From Gauge Theory To Khovanov Homology Via Floer Theory

Edward Witten, IAS

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A number of years later, I re-expressed this type of construction in terms of gauge theory and the counting of solutions of PDE's (see "Fivebranes and Knots," arXiv:1101.3216). That is the story I will describe today. A number of my previous lectures are available online (see arXiv:1603.03854, arXiv:1401.6996, arXiv:1108.3103) and what I will explain here is closest to the most recent of those.

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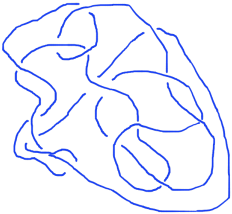

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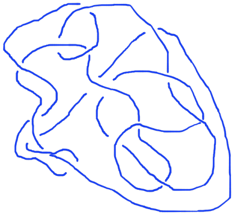

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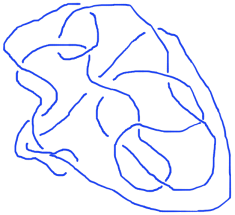

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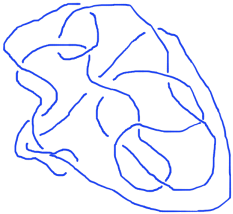

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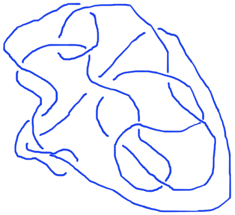

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In the simplest version, the invariants will be obtained by simply counting, with signs, the solutions of an equation. The solutions will have an integer-valued topological invariant P and if a_n is the "number" (counted algebraically) of solutions with $P = n$, then the Jones polynomial of the knot will be

$$J(q) = \sum_n a_n q^n.$$

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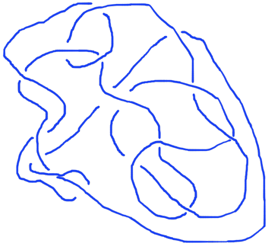
Going all the way back to the original work of Vaughn Jones in 1983, most approaches to the Jones polynomial define an invariant in terms of some sort of presentation of a knot, for example a projection to a plane



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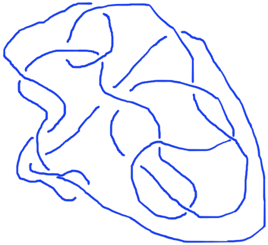

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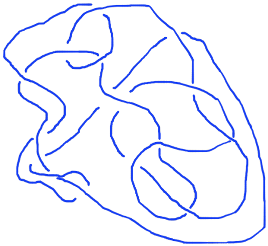

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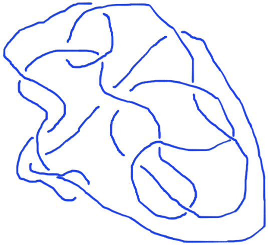

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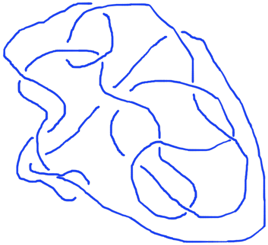

or some other “knot presentation.” One defines something that is manifestly well-defined once such a presentation is given. What one defines is not obviously independent of the knot presentation, but turns out to be. That step is where the magic is. And there always is some magic.

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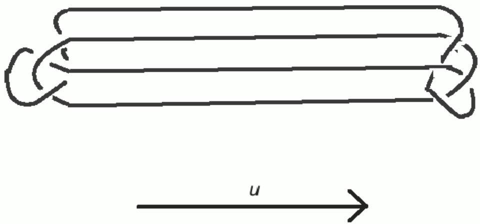

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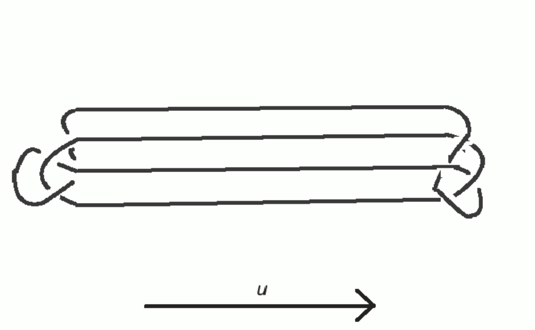

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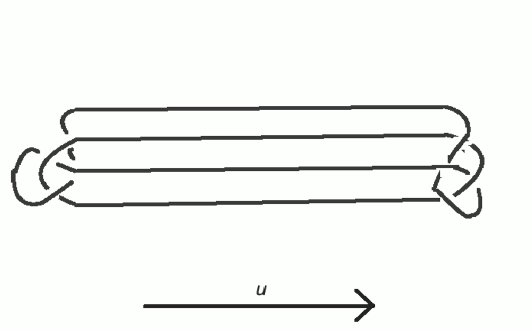

Then one wants it to be the case that except near the ends, the solutions are independent of u. (This is not automatically the case and we had to make a perturbation to get to a situation in which this would be true.)

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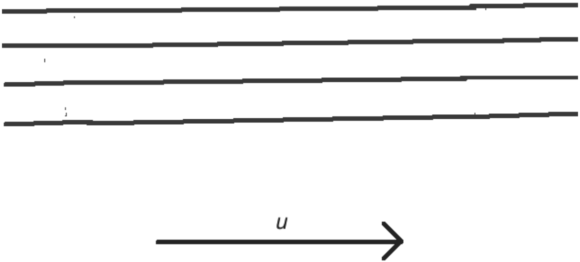

In $\mathcal{M}$, we define two "subspaces" $\mathcal{L}_\ell$ and $\mathcal{L}_r$ consisting of solutions that extend over the left or over the right. (For simplicity in my terminology, I will assume a given solution extends in at most one way, but this assumption is not necessary.)

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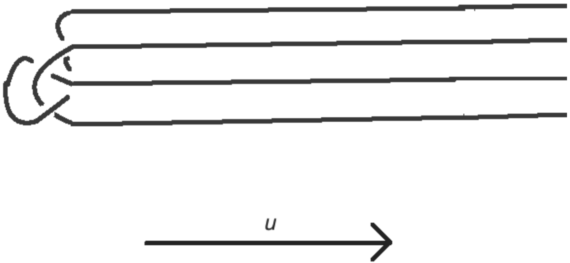

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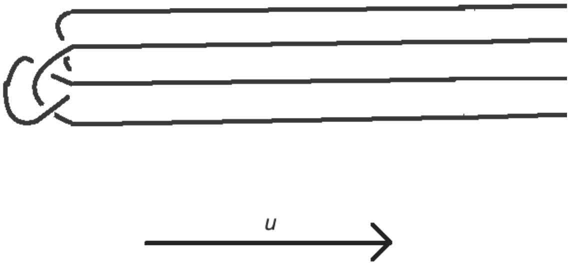
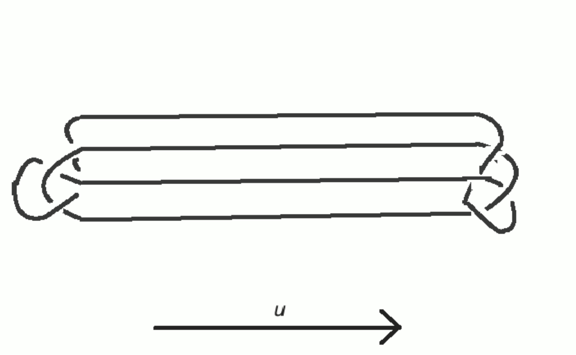


Likewise $\mathcal{L}_r$ parametrizes solutions that extend over the right end.

For a global knot with the strands ending on both ends



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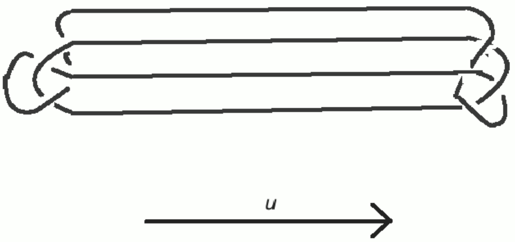

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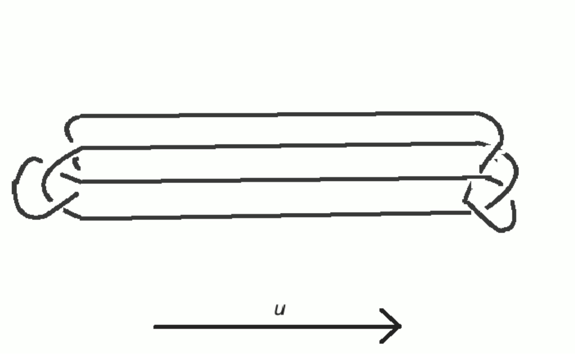

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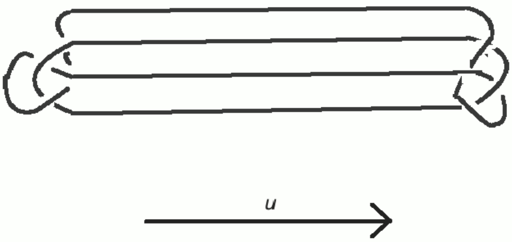

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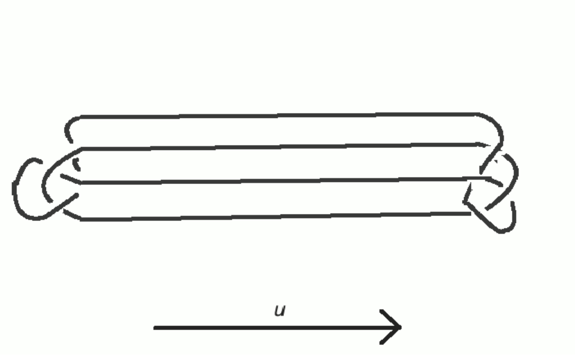

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(To be more exact, a_n is this intersection number computed by counting only intersections with $P = n$.)

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$$\varphi : E'|_{C \backslash p} \cong E|_{C \backslash p}.$$

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$$\mathcal{L}' = \mathcal{L}(np) = \mathcal{L} \otimes \mathcal{O}(p)^n$$

for some integer n. Here the integer n can be thought of as a weight of the Langlands-GNO dual group of $\mathbb{C}^*$, which is another copy of $\mathbb{C}^*$.

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Because of the dependence on the decomposition of E, or more accurately on the choice of a subbundle $\mathcal{O} \subset E$ that is going to be replaced by $\mathcal{O}(p)$, the Hecke modifications of this type at p form a family, parametrized by $\mathbb{CP}^1$.

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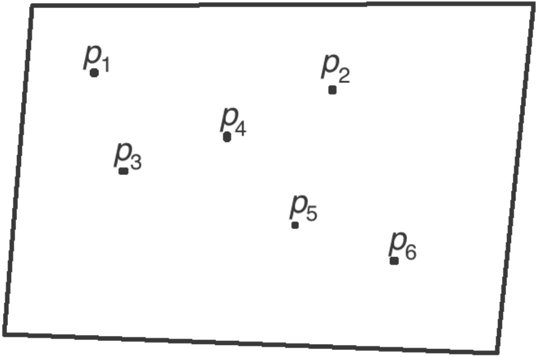

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Symplectic Khovanov homology is constructed by considering intersections of Lagrangian submanifolds of the space I just described – the space $\mathcal{M}$ of multiple Hecke modifications from a trivial bundle to itself.

We want to reinterpret this in terms of gauge theory PDE's.

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$$F = \star \mathrm{d}_A \phi.$$

($\star$ is the Hodge star and d_A is the gauge-covariant extension of the exterior derivative.)

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$$\left[\frac{D}{Dy} - \mathrm{i}\phi, \bar{\partial}_A \right] = 0.$$

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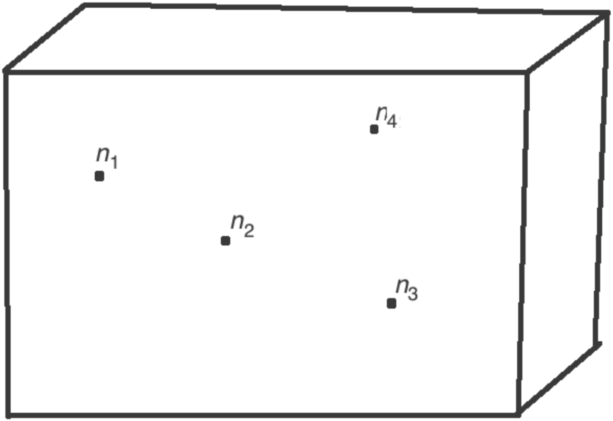
I have only defined F and not the connection A whose curvature is F or the line bundle $\mathcal{L}$ on which A is connection, but such an $\mathcal{L}$ and A exist (and are essentially unique) if $n \in \mathbb{Z}$.

For $G = U(1)$, since the Bogomolny equations are linear, they have a unique solution with singularities labeled by specified integers $n_1, n_2, \ldots$ at specified points in $\mathbb{R}^3$:



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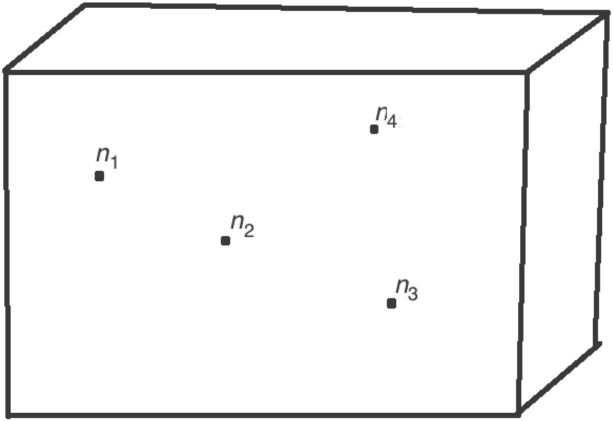
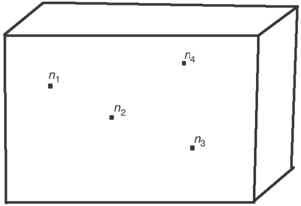


We assume that $\sum_i n_i = 0$, which ensures that the given solution vanishes at infinity faster than $1/|\vec{x}|$.

We pick a decomposition $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$ where we identify $\mathbb{R}^2$ as $\mathbb{C}$. Suppose that the singularities are at $y_i \times p_i$, with $y_i \in \mathbb{R}$, $p_i \in \mathbb{C}$:



For each $y \notin \{y_1, \ldots, y_n\}$, the indicated solution of the Bogomolny equations determines a holomorphic line bundle $\mathcal{L}_y \to \mathbb{C}$, and this naturally extends to $\mathcal{L}_y \to \mathbb{CP}^1$.

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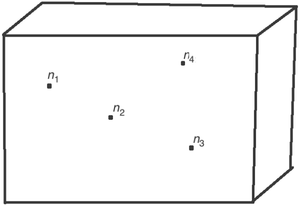

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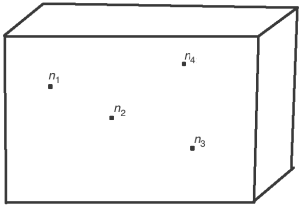

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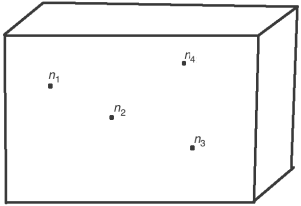

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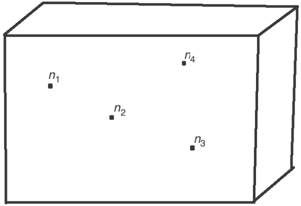

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Then we look for solutions of the Bogomolny equations for G with singularities of this type at specified points $y_i \times p_i \in \mathbb{R}^3$.

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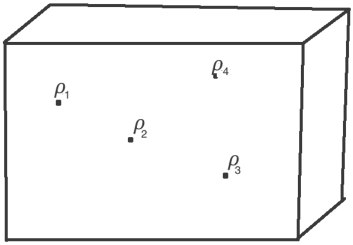

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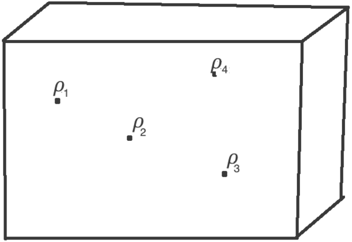

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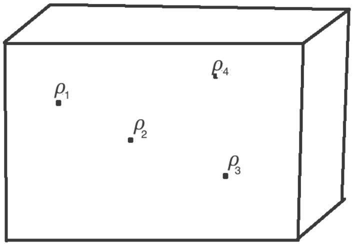

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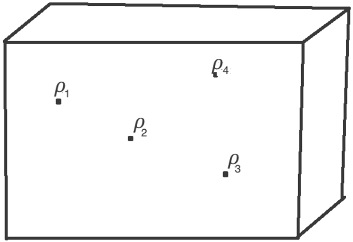

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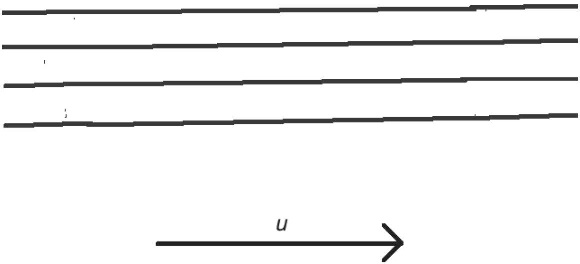
The moduli space $\mathcal{M}$ of solutions of the Bogomolny equations on $\mathbb{R}^3$ with the indicated singularities and vanishing at infinity faster than $1/r$ is actually a hyper-Kahler manifold, essentially first studied by P. Kronheimer in the 1980's.

The moduli space $\mathcal{M}$ of solutions of the Bogomolny equations on $\mathbb{R}^3$ with the indicated singularities and vanishing at infinity faster than $1/r$ is actually a hyper-Kahler manifold, essentially first studied by P. Kronheimer in the 1980's. If we pick a decomposition $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$, this picks one of the complex structures on the hyper-Kahler manifold and in that complex structure, $\mathcal{M}$ is the moduli space $\mathcal{M}_{p_1,\rho_1;p_2,\rho_2;\dots}$ of all Hecke modifications of the indicated types at the indicated points, mapping a trivial bundle over $\mathbb{CP}^1$ to itself.

This construction can be used to account for a number of properties of spaces of Hecke modifications, but for today we want to focus on the fact that for application to knot theory, we want $\mathcal{M}$ to be the space of u-independent solutions of some equations:



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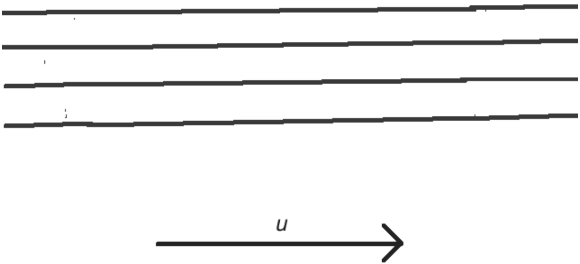

We already described $\mathcal{M}$ via solutions of some PDE's on $\mathbb{R}^3$, so now we have to think of $\mathcal{M}$ as a space of u-independent solutions on $\mathbb{R}^4 = \mathbb{R}^3 \times \mathbb{R}$, where the second factor is parametrized by u.

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In a special case $Y_4 = W_3 \times \mathbb{R}$, with A a pullback from W_3 and $\phi = \upphi \, \mathrm{d}u$ (where $\upphi$ is a section of $\mathrm{ad}(E)$ and u parametrizes the second factor in Y_4) these equations reduce to the Bogomolny equations:

$$F = \star \mathrm{d}_A \upphi.$$

Therefore, the singular solution of the Bogomolny equations that we have already studied can be embedded as a singular solution of the KW equation, but now the singularity is along a line rather than a point.

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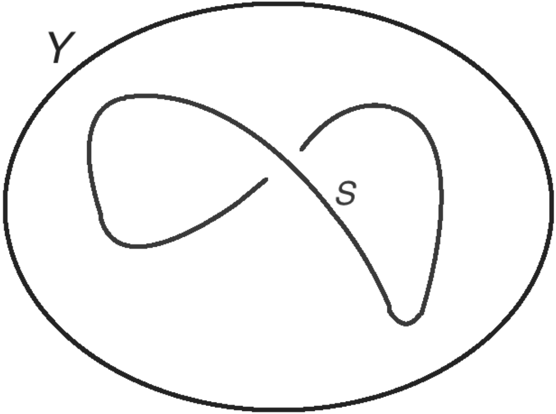
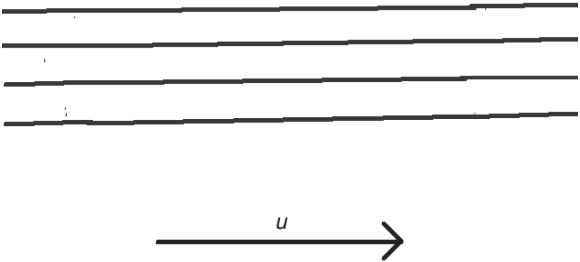


If we specialize to the case that $Y_4 = W_3 \times \mathbb{R}$, with $S = \cup_i S_i$, and $S_i = q_i \times \mathbb{R} \subset W_3 \times \mathbb{R}$ (q_i are points in $\mathbb{R}^3$)



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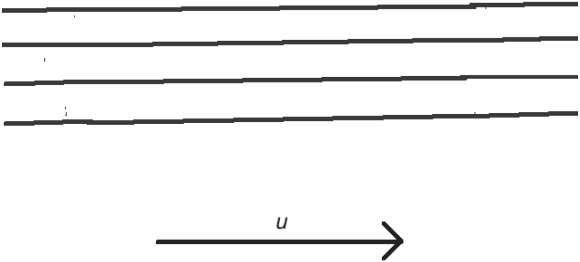

then the u-independent solutions of the KW equations are parametrized by $\mathcal{M}$; and indeed one can show that these are all solutions of the KW equations in this situation with reasonable behavior at infinity.

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So we have an elliptic PDE in four dimensions and we can specify in an interesting way what sort of singularity it should have on an embedded circle $S \subset Y_4$. But this sounds like a ridiculous framework for knot theory, because there is no knottedness of a 1-manifold in a 4-manifold!

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A couple of things are missing from what I have said so far. There are a few directions that we could go next but I think I will head for categorification, which will also resolve the point I just mentioned.

Let us practice with an ordinary equation rather than a partial differential equation.

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Concretely the differential is defined by

$$\mathrm{d}\psi_p = \sum_q n_{pq}\psi_q$$

where the sum runs over all critical points q whose Morse index exceeds by 1 that of p, and the integer n_{pq} is defined by counting flows from p to q:

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where the sum runs over all critical points q whose Morse index exceeds by 1 that of p, and the integer n_{pq} is defined by counting flows from p to q:

A "flow" is a solution fo the gradient flow equation

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = -\vec{\nabla} h.$$

(To define this equation, one has to pick a Riemannian metric on the manifold N. The complex that one gets is independent of the metric up to quasi-isomorphism. What one actually counts are 1-parameter families of flow, related by time translations.)

This tells us what we need in order to be able to categorify a problem of counting solutions of the KW equations:

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And the associated gradient flow equation, which will be a PDE in 5 dimensions on $X_5 = \mathbb{R} \times Y_4$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = -\frac{\delta \Gamma}{\delta A}, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}t} = -\frac{\delta \Gamma}{\delta \phi}$$

has to be elliptic, so that it will makes sense to try to count its solutions.

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Naively, this leads to “categorified” knot invariants for any three-manifold W_3, but to justify this claim one needs some compactness properties for solutions of the equations under consideration.

Naively, this leads to “categorified” knot invariants for any three-manifold W_3, but to justify this claim one needs some compactness properties for solutions of the equations under consideration. It is not really clear what conditions on W_3 will make the analysis work, and it is conceivable that a proper proof of the compactness properties requires that the Ricci tensor of W_3 is nonnegative, a very restrictive condition.

What I have described so far is supposed to correspond (for $W_3 = \mathbb{R}^3$, $G = PGL(2)$ and ρ corresponding to the 2-dimensional representation of $G^\vee = SL(2)$) to "singly-graded Khovanov homology."

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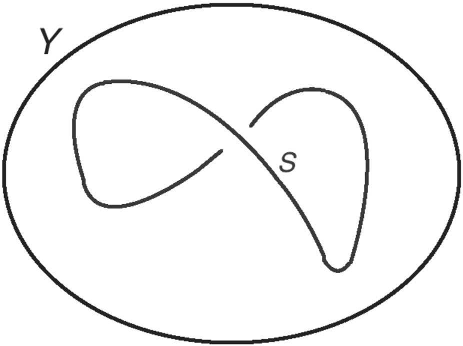
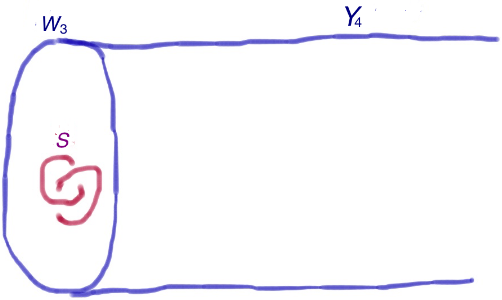


The physical picture tells us what we have to do to get the q-grading: Y_4 should be a manifold with boundary, with the knot placed in its boundary:



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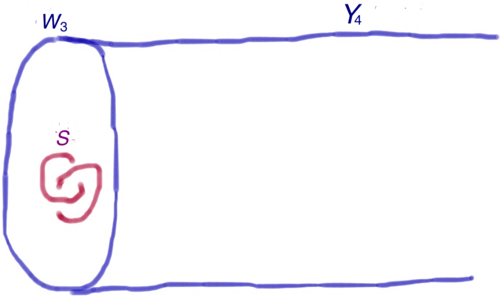

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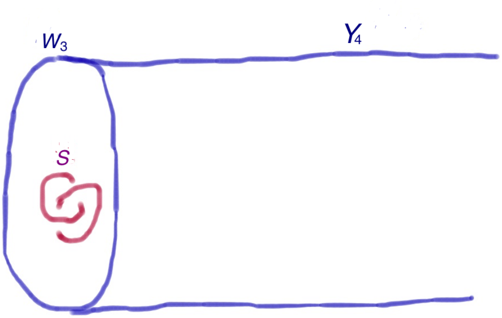

The boundary condition is a subtle one that was described this morning in Rafe Mazzeo's lecture. It has the property that the bundle is trivialized on the boundary, so the second Chern class can be defined.

In work I cited earlier, Gaiotto and I analyzed this situation (in the uncategorified situation, meaning that we counted solutions in 4 dimensions, not 5) with the aim of showing directly, without referring to the physical picture, that the Jones polynomial is

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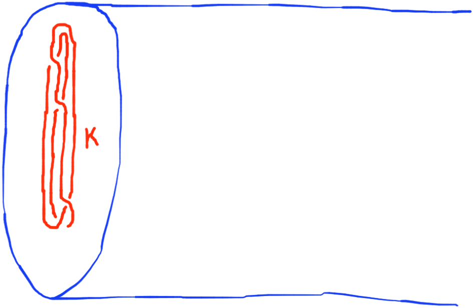
where a_n is the number of solutions with second Chern class n. As usual, the starting point was to stretch the knot in one direction, reducing to equations in one dimension less:



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