جامعة الملك عبدالله للعلوم والتقنية
King Abdullah University of Science and Technology

Monotonicity methods

Diogo A. Gomes

Monotone operators

Let H be a Hilbert space. $A : D \subset H \to H$ is a monotone operator if

$$(A(w) - A(z), w - z) \geq 0, \qquad \forall w, z \in D.$$

A variational inequality is the problem: find $w \in D$ such that

$$(A(w), z - w) \geq 0, \qquad \forall z \in D.$$

Examples of monotone operators

- For $H=\mathbb{R}$, monotone operators are increasing functions
- Gradients of convex functions are monotone operators

Stationary MFGs

Then, if $H(x,p)$ is convex in p and g is increasing, the operator

$$A\begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} -u - H(x,Du) + g(m) \\ m - \operatorname{div}(D_pHm) - 1 \end{bmatrix}$$

monotone in its domain $D \subset L^2 \times L^2$.

Time-dependent MFGs

Then, if $H(x,p)$ is convex in p and g is increasing, the operator

$$A\begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} u_t - H(x,Du) + g(m) \\ m_t - \operatorname{div}(D_pHm) - 1 \end{bmatrix}$$

monotone in its domain $D \subset L^2 \times L^2$.

Uniqueness

Often, monotonicity gives uniqueness. Given two solutions (m, u) and $(\tilde{m}, \tilde{u})$, we have

$$0 = \left(A \begin{bmatrix} m \\ u \end{bmatrix} - A \begin{bmatrix} \tilde{m} \\ \tilde{u} \end{bmatrix}, \begin{bmatrix} m \\ u \end{bmatrix} - \begin{bmatrix} \tilde{m} \\ \tilde{u} \end{bmatrix} \right) \geq 0.$$

Example

For

$$A\begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} u_t - \frac{u_x^2}{2} + m \\ m_t - (mu_x)_x \end{bmatrix},$$

we get

$$0 = \int_0^T \int (m + \tilde{m}) \frac{(u_x - \tilde{u}_x)^2}{2} + (m - \tilde{m})^2 \geq 0.$$

This implies $m = \tilde{m}$ and then, uniqueness of solution of

$$-u_t + \frac{u_x^2}{2} = m$$

gives $u = \tilde{u}$.

Variational inequalities

If $A : H \to H$ is monotone, then $A(w) = 0$ if and only if w satisfies

$$(A(w), z - w) \geq 0, \qquad \forall z \in H.$$

Weak solutions to variational inequalities

w is a weak solution of the variational inequality if

$$(A(z), z - w) \geq 0$$

for all $z \in D$.
Solutions of the variational inequality are weak solutions.
Under continuity assumptions and if D is large enough, weak solutions are solutions.

Weak solutions - an example

If $H = \mathbb{R}$, a monotone operator, A, is an increasing function. If A is continuous,

$$A(0) = 0$$

if and only if

$$A(z)(z - 0) = A(z)z \geq 0$$

for all z.

MFGs and variational inequalities

Consider the MFG

$$\begin{cases} u - \Delta u + H(x, Du) = g(m) \\ m - \Delta m - \operatorname{div}(D_p H m) = 1. \end{cases}$$

Then, if $H(x, p)$ is convex in p and g is increasing, the operator

$$A \begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} -u + \Delta u - H(x, Du) + g(m) \\ m - \Delta m - \operatorname{div}(D_p H m) - 1 \end{bmatrix}$$

is monotone in its domain $D \subset L^2 \times L^2$.

Weak solutions

A weak solution of the MFG is a pair (m, u), $m \geq 0$, such that

$$\left\langle \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m \\ u \end{bmatrix}, A \begin{bmatrix} \eta \\ v \end{bmatrix} \right\rangle_{\mathcal{D}'(\mathbb{T}^d)\times\mathcal{D}'(\mathbb{T}^d), C^\infty(\mathbb{T}^d)\times C^\infty(\mathbb{T}^d)} \geq 0$$

for all $(\eta, v) \in \mathcal{C}^\infty(\mathbb{T}^d; \mathbb{R}^+) \times \mathcal{C}^\infty(\mathbb{T}^d)$.

Existence of weak solutions

Main Theorem (Ferreira, G.)

Under suitable but general Assumptions, there exists a weak solution, $(m, u) \in \mathcal{D}'(\mathbb{T}^d) \times \mathcal{D}'(\mathbb{T}^d)$, $m \geq 0$, to the MFG

$$A \begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Moreover, $(m, u) \in \mathcal{M}_{ac} \times W^{1,\gamma}$ for some $\gamma > 1$ and $\int_{\mathbb{T}^d} m\,dx = 1$.

Scope

First-order, second-order, degenerate elliptic, and congestion problems satisfying monotonicity conditions.

Existence of weak solutions

Main Theorem (Ferreira, G.)

Under suitable but general Assumptions, there exists a weak solution, $(m, u) \in \mathcal{D}'(\mathbb{T}^d) \times \mathcal{D}'(\mathbb{T}^d)$, $m \geq 0$, to the MFG

$$A \begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Moreover, $(m, u) \in \mathcal{M}_{ac} \times W^{1,\gamma}$ for some $\gamma > 1$ and $\int_{\mathbb{T}^d} m\, dx = 1$.

Scope

First-order, second-order, degenerate elliptic, and congestion problems satisfying monotonicity conditions.

Example

Theorem (Ferreira, G.)

Let κ be a standard mollifier, $\alpha > 0$. Then, there exists a weak solution $u \in H^1$, $m \in L^{\alpha+1}$, $m \geq 0$ to

$$\begin{cases} u + \frac{|Du|^2}{2} + V(x) = m^\alpha + \kappa * m \\ m - \operatorname{div}(mDu) = 1. \end{cases}$$

That is, for all $(\eta, v) \in C^\infty$, $\eta > 0$, we have

$$\int \left(v + \frac{|Dv|^2}{2} + V(x) - \eta^\alpha - \kappa * m\right)(\eta - m) + \int (\eta - \operatorname{div}(\eta Dv) - 1)(v - u) \geq 0.$$

Example - further properties

Theorem (Ferreira, G.)

There exists a weak solution (u, m) such that

$$\begin{cases} -u - \frac{|Du|^2}{2} + V(x) + m^\alpha + \kappa * m \geq 0, & \text{in } \mathcal{D}' \\ m - \operatorname{div}(mDu) - 1 = 0, \text{a.e.}. \end{cases}$$

Moreover, if $\alpha > \max\left(\frac{d-4}{2}, 0\right)$

$$\left(-u - \frac{|Du|^2}{2} + V(x) + m^\alpha + \kappa * m \right) m = 0$$

almost everywhere.

The contracting flow

If A is a monotone operator in a Hilbert space, then the flow

$$\dot{w} = -A(w)$$

is a contraction in H.

Monotone flow

We introduced the dynamic approximation

$$\begin{cases} \dot{m} = \frac{u_x^2}{2} + V(x) - \ln m \\ \dot{u} = (m u_x)_x. \end{cases}$$

If (u, m) and $(\tilde{u}, \tilde{m})$ are solutions of the previous flow, then

$$\frac{d}{dt} \int |m - \tilde{m}|^2 + |u - \tilde{u}|^2 \leq 0,$$

provided $m, \tilde{m} \geq 0$.

u and m evolution by monotone flow. $V(x) = \sin(2\pi x)$.

The congestion problem

$$\begin{cases} \dot{m} = -\frac{|u_x|^2}{m^{1/2}} - \sin(2\pi x) + \ln m \\ \dot{u} = \operatorname{div}(m^{1/2} u_x). \end{cases}$$

Figure: m error evolution.

A two-dimensional example

u and m error - monotone flow. $V(x, y) = \sin(2\pi x) + \sin(2\pi y)$.