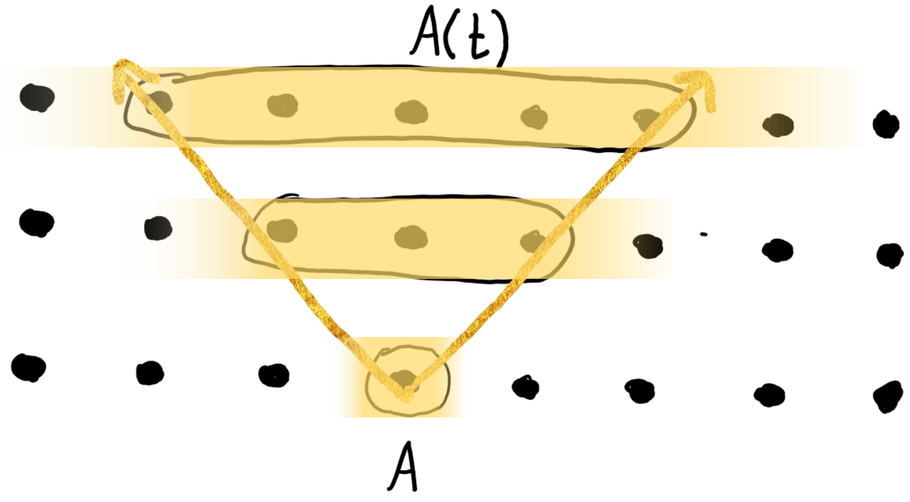
QCAs and approximate locality

IPAM summer school, 2021

Speaker: Daniel Ranard @ MIT



Please ask questions!

Ex. 1: “What’s the Heisenberg picture?”

Ex. 2: “What’s the operator norm?”

Goals

- Review QCAs
- Convince you to think about approximate locality
- Learn a little operator algebra
- Understand index theory of approximate QCAs in 1D
- Discuss open problems (help!)

References

1. Classification of 1D QCAs: *Index theory of one dimensional quantum walks and cellular automata*, GNVW
2. Classification of 1D approx. QCAs: *A converse to Lieb-Robinson bounds in one dimension using index theory*, Michael Walter, Freek Witteveen, DR

University of Amsterdam

3. Operator algebra tools: *Perturbations of operator algebras…,* Erik Christensen

Quick summary

- Physics question: Are local dynamics generated by local Hamiltonians? Find converse to Lieb-Robinson bounds.
- Mathematical physics question: Classify approximate QCAs.
- Results so far: Classification of 1D *approximate* QCAs resembles the classification for the *exact* case.
- Open questions: Classify approximate QCAs in high dimensions?

Setup: Quantum lattice systems

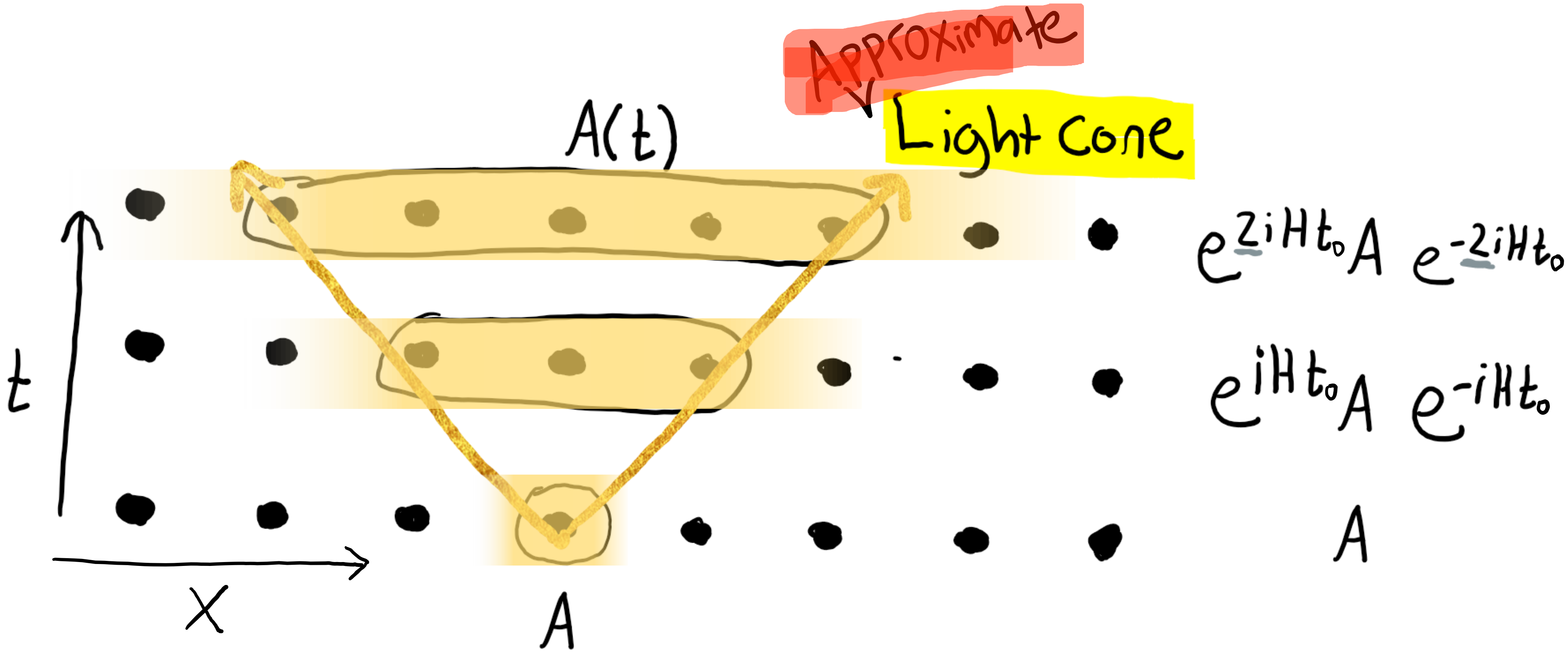
$$\mathcal{H} = \bigotimes_i \mathcal{H}_i$$ Hilbert space for lattice system.

$$\mathcal{H}_i = \mathbb{C}^d$$ Local Hilbert space on each site.

$$H = \sum_{\langle i,j \rangle} h_{ij}$$ Local Hamiltonian, e.g. nearest neighbor interactions.

$$A = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes A_i \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}$$ Operator local to site *i.*

$$\mathrm{support}(A) = \{i\}$$ “Support” of an operator.



Lieb-Robinson: Local Hamiltonian evolution obeys approximate lightcone.

Quantum lattice systems

Local interactions

Lattice Hamiltonian with short-range interactions.

Lieb and Robinson (1972)
Review: Hastings, arXiv:1008.5137

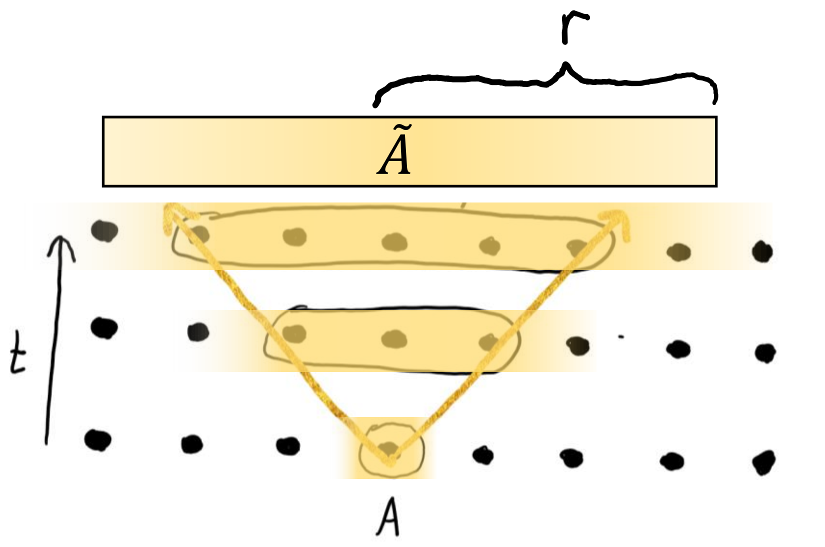
? ? ? ?

Present discussion.

Approximate light cone

Unitary evolution with an approximate lightcone, satisfying "Lieb-Robinson bounds."

Lieb-Robinson bound



If r bigger than vt, then $\tilde{A}$ is good truncated approximation to $A(t)$.

Consider a local lattice Hamiltonian with short-range interactions.

Lieb-Robinson bound (1972): A

Given H, there exist constants v ("L.R. velocity") and $c_1, c_2 > 0$
such that for any operator A,
for any time t and distance r,
there exists "truncated" operator $\tilde{A}$ local to $\mathrm{Ball_r}(\mathrm{supp}(A))$ with

$$\left\| A(t) - \widetilde{A} \right\| \leq \|A\| \, c_1 e^{-c_2 |r - vt|}.$$

(Strict) QCAs are not enough

Traditional QCA maps local operators to local operators.

For local Hamiltonian H and time t, consider map

$$X \mapsto e^{-iHt} X e^{iHt}.$$

Note *quite* a QCA: For X local, $e^{-iHt} X e^{iHt}$ is only *approximately* local.

Want relaxed notion of QCA to accommodate this approximate locality.

Operator algebras, QCAs, and approximate QCAs

Operator algebra review

A *C*-algebra* $\mathcal{A}$ on a Hilbert space H is a subset of linear operators on H,

$$\mathcal{A} \subset \mathrm{End}(H)$$

that is

- Linearly + multiplicatively closed
- Closed under adjoint (i.e. "star", "dagger")

In infinite dimension, also require:

- Bounded in operator norm, + complete w.r.t. norm

Can also define C*-algebra abstractly, without reference to operators acting on a Hilbert space. Just a Banach algebra with an involution.

Operator algebra review

Morphisms on C*-algebras given by

$$f : \mathcal{A} \mapsto \mathcal{B}$$

that are

- Linear
- Multiplicative homomorphism
- $f(x^*) = f(x)^*$

We'll study automorphisms.

Unitary operator vs. algebra automorphism

finite dim.

1. Evolution as unitary operator on Hilbert space,

$$U : \mathcal{H} \to \mathcal{H}$$

induces automorphism of algebra of operators of Hilbert space:

always

$$\alpha(X) = UXU^{-1}$$

2. Evolution as automorphism of algebra of operators:

$$\alpha : \mathcal{A} \to \mathcal{A}$$

Unitary always induces automorphism.

Automorphism always arises from unitary conjugation, in fin. dim.

Prefer algebra perspective!

QCA Review

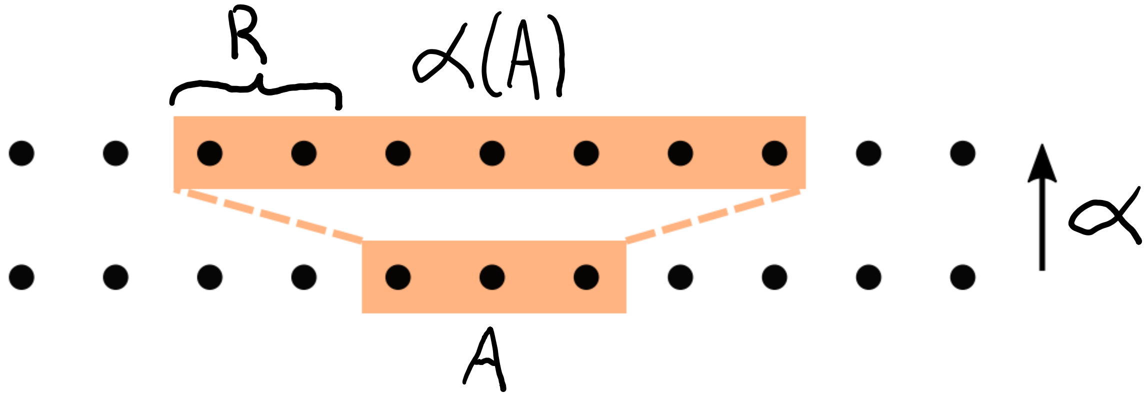
Algebra of operators on an infinite 1D lattice:

$\mathcal{A} = \bigotimes_{i\in\mathbb{Z}} \mathcal{A}_i \ , \quad \mathcal{A}_i = \mathrm{Mat_d}\,(\mathbb{C}).$

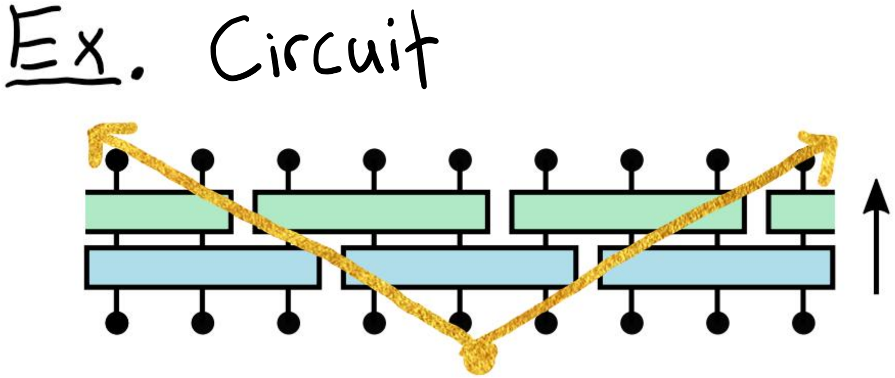
An automorphism $\alpha : \mathcal{A} \to \mathcal{A}$ is called a **quantum cellular automaton (QCA)** or **locality-preserving unitary (LPU)** if it preserves locality, i.e. there exists finite radius $R > 0$ such that

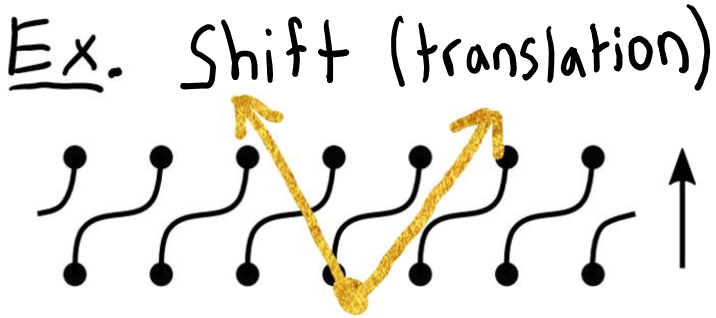
$$\mathrm{supp}(\alpha(A)) \subseteq \mathrm{Ball_R}\big(\mathrm{supp(A)}\big)$$

for all operators $A \in \mathcal{A}$.



Classification of 1D QCAs





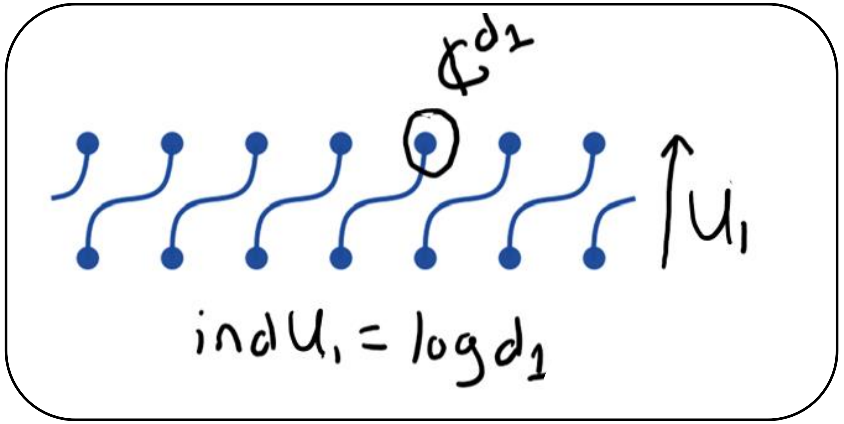
Gross, Nesme, Vogts, and Werner ("GNVW," 2009):

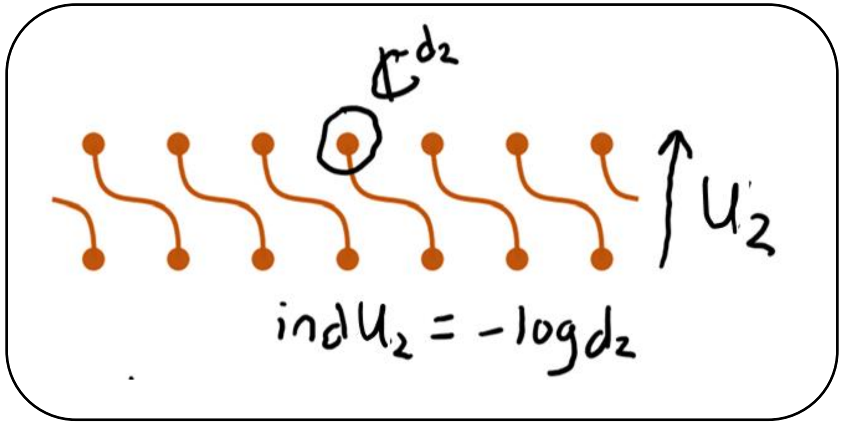
- All 1D QCAs are compositions of circuits and shifts.
- Shifts cannot be achieved by circuits.
- QCAs modulo circuits characterized by a quantized *index* (GNVW index).

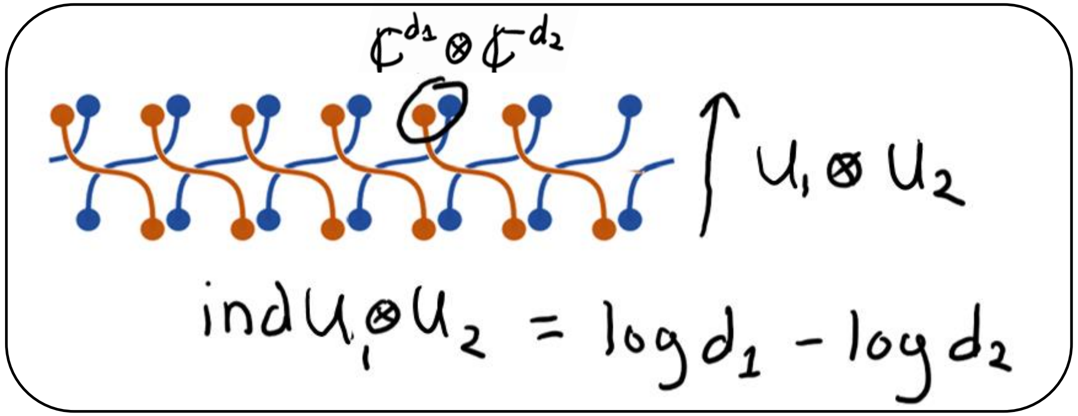
Classification of 1D QCAs

$$\mathrm{ind(U)} \equiv [\mathit{amount\ of\ quantum\ information\ flowing\ right}] - [\mathit{amount\ flowing\ left}]$$

How to define the index for shift QCAs:

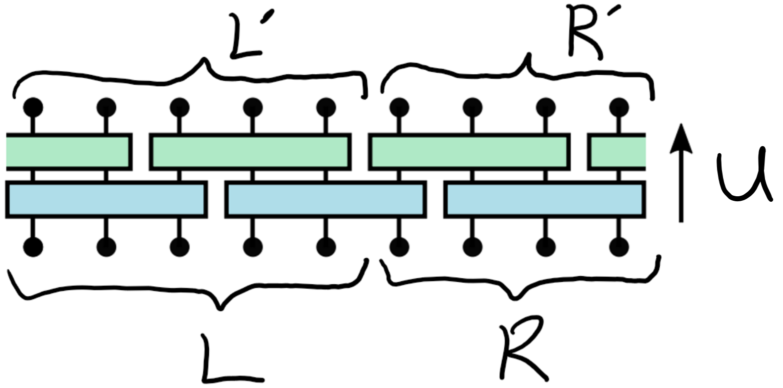






(Re-)defining the GNVW index

(Doesn't need to be circuit)



$$\boxed{\operatorname{ind} U \equiv \frac{1}{2}\left(I(L,R') - I(L';R)\right)}$$

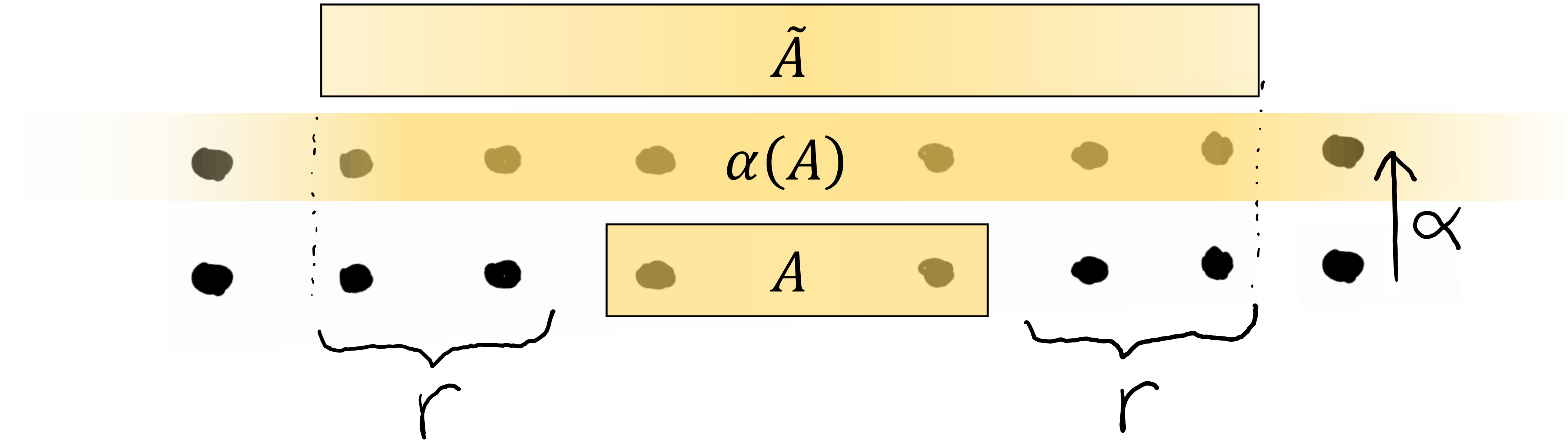
i.e. mutual informations in Choi state of unitary channel

$$\operatorname{ind} U \in \mathbb{Z}[\{\log p_i\}_i]$$

Prime factors in local dimensions

integer linear combos

ALPU: "Approximately locality-preserving unitary"



"α is an ALPU with $f(r)$-tails" means:

Can always find $\tilde{A}$ s.t. $\|\alpha(A)-\tilde{A}\| \leq f(r)$.

(for $\|A\|=1$)

Helpful notation + concept:

$A \subseteq_{\varepsilon} B$ "ε-near inclusion"

means:

$$\forall a \in A, \ \exists\, b \in B \text{ s.t. } \|a-b\| \leq \varepsilon \|a\|.$$

ALPU: "Approximately locality-preserving unitary"

An automorphism $\alpha : \mathcal{A} \to \mathcal{A}$ on a 1D lattice is an **ALPU with "$\boldsymbol{f(r)}$-tails"** if there exists a tail function

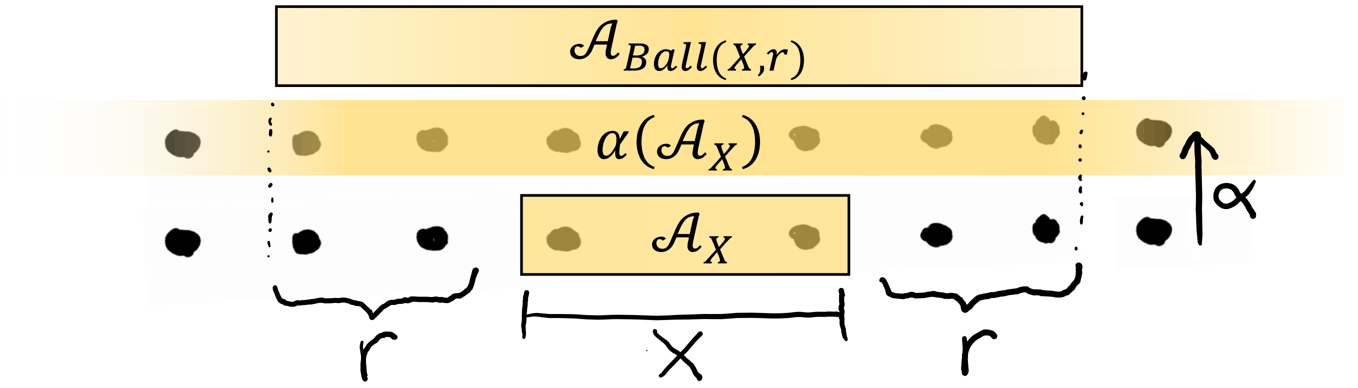
$$f : \mathbb{N} \to \mathbb{R}$$

that decays to zero, such that for all intervals $X \subset \mathbb{Z}$, for all $r > 0$,

$$\alpha(\mathcal{A}_X) \underset{\epsilon}{\subseteq} \mathcal{A}_{\mathrm{Ball}(X,r)} \qquad \epsilon = f(r)$$

Ex. 1: QCAs

Ex. 2: Local Hamiltonian evolution



Classification and index theory of ALPUs

Want to classify space of ALPUs $\{\alpha : \mathcal{A} \to \mathcal{A}\}$ modulo local Hamiltonian evolution.

- Why not obvious?
  - Previous classification (GNVW) only treats exact QCAs
  - GNVW index not obviously robust
  - GNVW proof uses algebra, appears very “brittle” when you add noise
  - Hamiltonian evolutions are *not* (exactly) circuits
  - Many examples where “exact” cases *don’t* generalize to “approximate cases”
  - Should index theory still give *quantized* index?

- Why do we care?
  - Hamiltonian evolutions are ALPUs, not QCAs!
  - Converse to Lieb-Robinson bounds?
  - Existence of lattice momentum density?
  - Stability of chiral many-body localized floquet phases? (cf. Lukasz Fidkowski’s talk).

Classification and index theory of 1D ALPUs

It turns out…

Classification of 1D ALPUs modulo evolution time-dependent, quasi-local Hamiltonians

is very similar to…

Classification of 1D QCAs modulo local circuits
(GNVW, 2009)

In particular:

- ALPUs characterized by quantized index
- Every ALPU can be approximated by a sequence of QCAs
- Every ALPU is composition of time-dep. local Hamiltonian evolution with a shift.
- For finite open chain, every ALPU given by local time-dep. Hamiltonian evolution.
- Translations *cannot* be well-approximated by quasi-local Hamiltonian evolution for finite time. (No momentum density on the lattice!)

Proof technique

ALPUs: (Approximately locality-preserving unitaries)

$\alpha: \mathcal{A} \to \mathcal{A}$ automorphism

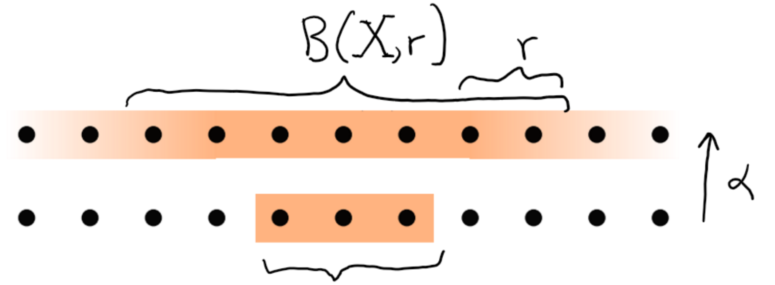
α is an ALPU with "$f(r)$-tails" whenever, $\forall$ intervals $X \subset \mathbb{Z}$, $\forall r$,

$$\alpha(\mathcal{A}_X) \overset{f(r)}{\subset} \mathcal{A}_{B(X,r)}$$

} Ball of radius r around X

"Near inclusion" with error $f(r)$

$$\boxed{\begin{array}{l} f: \mathbb{N} \to \mathbb{R}^+ \\ \lim_{r\to\infty} f(r) = 0 \end{array}}$$



$X \subset \mathbb{Z}$, algebra $\mathcal{A}_X$

Near inclusions of algebras:

operator algebra

$A \overset{\varepsilon}{\subset} B$ if $\forall a \in A$, $\exists\, b \in B$ s.t.
$\|a - b\| \leq \varepsilon \|a\|$.

Perturbing (via small rotation) near inclusions to make them exact incusions:

Theorem (Christensen, 1980):

For algebras $A, B \subset B(\mathcal{H})$,

if $A \overset{\varepsilon}{\subset} B$ for some $\varepsilon < 1/8$

then $\exists$ unitary $u \in B(\mathcal{H})$ s.t.

$uAu^* \subset B$, $\|u - \mathbb{1}\| \leq 12\varepsilon$.

Also: $u \in \underbrace{\langle\{A, B\}\rangle}_{\text{Alg. gen. by } A,\, B}$.

Application: Strictly localize image of a region.

$$\tilde{\alpha}(\cdot) \equiv u\alpha(\cdot)u^*$$

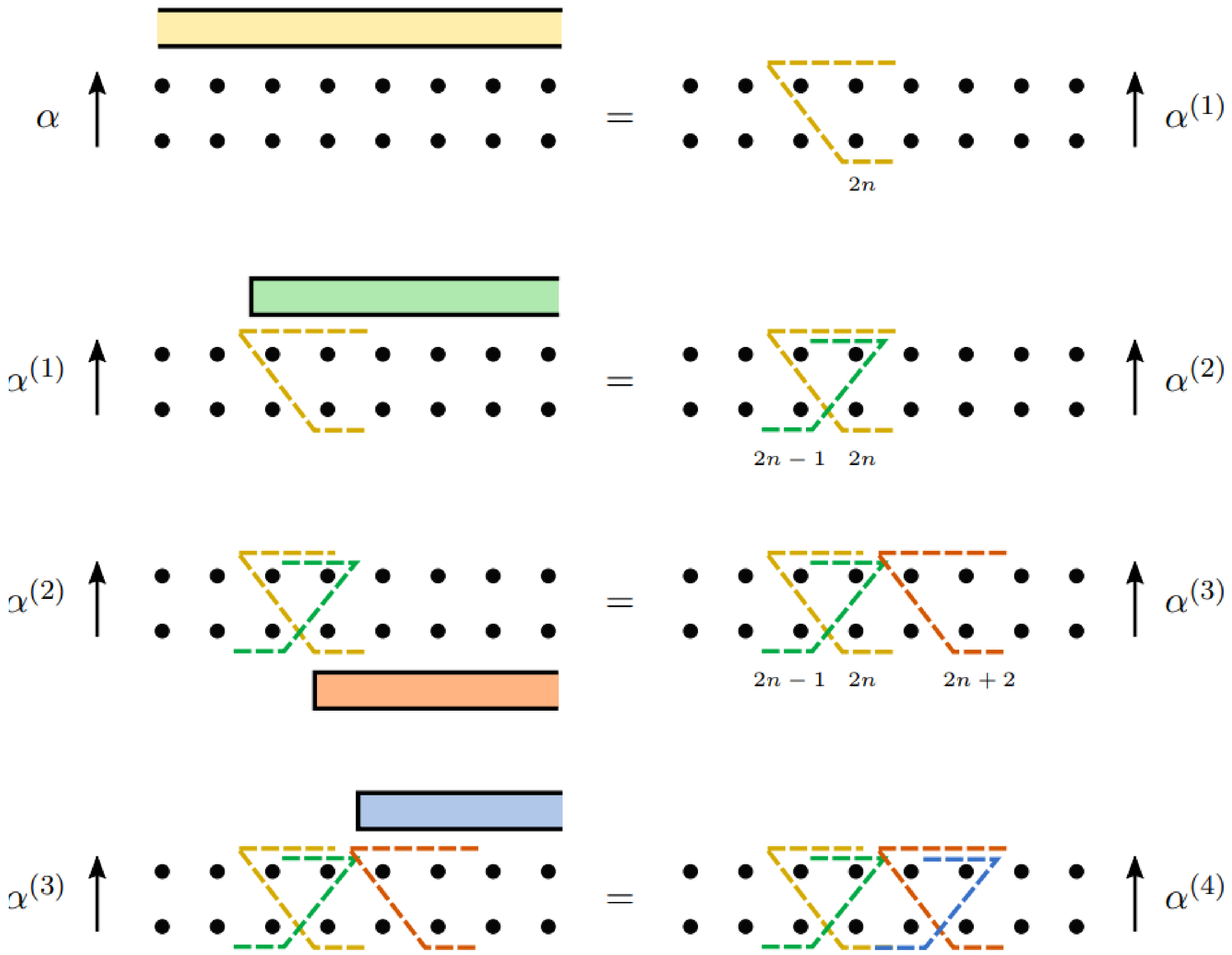
$$\alpha(A_X) \overset{f(r)}{\subset} A_{B(X,r)} \rightsquigarrow \tilde{\alpha}(A_X) \subset A_{B(X,r)}$$

We use these tools to

$\boxed{\text{Approximate any ALPU with a QCA.}}$

Then find a sequence of QCAs β_j of radius j s.t. $\beta_j \xrightarrow[j\to\infty]{} \alpha$.

Approximate any ALPU with QCA:

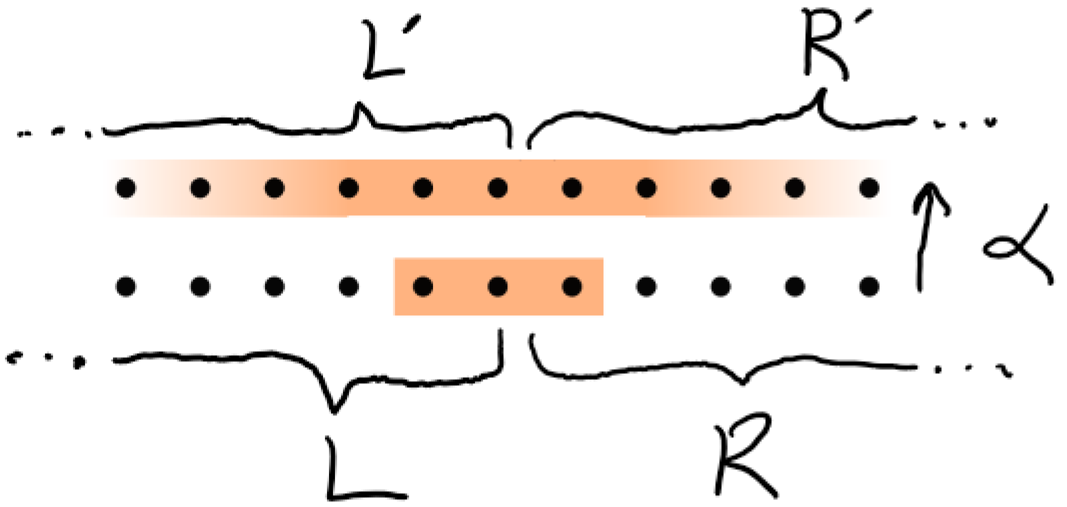


Allows one to prove the <u>index for ALPUs</u>

$$\boxed{\operatorname{ind}\alpha \equiv \frac{1}{2}\left(I(L,R') - I(L';R)\right)}$$

i.e. mutual informations in Choi state of unitary channel

is well-defined and quantized for L, R = half-chains.



We find similar structure theory as for QCAs but with quasilocal Ham. evolution replacing circuits.

time-dependent!

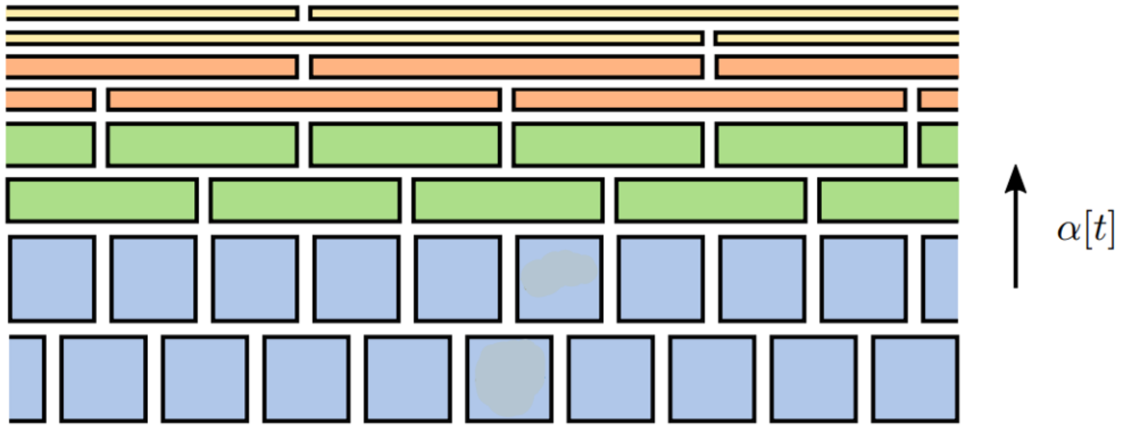
Theorem (ALPU index theory)

- $\operatorname{ind}\alpha = \operatorname{ind}\beta \iff \alpha = \overset{\text{loc. Ham. evol}}{\gamma} \circ \beta \iff$ α and β can be "blended"

- $\operatorname{ind}\alpha\circ\beta = \operatorname{ind}\alpha + \operatorname{ind}\beta$, $\operatorname{ind}\left(\text{loc. Ham. evol}\right) = 0$.
- $\operatorname{ind}\alpha\otimes\beta = \operatorname{ind}\alpha + \operatorname{ind}\beta$
- $\operatorname{ind}\alpha$ continuous w.r.t. α

$\boxed{\bullet\ \alpha = (\text{Loc. Ham. evol}) \circ \text{Shift}}$ Structure theorem for ALPUs.

What do the quasilocal time-dependent Hamiltonians look like?



- Time-dep. Ham.
- Evolves for unit time
- $H(t)$ is piecewise-constant
- $H(t)$ at fixed t has geometrically local range-k interactions

depends on t

$$H = \sum_{\substack{X \subset \mathbb{Z} \\ |X| = k}} H_X, \quad \|H_X\| = \mathcal{O}(f(k) \log k)$$

Open questions

- Classify ALPUs in higher dimensions
- Other applications for perturbing algebras?
- Prove some conjectures about “approximate” algebras, Ulam stability
- Formulate in QFT setting
  - Anomalies
  - Given abstract charge, find local current?
  - “Splittable” symmetries

Thank you!

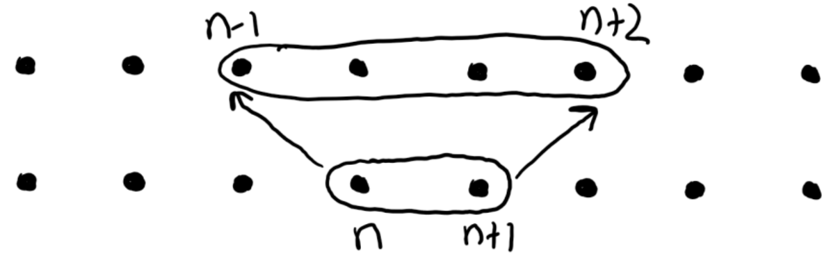
Supplementary notes

Theorem (GNVW)

- $\operatorname{ind}\alpha = \operatorname{ind}\beta \iff \alpha = \overset{\text{circuit}}{\gamma}\circ\beta \iff$ α and β can be "blended,"

- $\operatorname{ind}\alpha\circ\beta = \operatorname{ind}\alpha + \operatorname{ind}\beta$, $\operatorname{ind}(\text{circuit}) = 0$
- $\operatorname{ind}\alpha\otimes\beta = \operatorname{ind}\alpha + \operatorname{ind}\beta$
- $\operatorname{ind}\alpha$ continuous w.r.t. α
- $\boxed{\alpha = \text{Circuit}\circ\text{Shift}}$ Structure theorem for QCAs

Factorization result for QCAs.



$\uparrow \alpha$ $\quad \alpha(A_n \otimes A_{n+1}) \subset A_{n-1} \otimes \cdots \otimes A_{n+2}$

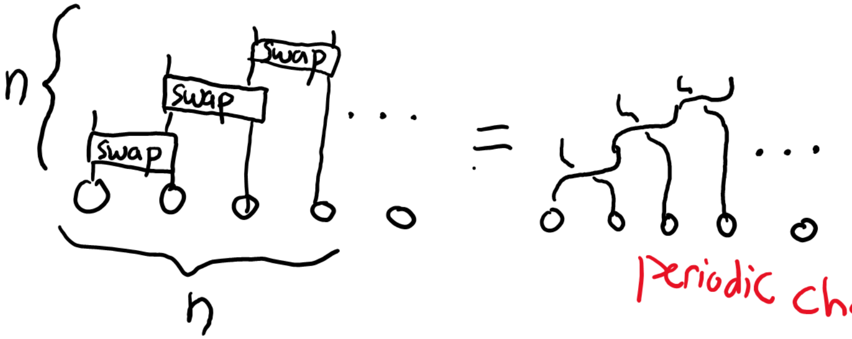
$$\mathcal{L} = \alpha^{-1}\big(\alpha(A_n \otimes A_{n+1}) \cap A_{n-1} \otimes A_n\big)$$

$$\mathcal{R} = \alpha^{-1}\big(\alpha(A_n \otimes A_{n+1}) \cap A_{n+1} \otimes A_{n+2}\big)$$

$$\boxed{A_n \otimes A_{n+1} = \mathcal{L} \otimes \mathcal{R}}$$ Factorization

Can a (constant-depth) circuit implement a translation?

Consider the Shift QCA (i.e. translation) 

Try circuit of swaps? 

Works, but needs depth $\mathcal{O}(n)$

Periodic chain

Can we achieve w/ constant depth circuit?

What about using $U = e^{iHt}$ for $t = \mathcal{O}(1)$, H = local Ham.?

Like asking if $\exists$ local momentum density P on the lattice, $e^{iP} = \text{Shift}$.

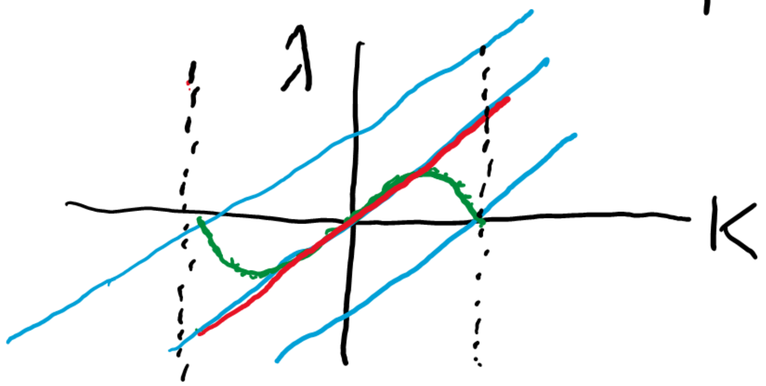
An aside: First quantized case: No local momentum op.

For first-quantized, discretized ring, $\mathcal{H} = \text{Span}\{|i\rangle\}_{i=1}^{L}$

Unitary shift operator T, $T^L = \mathbb{1}$

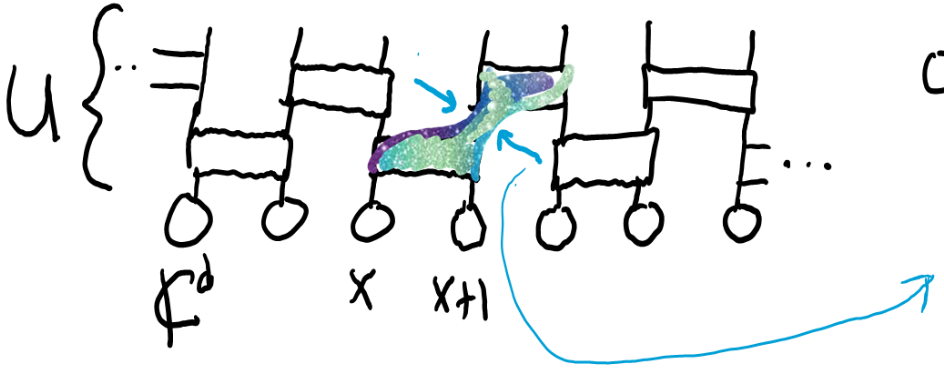
$$\text{Spec}(T) = \{e^{i\theta}\} \quad \theta = \frac{m}{L}, \quad m = 0, 1, \dots, L-1$$

Let $T = e^{iP}$ $\quad \text{Spec}(P) = \frac{m}{L} + 2\pi n, \; n \in \mathbb{Z}.$



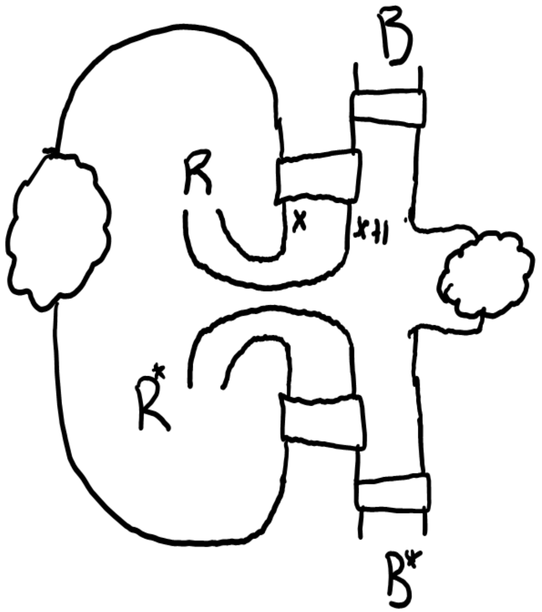
If p is local operator on ring, green curve must be smooth.

Information bottleneck argument (that constant depth circuit cannot approximate a shift)



Obstruction to obtaining the shift:
There seems to be an information bottleneck.

To illustrate, act circuit on state max. entangled between x, x+1 and reference R.



$= \rho_{RB} \overset{?}{=}$ max. entangled, if $U = \text{Shift}$

$\hookrightarrow = \sum p_i |\psi_i^{RB}\rangle\langle\psi_i^{RB}|$, for rank $\psi_i^{RB} \le d$

$F(\rho_{RB}, \rho_{RB}^{max.ent.}) = \langle\varphi_{max.ent.}|\rho_{RB}|\varphi_{max.ent.}\rangle$

$= \sum_i p_i |\langle \psi_i^{RB} | \varphi_{max\,ent} \rangle|^2$

rank d rank d^2

$\le \sum_i p_i \frac{d}{d^2} = \frac{1}{d}$