Singularity formation in Black hole interiors.

Spyros Alexakis

IPAM, January 2019.

Strong Cosmic Censorship Conjecture–SCC

Conjecture (Strong Cosmic censorship–Penrose 1970's)

Consider an initial data (Σ^3, h, k) *with reasonable matter fields. Consider the maximal hyperbolic development* $(\mathcal{M}^{3+1}, \mathbf{g})$ *of this initial data set. Assume that* $\mathcal{M}^{3+1}$ *contains a black hole region* $\mathcal{M}_{\mathrm{b.h.}} \subset \mathcal{M}^{3+1}$*.*
Then **generically** *inside* $\mathcal{M}_{\mathrm{b.h.}}$ $\mathbf{g}$ *will end at a* **terminal** *singularity.*

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Then **generically** *inside* $\mathcal{M}_{\mathrm{b.h.}}$ $\mathbf{g}$ *will end at a* **terminal** *singularity.*

Terminal means that $\mathbf{g}$ cannot be extended past the singularity and still solve the Einstein equations (even in a weak sense).

The nature of the singularity; (strengthened SCC)

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The validity of the above was widely believed, eg.

BBC Black Holes Quiz

How much do you know about black holes?

Share

Stephen Hawking and black holes

Inside the Mind of Prof Stephen Hawking

Take a cosmic journey with the world's most famous physicist.

Prof Hawking on Desert Island Discs

First broadcast on Christmas Day in 1992.

In Our Time: Black Holes

Melvyn Bragg discusses black holes, the dead collapsed ghosts of massive stars.

How much do you know about black holes?

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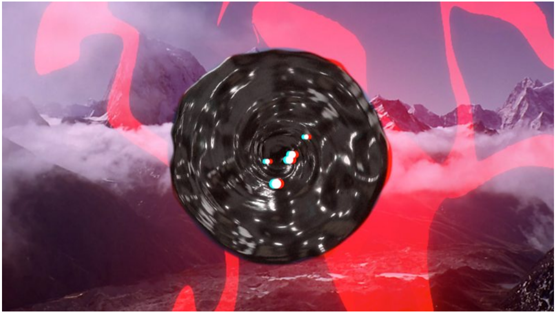

Question 3 of 9

When a star collapses into a black hole all its mass gets squeezed into:

- The singularity
- The event horizon
- Another dimension

Nature of the singularity–very strong SCC conjecture.

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Nature of Asymptotically Velocity Term Dominated behaviour:
At each point on the singularity the solution approaches a different Kasner solution: $-dt^2 + t^{p_1}(dx_1)^2 + t^{p_2}(dx_2)^2 + t^{p_3}(dx_3)^2$.

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Fuchsian techniques (i.e. examples). 2 or 3 degrees of symmetry essential.

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Theorem (Dafermos 2012)

For **two-ended** *initial data and with charged scalar field,* **generically** *there exists* **no** *space-like singularity. There exists a weaker* **null** *singularity.*

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Theorem (Dafermos-Luk, 2017+)

In vacuum: For generic perturbations of a Kerr solution exterior, $\exists$ a **portion** *of weak null singularity inside black hole.*

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Crucially relies on scalar field as a stabilizing force. Vacuum results in very high dimension ($d \geq 30$).

Our result: Stability under polarized axial symmetry.

$$\mathbf{g}_{\text{Schw}} = (1 - \frac{2M}{r})^{-1} dr^2 - (1 - \frac{2M}{r}) dt^2 + r^2 (d\theta^2 + sin^2\theta d\phi^2).$$

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Consider **axi-symmetric**, **polarized** *perturbations of the Schwarzschild data on $r = M, t \in [0, M]$. Then the solution $\mathbf{g}_{\mathrm{perturb}}$ of the vacuum Einstein equations develops a space-like singularity, with (gauge-normalized) asymptotics of the form:*

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In fact $\alpha(t,\theta) \sim 1$, $\delta = \delta(\alpha) \sim 1, \beta = \beta(\alpha) \sim -1, \epsilon = \epsilon(\alpha) \sim \frac{5}{2}$.

The reduction of the Einstein equations

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$$ \Box_g \gamma = 0, Ric_{ij}({}^{2+1}h) = \nabla_{ij}\gamma + \nabla_i\gamma\nabla_j\gamma. $$

Connection $K_{ij} = < \nabla_{e^i} e^0, e^j >, A_{ij,k} = < \nabla_{e^i} e^j, e^k >$ of ${}^{2+1}h$ given by Connection of ${}^{3+1}g$ and γ.

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In 2+1 dim's $R_{ijkl} = Ric \otimes g + W_{ijkl}$. But $W_{ijkl} = 0$.

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$$\begin{aligned} K_{11} &= \beta(t,\theta) r^{-3/2} + O(r^{-1/2}) + \bar{y}(t,\theta) r^{\epsilon(t,\theta)}, \\ K_{22} &= \delta(t,\theta) r^{-3/2} + O(r^{-1/2}), K_{12} = O(r^1). \end{aligned} \tag{4}$$

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$d_1(\alpha(t,\theta)), d_2(\alpha(t,\theta))$ are explicit and it turns out:

$$trK(r=\rho) = \frac{3}{2}\rho^{-3/2} + O(\rho^{-1/2}).$$

Math: Energy estimates, Gauge choice, singular branch.

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admits *free branch* like $r^{\epsilon(t,\theta)}$ (consistent with *undifferentiated Ricatti*).

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admits free branch $r^{-3+\epsilon(t,\theta)}$. **If** this is present then *no* possibility of deriving any estimates for the system. (In the iteration estimates would be getting exponentially worse at each step).