattering angle (ignoring radiation reaction).
Match overlap terms in known 4PN result.

Bini and Damour

2. Calculated amplitude using potentials.
Match on overlap terms in known 4PN Hamitonian.

Damour, Jaranowski, Schaefer

3. In test mass limit, $m_1 \ll m_2$, matches Schwarzschild Hamiltonian.

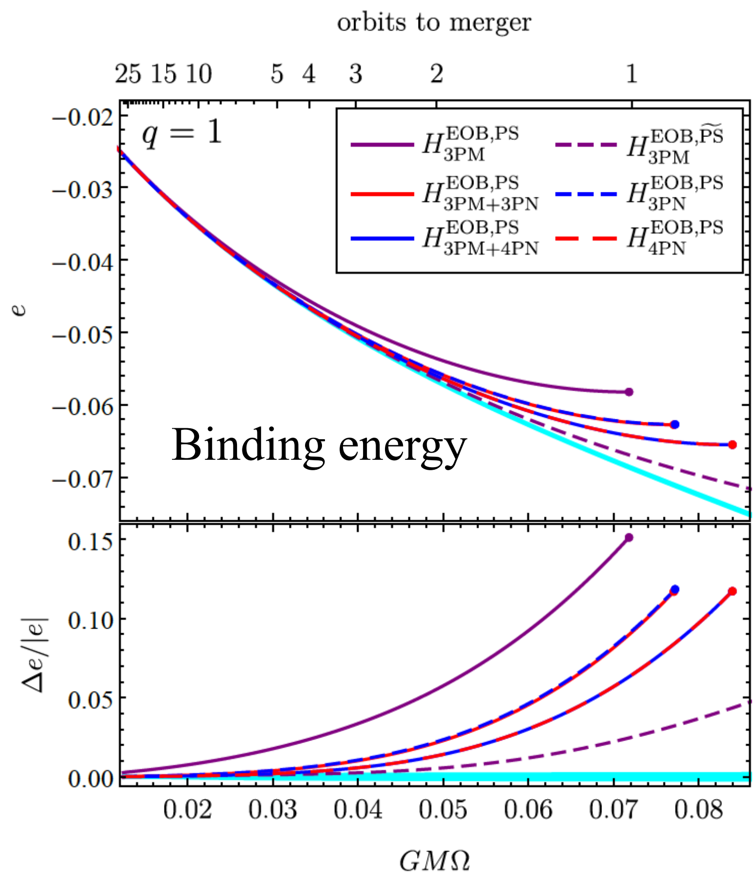
Wex and Schaefer

Gives us confidence that our Hamiltonian is correct and lines up with LIGO's template construction.

Tests of Our 3PM Hamiltonian for LIGO

Antonelli, Buonanno, Steinhoff, van de Meent, and Vines, arXiv:1901.07102 (last week)

(8 days after our paper)



Test against numerical relativity.

Note: Not conclusive, e. g. radiation not taken into accounted

Winning curve is based on 3PM.

numerical relativity taken as truth

"This rather encouraging result motivates a more comprehensive study…"

3PM + 4PN fed into EOB ➔ Most advanced 2 body Hamiltonian

Outlook for Gravitational Wave Physics

- **Methods are far from exhausted.**
- **Even more efficient methods seem likely.**
- **Methods should scale well to higher orders.**

Natural future questions to investigate:

- **Higher orders. Resummation in *G*.**
- **Radiation.**
- **Spin.**
- **Finite size effects.**

Hope to learn from the assembled experts about priorities.

Summary

- Remarkable connection between gauge and gravity theories:
  — color ⟷ kinematics.
  — gravity ~ (gauge theory)^2
- Double-copy idea gives us a powerful new way to think about gravity. Unified framework for gravity and gauge theory.
- Obtained the 3PM conservative 2-body potential.
- Methods nowhere close to exhausted.
- Spin, finite size effects, radiation, and higher orders in G obvious possibilities to investigate.

Expect many more advances in coming years, not only for gravitational waves but also for understanding gravity and its relation to the other forces via double copy.

Extra Slides