Monotonicity methods
Diogo A. Gomes
Let $H$ be a Hilbert space. $A : D \subset H \rightarrow H$ is a monotone operator if

$$(A(w) - A(z), w - z) \geq 0, \quad \forall w, z \in D.$$ 

A variational inequality is the problem: find $w \in D$ such that

$$(A(w), z - w) \geq 0, \quad \forall z \in D.$$
Examples of monotone operators

- For $H = \mathbb{R}$, monotone operators are increasing functions
- Gradients of convex functions are monotone operators
Then, if $H(x, p)$ is convex in $p$ and $g$ is increasing, the operator

$$A \begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} -u - H(x, Du) + g(m) \\ m - \text{div}(D_p H m) - 1 \end{bmatrix}$$

monotone in its domain $D \subset L^2 \times L^2$. 
Then, if $H(x, p)$ is convex in $p$ and $g$ is increasing, the operator

$$A \begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} u_t - H(x, Du) + g(m) \\ m_t - \text{div}(D_p H m) - 1 \end{bmatrix}$$

is monotone in its domain $D \subset L^2 \times L^2$. 

**Time-dependent MFGs**
Often, monotonicity gives uniqueness. Given two solutions \((m, u)\) and \((\tilde{m}, \tilde{u})\), we have

\[
0 = \left( A \begin{bmatrix} m \\ u \end{bmatrix} - A \begin{bmatrix} \tilde{m} \\ \tilde{u} \end{bmatrix}, \begin{bmatrix} m \\ u \end{bmatrix} - \begin{bmatrix} \tilde{m} \\ \tilde{u} \end{bmatrix} \right) \geq 0.
\]
Example

For

\[
A \begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} u_t - \frac{u_x^2}{2} + m \\ m_t - (mu_x)_x \end{bmatrix},
\]

we get

\[
0 = \int_0^T \int (m + \tilde{m}) \frac{(u_x - \tilde{u}_x)^2}{2} + (m - \tilde{m})^2 \geq 0.
\]

This implies \( m = \tilde{m} \) and then, uniqueness of solution of

\[
-u_t + \frac{u_x^2}{2} = m
\]

gives \( u = \tilde{u} \).
Variational inequalities

If \( A : H \to H \) is monotone, then \( A(w) = 0 \) if and only if \( w \) satisfies

\[
(A(w), z - w) \geq 0, \quad \forall z \in H.
\]
$w$ is a weak solution of the variational inequality if

$$(A(z), z - w) \geq 0$$

for all $z \in D$. Solutions of the variational inequality are weak solutions. Under continuity assumptions and if $D$ is large enough, weak solutions are solutions.
If $H = \mathbb{R}$, a monotone operator, $A$, is an increasing function. If $A$ is continuous,

$$A(0) = 0$$

if and only if

$$A(z)(z - 0) = A(z)z \geq 0$$

for all $z$. 
Consider the MFG

\[
\begin{align*}
    u - \Delta u + H(x, Du) &= g(m) \\
    m - \Delta m - \text{div}(D_p H m) &= 1.
\end{align*}
\]

Then, if \( H(x, p) \) is convex in \( p \) and \( g \) is increasing, the operator

\[
A \begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} -u + \Delta u - H(x, Du) + g(m) \\ m - \Delta m - \text{div}(D_p H m) - 1 \end{bmatrix}
\]

is monotone in its domain \( D \subset L^2 \times L^2 \).
Weak solutions for monotone MFGs

Weak solutions

A weak solution of the MFG is a pair \((m, u), m \geq 0\), such that

\[
\left\langle \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m \\ u \end{bmatrix}, A \begin{bmatrix} \eta \\ v \end{bmatrix} \right\rangle_{\mathcal{D}'(T^d) \times \mathcal{D}'(T^d), \mathcal{C}^\infty(T^d) \times \mathcal{C}^\infty(T^d)} \geq 0
\]

for all \((\eta, v) \in \mathcal{C}^\infty(T^d; \mathbb{R}^+) \times \mathcal{C}^\infty(T^d)\).
Existence of weak solutions

**Main Theorem (Ferreira, G.)**

Under suitable but general Assumptions, there exists a weak solution, \((m, u) \in D'(\mathbb{T}^d) \times D'(\mathbb{T}^d), m \geq 0\), to the MFG

\[
A \begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

Moreover, \((m, u) \in M_{ac} \times W^{1,\gamma}\) for some \(\gamma > 1\) and \(\int_{\mathbb{T}^d} m \, dx = 1\).

**Scope**

First-order, second-order, degenerate elliptic, and congestion problems satisfying monotonicity conditions.
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Example

Theorem (Ferreira, G.)

Let $\kappa$ be a standard mollifier, $\alpha > 0$. Then, there exists a weak solution $u \in H^1$, $m \in L^{\alpha+1}$, $m \geq 0$ to

\[
\begin{align*}
&\quad u + \frac{|Du|^2}{2} + V(x) = m^\alpha + \kappa \ast m \\
&\quad m - \text{div}(mDu) = 1.
\end{align*}
\]

That is, for all $(\eta, v) \in C^\infty$, $\eta > 0$, we have

\[
\int (v + \frac{|Dv|^2}{2} + V(x) - \eta^\alpha - \kappa \ast m)(\eta - m) \\
+ \int (\eta - \text{div}(\eta Dv) - 1)(v - u) \geq 0.
\]
Example - further properties

Theorem (Ferreira, G.)

There exists a weak solution \((u, m)\) such that

\[
\begin{cases}
-u - \frac{|Du|^2}{2} + V(x) + m^\alpha + \kappa \ast m \geq 0, & \text{in } \mathcal{D}' \\
m - \text{div}(mDu) - 1 = 0, & \text{a.e.}
\end{cases}
\]

Moreover, if \(\alpha > \max \left(\frac{d-4}{2}, 0\right)\)

\[
\left(-u - \frac{|Du|^2}{2} + V(x) + m^\alpha + \kappa \ast m\right)m = 0
\]

almost everywhere.
If $A$ is a monotone operator in a Hilbert space, then the flow

$$\dot{w} = -A(w)$$

is a contraction in $H$. 
Monotone flow

We introduced the dynamic approximation

\[
\begin{align*}
\dot{m} &= \frac{u_x^2}{2} + V(x) - \ln m \\
\dot{u} &= (m u_x)_x.
\end{align*}
\]

If \((u, m)\) and \((\tilde{u}, \tilde{m})\) are solutions of the previous flow, then

\[
\frac{d}{dt} \int |m - \tilde{m}|^2 + |u - \tilde{u}|^2 \leq 0,
\]

provided \(m, \tilde{m} \geq 0\).
$u$ and $m$ evolution by monotone flow. $V(x) = \sin(2\pi x)$. 
The congestion problem

\[ \begin{cases} \dot{m} = -\frac{|u_x|^2}{m^{1/2}} - \sin(2\pi x) + \ln m \\ \dot{u} = \text{div}(m^{1/2}u_x). \end{cases} \]

Figure: $m$ error evolution.
A two-dimensional example

$u$ and $m$ error - monotone flow. $V(x, y) = \sin(2\pi x) + \sin(2\pi y)$. 