Singularity formation in Black hole interiors.

Spyros Alexakis

IPAM, January 2019.
Conjecture (Strong Cosmic censorship–Penrose 1970’s)

Consider an initial data $(\Sigma^3, h, k)$ with reasonable matter fields. Consider the maximal hyperbolic development $(\mathcal{M}^{3+1}, g)$ of this initial data set. Assume that $\mathcal{M}^{3+1}$ contains a black hole region $\mathcal{M}_{b.h.} \subset \mathcal{M}^{3+1}$.

Then generically inside $\mathcal{M}_{b.h.}$, $g$ will end at a terminal singularity.
Conjecture (Strong Cosmic censorship–Penrose 1970’s)

Consider an initial data \((\Sigma^3, h, k)\) with reasonable matter fields. Consider the maximal hyperbolic development \((\mathcal{M}^{3+1}, g)\) of this initial data set. Assume that \(\mathcal{M}^{3+1}\) contains a black hole region \(\mathcal{M}_{b.h.} \subset \mathcal{M}^{3+1}\). Then generically inside \(\mathcal{M}_{b.h.}\) \(g\) will end at a terminal singularity.

Terminal means that \(g\) cannot be extended past the singularity and still solve the Einstein equations (even in a weak sense).
The nature of the singularity; (strengthened SCC)

Conjecture (Penrose)

**Generically** the singularity will be **space-like**, and involve collapsing in spatial directions.
The nature of the singularity; (strengthened SCC)

Conjecture (Penrose)

**Genericall**y *the singularity will be space-like, and involve collapsing in spatial directions.*

The validity of the above was widely believed, eg.
How much do you know about black holes?

Stephen Hawking and black holes

Inside the Mind of Prof Stephen Hawking
Take a cosmic journey with the world's most famous physicist.

Prof Hawking on Desert Island Discs

In Our Time: Black Holes
Melvyn Bragg discusses black holes, the dead collapsed ghosts of massive stars.
How much do you know about black holes?

Question 3 of 9

When a star collapses into a black hole all its mass gets squeezed into:

- The singularity
- The event horizon
- Another dimension
Nature of the singularity—very strong SCC conjecture.

Conjecture (Belinskii-Khalatnikov-Lifshitz)

**Generically** the space-time metric will oscillate wildly prior to the singularity.
Nature of the singularity–very strong SCC conjecture.

Conjecture (Belinskii-Khalatnikov-Lifshitz)

**Generically** the space-time metric will oscillate wildly prior to the singularity.

**Remark:** This is an extrapolation from an analogous conjecture concerning the *initial, Big Bang* singularity of space-time.
Nature of the singularity–very strong SCC conjecture.

**Conjecture (Belinskii-Khalatnikov-Lifshitz)**

**Generically** the space-time metric will oscillate wildly prior to the singularity.

**Remark:** This is an extrapolation from an analogous conjecture concerning the **initial, Big Bang** singularity of space-time.

Heuristic argument based on linearization;
Conjecture (Belinskii-Khalatnikov-Lifshitz)

**Generically** the space-time metric will oscillate wildly prior to the singularity.

**Remark:** This is an extrapolation from an analogous conjecture concerning the initial, Big Bang singularity of space-time.

Heuristic argument based on linearization; **very little evidence** in favour of this.
Conjecture (Belinskii-Khalatnikov-Lifshitz)

**Generically** the space-time metric will oscillate wildly prior to the singularity.

**Remark:** This is an extrapolation from an analogous conjecture concerning the initial, Big Bang singularity of space-time.

Heuristic argument based on linearization; **very little evidence** in favour of this. **Only** in the Big Bang setting. cf. Asthekar, Misner, Ringström.
AVD in Big Bang space-times.

Many examples of AVD behaviour constructed for cosmological singularities.
AVD in Big Bang space-times.

Many examples of AVD behaviour constructed for cosmological singularities.
Mostly $\mathbb{T}^3$-Gowdy; also $\mathbb{T}^2$-Gowdy.
AVD in Big Bang space-times.

Many examples of AVD behaviour constructed for cosmological singularities.
Mostly \( \mathbb{T}^3 \)-Gowdy; also \( \mathbb{T}^2 \)-Gowdy.

Many examples of AVD behaviour constructed for cosmological singularities. Mostly $T^3$-Gowdy; also $T^2$-Gowdy.


Nature of Asymptotically Velocity Term Dominated behaviour:
At each point on the singularity the solution approaches a different Kasner solution: $-dt^2 + t^{p_1}(dx_1)^2 + t^{p_2}(dx_2)^2 + t^{p_3}(dx_3)^2$. 

Fuchsian techniques (i.e. examples). 2 or 3 degrees of symmetry essential.
AVD in Big Bang space-times.

Many examples of AVD behaviour constructed for cosmological singularities.
Mostly $T^3$-Gowdy; also $T^2$-Gowdy.


Nature of Asymptotically Velocity Term Dominated behaviour: At each point on the singularity the solution approaches a different Kasner solution: $-dt^2 + t^{p_1}(dx_1)^2 + t^{p_2}(dx_2)^2 + t^{p_3}(dx_3)^2$.

AVD also captured in energy behaviour of the fields:

$$\lim_{T \to 0^+} \frac{\int_{t=T} (\partial_t g_{ii})^2}{\int_{t=T} |\nabla_x g_{ii}|^2} \to +\infty.$$  (1)
AVD in Big Bang space-times.

Many examples of AVD behaviour constructed for cosmological singularities. Mostly $\mathbb{T}^3$-Gowdy; also $\mathbb{T}^2$-Gowdy.


Nature of Asymptotically Velocity Term Dominated behaviour:
At each point on the singularity the solution approaches a different Kasner solution: $-dt^2 + t^{p_1}(dx_1)^2 + t^{p_2}(dx_2)^2 + t^{p_3}(dx_3)^2$.

AVD also captured in energy behaviour of the fields:

$$\lim_{T \to 0^+} \frac{\int_{t=T} (\partial_t g_{ii})^2}{\int_{t=T} |\nabla g_{ii}|^2} \to +\infty.$$  

(1)

Fuchsian techniques (i.e. examples). 2 or 3 degrees of symmetry essential.
All known results prior to ours have *two* degrees of symmetry imposed. (Mathematical, physical and Numerical).
All known results prior to ours have *two* degrees of symmetry imposed. (Mathematical, physical and Numerical). There exists a *very* extensive literature in this setting. Some pertinent results in * spherical symmetry*: 

**Theorem (Christodoulou. Mid 80’s–late 90’s)**

*For Einstein-massless scalar field strengthened strong cosmic censorship is *generically* true.*
All known results prior to ours have two degrees of symmetry imposed. (Mathematical, physical and Numerical). There exists a very extensive literature in this setting. Some pertinent results in spherical symmetry:

**Theorem (Christodoulou. Mid 80’s–late 90’s)**

*For Einstein-massless scalar field strengthened strong cosmic censorship is *generically* true. ∃ counterexamples of six types, but they are *non* generic in Spherical symmetry.*
Singularities in black holes–Results: Spherical symmetry.

All known results prior to ours have two degrees of symmetry imposed. (Mathematical, physical and Numerical). There exists a very extensive literature in this setting. Some pertinent results in spherical symmetry:

Theorem (Christodoulou. Mid 80’s–late 90’s)

For Einstein-massless scalar field strengthened strong cosmic censorship is \textbf{generically} true. \exists counterexamples of six types, but they are \textbf{non} generic in Spherical symmetry.

Theorem (Dafermos 2012)

For \textbf{two-ended} initial data and with charged scalar field, \textbf{generically} there exists \textbf{no} space-like singularity. There exists a \textbf{weaker null} singularity.
Theorem (Dafermos-Luk, 2017+)

In vacuum: For generic perturbations of a Kerr solution exterior, \( \exists \) a \textbf{portion} of weak null singularity inside black hole.
Theorem (Dafermos-Luk, 2017+)

*In vacuum:* For generic perturbations of a Kerr solution exterior, ∃ a portion of weak null singularity inside black hole.

Theorem (Rodnianski-Speck, 2014, 2017)

For Big-Bang singularities in Einstein-scalar field, solving backwards towards the singularity: Perturbations of the data at \( \{ t = 1 \} \) lead to space-like singularity formation at \( \{ t = 0 \} \). AVD behaviour observed.
Theorem (Dafermos-Luk, 2017+)

In vacuum: For generic perturbations of a Kerr solution exterior, \( \exists \) a portion of weak null singularity inside black hole.

Theorem (Rodnianski-Speck, 2014, 2017)

For Big-Bang singularities in Einstein-scalar field, solving backwards towards the singularity: Perturbations of the data at \( \{ t = 1 \} \) lead to space-like singularity formation at \( \{ t = 0 \} \). AVD behaviour observed.

Crucially relies on scalar field as a stabilizing force. Vacuum results in very high dimension (\( d \geq 30 \)).
Our result: Stability under polarized axial symmetry.

\[ g_{\text{Schw}} = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - \left(1 - \frac{2M}{r}\right) dt^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]
Our result: Stability under polarized axial symmetry.

\[ g_{\text{Schw}} = (1 - \frac{2M}{r})^{-1} dr^2 - (1 - \frac{2M}{r}) dt^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]

\[ \partial_{\phi} \text{ is Killing.} \]
Our result: Stability under polarized axial symmetry.

\[ g_{\text{Schw}} = \left(1 - \frac{2M}{r}\right)^{-1}dr^2 - \left(1 - \frac{2M}{r}\right)dt^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \]

\[ \partial_\phi \text{ is Killing. Polarized Killing because } \partial_\phi \perp \partial_t, \partial_\theta, \partial_r. \]
Our result: Stability under polarized axial symmetry.

\[ g_{\text{Schw}} = \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - \left(1 - \frac{2M}{r}\right) dt^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]

\[ \partial_c \phi \text{ is Killing. Polarized Killing} \text{ because } \partial_c \phi \perp \partial_t, \partial_\theta, \partial_r. \]

**Theorem (A.–Fournodavlos)**

Consider axi-symmetric, polarized perturbations of the Schwarzschild data on \( r = M, t \in [0, M] \). Then the solution \( g_{\text{perturb}} \) of the vacuum Einstein equations develops a space-like singularity, with (gauge-normalized) asymptotics of the form:

\[ g_p \sim \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^{2\beta(t,\theta)} dt^2 + r^{2\delta(t,\theta)} d\theta^2 + r^{\epsilon(t,\theta)} dt d\theta \]

\[ + r^{2\alpha(t,\theta)} \sin^2 \theta d\phi^2. \]
Our result: Stability under polarized axial symmetry.

$$g_{\text{Schw}} = (1 - \frac{2M}{r})^{-1} dr^2 - (1 - \frac{2M}{r}) dt^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2).$$

$\partial_\phi$ is Killing. Polarized Killing because $\partial_\phi \perp \partial_t, \partial_\theta, \partial_r$.

**Theorem (A.–Fournodavlos)**

Consider axi-symmetric, polarized perturbations of the Schwarzschild data on $r = M, t \in [0, M]$. Then the solution $g_{\text{perturb}}$ of the vacuum Einstein equations develops a space-like singularity, with (gauge-normalized) asymptotics of the form:

$$g_{p} \sim (1 - \frac{2M}{r})^{-1} dr^2 + r^2 \beta(t, \theta) dt^2 + r^2 \delta(t, \theta) d\theta^2 + r \epsilon(t, \theta) dtd\theta$$

$$+ r^2 \alpha(t, \theta) \sin^2 \theta d\phi^2.$$  \hspace{1cm} (2)

In fact $\alpha(t, \theta) \sim 1, \delta = \delta(\alpha) \sim 1, \beta = \beta(\alpha) \sim -1, \epsilon = \epsilon(\alpha) \sim \frac{5}{2}$.
Express

\[ 3+1g = e^{2\gamma(r,t,\theta)} \sin^2 \theta \, d\phi^2 + 2+1 \, h(r, t, \theta) \]

(polarized \(3+1g\)).
The reduction of the Einstein equations

Express

$$\text{3+1} \, g = e^{2\gamma(r,t,\theta)} \sin^2 \theta d\phi^2 + \text{2+1} \, h(r, t, \theta)$$

(polarized $\text{3+1} \, g$). $\text{Ric}(\text{3+1} \, g) = 0$ equivalent to:

$$\Box_g \gamma = 0, \text{Ric}_{ij}(\text{2+1} \, h) = \nabla_{ij} \gamma + \nabla_{i} \gamma \nabla_{j} \gamma.$$ 

Connection $K_{ij} = \langle \nabla_{e^i} e^{i0}, e^j \rangle$, $A_{ij, k} = \langle \nabla_{e^i} e^j, e^k \rangle$ of $\text{2+1} \, h$
given by Connection of $\text{3+1} \, g$ and $\gamma$. 
Express

\[ 3+1g = e^{2\gamma(r,t,\theta)} \sin^2 \theta d\phi^2 + 2+1 h(r, t, \theta) \]

(polarized \(3+1g\)). \(Ric(3+1g) = 0\) equivalent to:

\[ \Box g \gamma = 0, \; Ric_{ij}(2+1h) = \nabla_{ij} \gamma + \nabla_i \gamma \nabla_j \gamma. \]

Connection \(K_{ij} = \langle \nabla_{e^i} e^0, e^j \rangle, \; A_{ij,k} = \langle \nabla_{e^i} e^j, e^k \rangle\) of \(2+1h\) given by Connection of \(3+1g\) and \(\gamma\). 2nd equation expressible as ODEs in \(K, A!\)
The reduction of the Einstein equations

Express

$$3+1 \, g = e^{2\gamma(r,t,\theta)} \sin^2 \theta d\phi^2 + 2+1 \, h(r, t, \theta)$$

(polarized $3+1 \, g$). $Ric(3+1 \, g) = 0$ equivalent to:

$$\Box g \gamma = 0, \, Ric_{ij}(2+1 \, h) = \nabla_{ij} \gamma + \nabla_i \gamma \nabla_j \gamma.$$ 

Connection $K_{ij} = \langle \nabla_{e_i} e^0, e^j \rangle$, $A_{ijk} = \langle \nabla_{e_i} e^j, e^k \rangle$ of $2+1 \, h$ given by Connection of $3+1 \, g$ and $\gamma$. 2nd equation expressible as ODEs in $K, A$! In geodesic frame $\nabla_{e^0} e^i = 0$,

$$e^0 K_{ij} = K \ast K + R_{0ij0}^h, \, e^0 A_{ijk} = K \ast A + R_{0ijk}^h.$$
Express
\[ 3+1g = e^{2\gamma(r,t,\theta)} \sin^2 \theta d\phi^2 + 2+1h(r, t, \theta) \]
(polarized \(3+1g\)). \(Ric(3+1g) = 0\) equivalent to:
\[ \Box g \gamma = 0, \quad Ric_{ij}(2+1h) = \nabla ij \gamma + \nabla i \gamma \nabla j \gamma. \]

Connection \(K_{ij} = \langle \nabla e^0 e^i, e^j \rangle, A_{ij,k} = \langle \nabla e^i e^j, e^k \rangle\) of \(2+1h\)
given by Connection of \(3+1g\) and \(\gamma\). 2nd equation expressible as ODEs in \(K, A!\) In geodesic frame \(\nabla e^0 e^i = 0,\)
\[ e^0 K_{ij} = K \ast K + R_{0ij0}^h, \quad e^0 A_{ijk} = K \ast A + R_{0ijk}^h. \]

In \(2+1\) dim’s \(R_{ijkl} = Ric \otimes g + W_{ijkl}. \) But \(W_{ijkl} = 0.\)
Wave-ODE system. Formal Asymptotics at singularity.

Our system reduces to:

\[
\begin{align*}
\square g = 0, \\
\epsilon_0 K &= K^* K + \nabla^2 \gamma + \nabla \gamma \nabla \gamma, \\
\epsilon_0 A &= K^* A + \nabla^2 \gamma + \nabla \gamma \nabla \gamma.
\end{align*}
\]

Formal asymptotics:

\[
\gamma \sim \alpha(t, \theta) \log r + B(t, \theta) + O(r)
\]

Assuming this for \( \gamma \) we have in principal frame for \( K \):

\[
K_{11} = \beta(t, \theta) r^{-3/2} + O(r^{-1/2}), \\
K_{22} = \delta(t, \theta) r^{-3/2} + O(r^{-1/2}), \\
K_{12} = O(r^1).
\]

\( d_1(\alpha(t, \theta)) \), \( d_2(\alpha(t, \theta)) \) are explicit and it turns out:

\[
\text{tr} K(r = \rho) = \frac{3}{2} \rho - \frac{3}{2} + O(\rho^{-1/2}).
\]
Wave-ODE system. Formal Asymptotics at singularity.

Our system reduces to:

\[ \Box g \gamma = 0, \]
\[ e^0 K = K \ast K + \nabla^2 \gamma + \nabla \gamma \nabla \gamma, \tag{3} \]
\[ e^0 A = K \ast A + \nabla^2 \gamma + \nabla \gamma \nabla \gamma. \]
Wave-ODE system. Formal Asymptotics at singularity.

Our system reduces to:

\[ \Box g \gamma = 0 , \]
\[ e^0 K = K * K + \nabla^2 \gamma + \nabla \gamma \nabla \gamma , \tag{3} \]
\[ e^0 A = K * A + \nabla^2 \gamma + \nabla \gamma \nabla \gamma . \]

Formal asymptotics: \( \gamma \sim \alpha(t, \theta) \log r + B(t, \theta) + O(r) . \)
Wave-ODE system. Formal Asymptotics at singularity.

Our system reduces to:

\[ \Box g \gamma = 0, \]
\[ e^0 K = K \ast K + \nabla^2 \gamma + \nabla \gamma \nabla \gamma, \quad (3) \]
\[ e^0 A = K \ast A + \nabla^2 \gamma + \nabla \gamma \nabla \gamma. \]

**Formal asymptotics:** \( \gamma \sim \alpha(t, \theta) \log r + B(t, \theta) + O(r). \) Assuming this for \( \gamma \) we have in principal frame for \( K \):

\[ K_{11} = \beta(t, \theta) r^{-3/2} + O(r^{-1/2}) + \overline{y}(t, \theta) r^{\epsilon(t, \theta)}, \]
\[ K_{22} = \delta(t, \theta) r^{-3/2} + O(r^{-1/2}), \quad K_{12} = O(r^1). \]
Wave-ODE system. Formal Asymptotics at singularity.

Our system reduces to:

\[ \square g \gamma = 0, \]
\[ e^0 K = K \ast K + \nabla^2 \gamma + \nabla \gamma \nabla \gamma, \]
\[ e^0 A = K \ast A + \nabla^2 \gamma + \nabla \gamma \nabla \gamma. \]  (3)

Formal asymptotics: \( \gamma \sim \alpha(t, \theta) \log r + B(t, \theta) + O(r). \) Assuming this for \( \gamma \) we have in principal frame for \( K \):

\[ K_{11} = \beta(t, \theta) r^{-3/2} + O(r^{-1/2}) + \bar{y}(t, \theta) r^\epsilon(t, \theta), \]
\[ K_{22} = \delta(t, \theta) r^{-3/2} + O(r^{-1/2}), K_{12} = O(r^1). \]  (4)

\( d_1(\alpha(t, \theta)), \ d_2(\alpha(t, \theta)) \) are explicit and it turns out:

\[ trK(r = \rho) = \frac{3}{2} \rho^{-3/2} + O(\rho^{-1/2}). \]
Ricatti equation for $K_{22}$ sees the *collapsing* direction $\partial_\theta$. Can well blow up *before* $r = 0$ for gauge reasons. Forced to solve the above by *iteration*.
Ricatti equation for $K_{22}$ sees the collapsing direction $\partial_\theta$. Can well blow up before $r = 0$ for gauge reasons. Forced to solve the above by iteration.

Forced to use energy estimates to produce real solution.
Ricatti equation for $K_{22}$ sees the *collapsing* direction $\partial \theta$. Can well blow up *before* $r = 0$ for gauge reasons. Forced to solve the above by *iteration*. Forced to use energy estimates to produce *real* solution. Asymptotically CMC of $r$ is *crucial*. 
Ricatti equation for \( K_{22} \) sees the \textit{collapsing} direction \( \partial_\theta \). Can well blow up \textit{before} \( r = 0 \) for gauge reasons. Forced to solve the above by \textit{iteration}.

Forced to use energy estimates to produce \textit{real} solution.

Asymptotically CMC of \( r \) is \textit{crucial}. \textit{Danger} in differentiated Ricatti: \( \partial = \partial_t, \partial_\theta \).

\[
e^0 \partial K_{11} + 2K_{11} \partial K_{11} + 2K_{12} \partial K_{12} = \partial[\nabla^2 \gamma + \nabla \gamma \nabla \gamma]
\]

admits \textit{free branch} like \( r^{\epsilon(t,\theta)} \) (consistent with undifferentiated Ricatti).
Ricatti equation for $K_{22}$ sees the *collapsing* direction $\partial_\theta$. Can well blow up before $r = 0$ for gauge reasons. Forced to solve the above by *iteration*.

Forced to use energy estimates to produce *real* solution.

Asymptotically CMC of $r$ is *crucial*. *Danger* in differentiated Ricatti: $\partial = \partial_t, \partial_\theta$.

\[
e^0 \partial K_{11} + 2K_{11} \partial K_{11} + 2K_{12} \partial K_{12} = \partial [\nabla^2 \gamma + \nabla \gamma \nabla \gamma]
\]

admits *free branch* like $r^{\epsilon(t,\theta)}$ (consistent with *undifferentiated Ricatti*). *But*

\[
e^0 \partial K_{22} + 2K_{22} \partial K_{22} + 2K_{12} \partial K_{12} = \partial [\nabla^2 \gamma + \nabla \gamma \nabla \gamma]
\]

admits free branch $r^{-3+\epsilon(t,\theta)}$. 
Ricatti equation for $K_{22}$ sees the *collapsing* direction $\partial_{\theta}$. Can well blow up *before* $r = 0$ for gauge reasons. Forced to solve the above by *iteration*. Forced to use energy estimates to produce *real* solution. Asymptotically CMC of $r$ is *crucial*. *Danger* in differentiated Ricatti: $\partial = \partial_t, \partial_{\theta}$.

\[
e^0 \partial K_{11} + 2K_{11} \partial K_{11} + 2K_{12} \partial K_{12} = \partial [\nabla^2 \gamma + \nabla \gamma \nabla \gamma]
\]

admits *free branch* like $r^{\epsilon(t,\theta)}$ (consistent with *undifferentiated* Ricatti). **But**

\[
e^0 \partial K_{22} + 2K_{22} \partial K_{22} + 2K_{12} \partial K_{12} = \partial [\nabla^2 \gamma + \nabla \gamma \nabla \gamma]
\]

admits free branch $r^{-3+\epsilon(t,\theta)}$. **If** this is present then *no* possibility of deriving any estimates for the system. (In the iteration estimates would be getting exponentially worse at each step).