جامعة الملك عبدالله
للعلوم والتقنية
King Abdullah University of
Science and Technology

Stationary mean-field games

Diogo A. Gomes

www.kaust.edu.sa

We consider is the periodic stationary MFG,

$$\begin{cases} -\epsilon\Delta u + \frac{|Du|^2}{2} + V(x) = g(m) + \overline{H} \\ -\epsilon\Delta m - \operatorname{div}(mDu) = 0, \end{cases} \tag{1}$$

where the unknowns are $u : \mathbb{T}^d \to \mathbb{R}$, $m : \mathbb{T}^d \to \mathbb{R}$, with $m \geq 0$ and $\int m = 1$, and $\overline{H} \in \mathbb{R}$.

We suppose that $V : \mathbb{T}^d \to \mathbb{R}$ is C^∞, $g : \mathbb{R}^+ \to \mathbb{R}$ (or $g : \mathbb{R}_0^+ \to \mathbb{R}$), C^∞ in the set $m > 0$, satisfying

$$\int_{\mathbb{T}^d} g(m) \leq C + \frac{1}{2} \int_{\mathbb{T}^d} mg(m).$$

We say that $(u, m, \overline{H})$ or (u, m) is a classical solution of, respectively, (1) if u and m are C^∞, $m > 0$, (u, m) solves (1).

Bounds for $\overline{H}$

Proposition

Let u be a classical solution of (1). Suppose that $g \geq 0$. Then,

$$\overline{H} \leq \sup_{\mathbb{T}^d} V.$$

Proof

Because u is periodic, it achieves a minimum at a point, x_0. At this point, $Du(x_0) = 0$ and $\Delta u \geq 0$. Consequently,

$$V(x_0) \geq \overline{H} + g(m) \geq \overline{H}.$$

Hence, $\overline{H} \leq \sup V$.

Proposition

There exists a constant, C, such that, for any classical solution, $(u, m, \overline{H})$, of (1), we have

$$\int_{\mathbb{T}^d} \frac{|Du|^2}{2}(1+m) + \frac{1}{2} g(m) m \mathrm{d}x \leq C.$$

Proof

Multiply the Hamilton-Jacobi equation by $(m-1)$ and Fokker Planck equation by $-u$, adding them , and integrating by parts gives

$$\int_{\mathbb{T}^d} \frac{|Du|^2}{2}(1+m) + mg(m)\mathrm{dx} = \int_{\mathbb{T}^d} V(m-1) + g(m)\mathrm{dx}.$$

Using the assumption on g, we obtain the result.

Corollary

Let $(u, m, \overline{H})$ *be a classical solution of* (1)*. Suppose that* $g \geq 0$*. Then, there exists a constant,* C*, not depending on the particular solution, such that*

$$|\overline{H}| \leq C.$$

Proof

We have:

- $\frac{|Du|^2}{2} \in L^1$.
- From the assumptions and the preceding estimate, $g(m) \in L^1$.

Therefore, integrating the Hamilton-Jacobi equation, we obtain the bound for $\overline{H}$.

Bernstein estimates

Here, we examine the Hamilton-Jacobi equation,

$$-\Delta u(x) + \frac{|Du(x)|^2}{2} + V(x) = \bar{H},$$

with $V \in L^p$. Our goal is to bound the norm of Du in L^q for some $q > 1$.

Lemma

Let $u \in C^3$ and $v = |Du|^2$. Suppose that $V \in C^1$. Then, there exist, $c, C > 0$, which do not depend on u or V such that, for every $p > 1$,

$$-\int_{\mathbb{T}^d} v^p \Delta v \mathrm{d}x \geq \frac{4pc}{(p+1)^2}\left[\left(\int_{\mathbb{T}^d} v^{\frac{(p+1)d}{d-2}} \mathrm{d}x\right)^{\frac{d-2}{d}} - C\left(\int_{\mathbb{T}^d} v^{p+2}\mathrm{d}x\right)^{\frac{p+1}{p+2}}\right]$$

and

$$-2\int_{\mathbb{T}^d} DV \cdot Du\, v^p \mathrm{d}x \leq \frac{1}{2}\int_{\mathbb{T}^d} \left|D^2 u\right|^2 v^p \mathrm{d}x + C_p \int_{\mathbb{T}^d} |V|^2 v^p \mathrm{d}x.$$

Proof

By integration by parts, we have the identity

$$-\int_{\mathbb{T}^d} v^p \Delta v \mathrm{d}x = \int_{\mathbb{T}^d} p v^{p-1} |Dv|^2 \mathrm{d}x = \frac{4p}{(p+1)^2} \int_{\mathbb{T}^d} |Dv^{\frac{p+1}{2}}|^2 \mathrm{d}x.$$

Next, we use Sobolev's inequality to obtain

$$\int_{\mathbb{T}^d} |Dv^{\frac{p+1}{2}}|^2 \mathrm{d}x + \int_{\mathbb{T}^d} v^{p+1} \mathrm{d}x \geq c \left\| v^{\frac{p+1}{2}} \right\|_{2^*}^2 = c \left(\int_{\mathbb{T}^d} v^{\frac{(p+1)d}{d-2}} \mathrm{d}x \right)^{\frac{d-2}{d}},$$

moreover, from Young's inequality,

$$\int_{\mathbb{T}^d} v^{p+1} \mathrm{d}x \leq \left(\int_{\mathbb{T}^d} v^{p+2} \mathrm{d}x \right)^{\frac{p+1}{p+2}},$$

Proof

For the second inequality, we integrate again by parts to get

$$-\int_{\mathbb{T}^d} DV \cdot Du\, v^p \mathrm{d}x = \int_{\mathbb{T}^d} V \Delta u\, v^p \mathrm{d}x + p \int_{\mathbb{T}^d} V\, v^{p-1} Dv \cdot Du \mathrm{d}x.$$

Next, we apply a weighted Cauchy inequality to each of the terms in the prior identity to get

$$\int_{\mathbb{T}^d} V \cdot \Delta u\, v^p \mathrm{d}x \leq \frac{1}{8} \int_{\mathbb{T}^d} \left|D^2 u\right|^2 v^p \mathrm{d}x + C \int_{\mathbb{T}^d} |V|^2\, v^p \mathrm{d}x.$$

Next, because $v = |Du|^2$, we have $Dv = 2D^2uDu$, hence

$$p \int_{\mathbb{T}^d} V\, v^{p-1} Dv \cdot Du \mathrm{d}x \leq 2p \int_{\mathbb{T}^d} |V|\, v^p |D^2 u| \mathrm{d}x \leq \frac{1}{8} \int_{\mathbb{T}^d} \left|D^2 u\right|^2 v^p \mathrm{d}x + C_p \int_{\mathbb{T}^d} |V|^2\, v^p \mathrm{d}x.$$

Using the two preceding bounds, we get the second estimate.

Bernstein estimate

Theorem

Let u be C^3 and $V \in C^1$. Then, for any $p > 1$, there exists a constant, $C_p > 0$, that depends only on $|\bar{H}|$, such that

$$\|Du\|_{L^{\frac{2d(p+1)}{d-2}}(\mathbb{T}^d)} \leq C_p \left(1 + \|V\|_{L^{\frac{2d(1+p)}{d+2p}}(\mathbb{T}^d)}\right).$$

Note that $\gamma_p = \frac{2d(1+p)}{d+2p} \to d$ when $p \to \infty$ and that γ_p is increasing when $d > 2$.

Proof

We set $v = |Du|^2$. Differentiating the Hamilton-Jacobi equation

$$\Delta u_{x_i} = \frac{1}{2} v_{x_i} + V_{x_i}.$$

Thus,

$$\begin{aligned} -\Delta v &= -2 \sum_{i,j=1}^{d} \left(u_{x_i x_j}\right)^2 - 2 \sum_{i=1}^{d} u_{x_i} \Delta u_{x_i} \qquad (2) \\ &= -2 \sum_{i,j=1}^{d} \left(u_{x_i x_j}\right)^2 - 2 \sum_{i=1}^{d} u_{x_i} \left(\frac{1}{2} v_{x_i} + V_{x_i}\right). \end{aligned}$$

Proof

By multiplying (2) by v^p and integrating over $\mathbb{T}^d$, we have

$$-\int_{\mathbb{T}^d} v^p \Delta v \mathrm{d}x + 2\int_{\mathbb{T}^d} \left|D^2 u\right|^2 v^p \mathrm{d}x = -\int_{\mathbb{T}^d} Du \cdot Dv\, v^p \mathrm{d}x - 2\int_{\mathbb{T}^d} DV \cdot Du\, v^p \mathrm{d}x.$$

The Lemma provides bounds for the first term on the left-hand side and the last term on the right-hand side.
For $\delta > 0$, there exists a constant, $C_\delta > 0$, such that

$$-\int_{\mathbb{T}^d} Du \cdot Dv\, v^p \mathrm{d}x \leq \delta \int_{\mathbb{T}^d} \left|D^2 u\right|^2 v^p \mathrm{d}x + \frac{C_\delta}{p+1}\int_{\mathbb{T}^d} v^{p+2} \mathrm{d}x,$$

for every $p > 1$.

Proof

Now, we claim that for any large enough $p > 1$, there exists $C_p > 0$ that does not depend on u, such that

$$\left(\int_{\mathbb{T}^d} v^{\frac{d(p+1)}{d-2}} \mathrm{d}x \right)^{\frac{(d-2)}{d(p+1)}} \leq C_p \left(\int_{\mathbb{T}^d} |V|^{2\beta_p} \mathrm{d}x \right)^{\frac{1}{\beta_p}} + C_p, \qquad (3)$$

where β_p is the conjugate exponent of $\frac{d(p+1)}{(d-2)p}$. Further, $\beta_p \to \frac{d}{2}$ when $p \to \infty$.

Proof

To prove the previous claim, we use the lemma to get

$$c_p \left(\int_{\mathbb{T}^d} v^{\frac{d(p+1)}{d-2}} \mathrm{d}x \right)^{\frac{d-2}{d}} + 2 \int_{\mathbb{T}^d} \left| D^2 u \right|^2 v^p \mathrm{d}x \leq$$

$$c_p \left(\int_{\mathbb{T}^d} v^{p+2} \mathrm{d}x \right)^{\frac{p+1}{p+2}} - \int_{\mathbb{T}^d} Du \cdot Dv \, v^p \mathrm{d}x - 2 \int_{\mathbb{T}^d} DV \cdot Du \, v^p \mathrm{d}x,$$

where $c_p := \frac{4p\tilde{C}}{(p+1)^2}$, for some constant $\tilde{C}$.

Proof

From the second estimate in the Lemma, and Young's inequality, ($z^{\frac{p+1}{p+2}} \leq \epsilon z + C_{p,\epsilon}$), we have

$$c_p \left(\int_{\mathbb{T}^d} v^{\frac{d(p+1)}{d-2}} \mathrm{d}x \right)^{\frac{d-2}{d}} + \left[2 - \left(\frac{1}{2} + \delta \right) \right] \int_{\mathbb{T}^d} \left| D^2 u \right|^2 v^p \mathrm{d}x \leq$$

$$C_p \int_{\mathbb{T}^d} |V|^2 v^p \mathrm{d}x + \left(\frac{C_\delta}{p+1} + \delta \right) \int_{\mathbb{T}^d} v^{p+2} \mathrm{d}x + C_{p,\delta}.$$

Proof

The key idea is now to use the Hamilton-Jacobi equation again:

$$\int_{\mathbb{T}^d} \left|D^2 u\right|^2 v^p \mathrm{d}x \geq \frac{1}{d}\int_{\mathbb{T}^d} |\Delta u|^2 v^p \mathrm{d}x = \frac{1}{d}\int_{\mathbb{T}^d} \left|\frac{v}{2} + V - \overline{H}\right|^2 v^p \mathrm{d}x$$

$$\geq \frac{1}{3d}\int_{\mathbb{T}^d} v^2 v^p \mathrm{d}x - \frac{1}{d}\int_{\mathbb{T}^d} V^2 v^p \mathrm{d}x - \frac{1}{d} C \int v^p \mathrm{d}x$$

$$\geq c\int_{\mathbb{T}^d} v^{p+2} \mathrm{d}x - C\int_{\mathbb{T}^d} V^2 v^p \mathrm{d}x - C_p,$$

where the second inequality follows from
$(a-b-c)^2 \geq \frac{1}{3}a^2 - b^2 - c^2$.

Proof

For a small δ and a large enough p, the preceding inequalities give

$$\left(\int_{\mathbb{T}^d} v^{\frac{d(p+1)}{d-2}} \mathrm{d}x\right)^{\frac{d-2}{d}} \leq C_p \int_{\mathbb{T}^d} |V|^2 v^p \mathrm{d}x \quad + C_p$$

$$\leq C_p \left(\int_{\mathbb{T}^d} v^{\frac{d(p+1)}{d-2}} \mathrm{d}x\right)^{\frac{(d-2)p}{d(p+1)}} \left(\int_{\mathbb{T}^d} |V|^{2\beta_p} \mathrm{d}x\right)^{\frac{1}{\beta_p}} + C_p.$$

Hence,

$$\left(\int_{\mathbb{T}^d} v^{\frac{d(p+1)}{d-2}} \mathrm{d}x\right)^{\frac{(d-2)}{d(p+1)}} \leq C_p \left(\int_{\mathbb{T}^d} |V|^{2\beta_p} \mathrm{d}x\right)^{\frac{1}{\beta_p}} + C_p.$$

This last estimate gives (3), and the theorem follows.

We fix a C^2 potential, $V : \mathbb{T}^d \to \mathbb{R}$, and look for a solution, $(u, m, \overline{H})$, of the MFG

$$\begin{cases} -\Delta u(x) + \frac{|Du(x)|^2}{2} + V(x) = \bar{H} + m^\alpha, \\ -\Delta m - \operatorname{div}(mDu) = 0, \\ \int u \mathrm{d}x = 0, \int m \mathrm{d}x = 1, \end{cases} \tag{4}$$

with $u, m : \mathbb{T}^d \to \mathbb{R}$ and $\overline{H} \in \mathbb{R}$.

Theorem

Let $(u, m, \overline{H})$ *solve* (4) *and* $0 < \alpha \leq \frac{1}{d-1}$. *Suppose that* $u, m \in C^2(\mathbb{T}^d)$. *Then, for every* $q > 1$, *there exists a constant,* $C_q > 0$, *that depends only on* $\|V\|_{L^{1+\frac{1}{\alpha}}(\mathbb{T}^d)}$, *such that*
$\|Du\|_{L^q(\mathbb{T}^d)} \leq C_q$.

Proof

- We use Bernstein's estimate with V replaced by $V(x) - m^\alpha$.
- By the previous results, we have

$$|\bar{H}| \leq C, \ \|m^\alpha\|_{L^{1+\frac{1}{\alpha}}(\mathbb{T}^d)} \leq C.$$

- Because $d \leq 1 + \frac{1}{\alpha}$, $\gamma_p \leq 1 + \frac{1}{\alpha}$, Bernstein estimate gives

$$\|Du\|_{L^q(\mathbb{T}^d)} \leq C_q \text{ for every } q > 1.$$

Proposition

Let $(u, m, \overline{H})$ *solve* (4) *with* u *and* m *in* $C^\infty(\mathbb{T}^d)$, *and let* $m > 0$. *Then, there exists a constant,* $C > 0$, *such that*

$$\|\ln m\|_{W^{1,q}(\mathbb{T}^d)} \leq C.$$

Hence, $m, \frac{1}{m} \in L^\infty$.

Proof

Standard elliptic regularity theory applied to Hamilton-Jacobi equation yields

$$\|u\|_{W^{2,q}(\mathbb{T}^d)} \leq C_q,$$

for every $q > 1$. Therefore, Morrey's Embedding Theorem implies that $u \in C^{1,\beta}(\mathbb{T}^d)$, for some $\beta \in (0,1)$.

Proof

Next, set $w = -2\ln m$. Straightforward computations show that w satisfies

$$-\Delta w + \frac{1}{2}|Dw|^2 - Du \cdot Dw + 2\,\mathrm{div}(Du) = 0.$$

Integrating, we conclude that $Dw \in L^2$.

Proof

The Bernstein estimate gives

$$\|Dw\|_{L^{\frac{2d(p+1)}{d-2}}(\mathbb{T}^d)} \leq C_p \left(C + \|Du \cdot Dw\|_{L^{\frac{2d(1+p)}{d+2p}}(\mathbb{T}^d)} + \|\operatorname{div}(Du)\|_{L^{\frac{2d(1+p)}{d+2p}}(\mathbb{T}^d)} \right).$$

Since $\frac{2d(p+1)}{d-2} > \frac{2d(1+p)}{d+2p}$, we get $Dw \in L^q$ for any $q \geq 1$.

Proof

Hence, $\ln m$ is a Hölder continuous function. Because $\int_{\mathbb{T}^d} m \mathrm{d}x = 1$, m is bounded from above and from below. Consequently, $\|\ln m\|_{L^q(\mathbb{T}^d)}$ is a priori bounded by some universal constant that depends only on q.

Proposition

Let $(u, m, \overline{H})$ *solve* (4) *with* u *and* m *in* $C^\infty(\mathbb{T}^d)$*, and let* $m > 0$*. For any* $k \geq 1$ *and* $q > 1$*, there exists a constant,* $C_{k,q} > 0$*, such that*

$$\left\|D^k u\right\|_{L^q(\mathbb{T}^d)}, \left\|D^k m\right\|_{L^q(\mathbb{T}^d)} \leq C_{k,q}.$$

Proof

The preceding results give

$$\|u\|_{W^{2,a}(\mathbb{T}^d)} \leq C_a$$

for every $1 < a < \infty$. Also, Proposition 7 gives

$$\|\ln m\|_{W^{1,a}(\mathbb{T}^d)} \leq C$$

for any $1 < a < \infty$. By differentiating the first equation in (4), we obtain

$$-D_x \Delta u = D_x g(m) - D^2 u Du. \tag{5}$$

Finally, we observe that the right-hand side of (5) is bounded in $L^a(\mathbb{T}^d)$. Thus,

$$\|u\|_{W^{3,a}(\mathbb{T}^d)} \leq C_{3,a},$$

which leads to

$$\|m\|_{W^{2,a}(\mathbb{T}^d)} \leq C_{2,a}.$$

The proof proceeds by iterating this procedure up to order k.

Here, we illustrate the continuation method by proving the existence of smooth solutions of

$$\begin{cases} -\Delta u + \frac{|Du|^2}{2} + V(x) = \overline{H} + g(m), \\ -\Delta m - \operatorname{div}(Dum) = 0, \\ \int_{\mathbb{T}^d} u = 0, \quad \int_{\mathbb{T}^d} m = 1. \end{cases} \tag{6}$$

First, for $0 \leq \lambda \leq 1$, we consider the family of problems

$$\begin{cases} -\Delta m_\lambda - \operatorname{div}(Du_\lambda m_\lambda) = 0, \\ \Delta u_\lambda - \frac{|Du_\lambda|^2}{2} - \lambda V + \overline{H}_\lambda + g(m_\lambda) = 0, \\ \int_{\mathbb{T}^d} u_\lambda = 0, \quad \int_{\mathbb{T}^d} m_\lambda = 1. \end{cases} \tag{7}$$

- Set

$$\dot{H}^k(\mathbb{T}^d,\mathbb{R}) = \left\{ f \in H^k(\mathbb{T}^d,\mathbb{R}) : \int_{\mathbb{T}^d} f\,\mathrm{d}x = 0 \right\}$$

and consider $F^k = \dot{H}^k(\mathbb{T}^d,\mathbb{R}) \times H^k(\mathbb{T}^d,\mathbb{R}) \times \mathbb{R},$, which is a Hilber space with norm

$$\|w\|^2_{F^k} = \|\psi\|^2_{\dot{H}^k(\mathbb{T}^d,\mathbb{R})} + \|f\|^2_{H^k(\mathbb{T}^d,\mathbb{R})} + |h|^2$$

for $w = (\psi, f, h) \in F^k$.
- $H^k_+(\mathbb{T}^d,\mathbb{R})$, for $k > \frac{d}{2}$is the set of (everywhere) positive functions in $H^k(\mathbb{T}^d,\mathbb{R})$.
- For any $k > \frac{d}{2}$, let

$$F^k_+ = \dot{H}^k(\mathbb{T}^d,\mathbb{R}) \times H^k_+(\mathbb{T}^d,\mathbb{R}) \times \mathbb{R}.$$

A classical solution is a tuple, $(u_\lambda, m_\lambda, \overline{H}_\lambda) \in \bigcap_{k\geq 0} F^k_+$.

Theorem

Assume that $g, V \in C^\infty(\mathbb{T}^d)$ *with* $g'(z) > 0$ *for* $z \in (0, +\infty)$ *and that we have the a priori estimate for any solution of* (7)*:*

$$|\overline{H}| + \left\| \frac{1}{m_\lambda} \right\|_{L^\infty(\mathbb{T}^d)} + \|u_\lambda\|_{W^{k,p}(\mathbb{T}^d)} + \|m_\lambda\|_{W^{k,p}(\mathbb{T}^d)} \leq C_{k,p}.$$

Then, there exists a classical solution to (6)*.*

Proof

For large enough k, define $E\colon \mathbb{R} \times F_+^k \to F^{k-2}$ by

$$E(\lambda, u, m, \overline{H}) = \begin{pmatrix} -\Delta m - \operatorname{div}(Dum) \\ \Delta u - \frac{|Du|^2}{2} - \lambda V + \overline{H} + g(m) \\ -\int_{\mathbb{T}^d} m + 1 \end{pmatrix}.$$

Our system equivalent to $E(\lambda, v_\lambda) = 0$, where $v_\lambda = (u_\lambda, m_\lambda, \overline{H}_\lambda)$.

Proof

The partial derivative of E in the second variable at $v_\lambda = (u_\lambda, m_\lambda, \overline{H}_\lambda)$,

$$\mathcal{L}_\lambda = D_2E(\lambda, v_\lambda)\colon F^k \to F^{k-2},$$

is

$$\mathcal{L}_\lambda(w)(x) = \begin{pmatrix} -\Delta f(x) - \operatorname{div}(Du_\lambda f(x) + m_\lambda D\psi) \\ \Delta\psi(x) - Du_\lambda D\psi + g'(m_\lambda(x))f(x) + h \\ -\int_{\mathbb{T}^d} f \end{pmatrix},$$

where $w = (\psi, f, h) \in F^k$. In principle, $\mathcal{L}_\lambda$ is a linear map on F^k for a large enough k. However, it is easy to see that it admits a unique extension to F^k for any $k > 1$.

Proof

- Let

$$\Lambda := \{\, \lambda \mid \quad 0 \le \lambda \le 1, \text{ (7) has a classical solution } (u_\lambda, m_\lambda, \overline{H}_\lambda) \,\}.$$

- Note that $0 \in \Lambda$ as $(u_0, m_0, \overline{H}_0) \equiv (0, 1, -g(1))$ is a solution to (7) for $\lambda = 0$.
- Our goal is to prove $\Lambda = [0, 1]$.
- The a priori bounds in the statement mean that Λ is a closed set.
- To prove that Λ is open, we s apply the implicit function theorem.

Proof

Let $F = F^1$. For $w_1, w_2 \in F$ with smooth components, set

$$B_\lambda[w_1, w_2] = \int_{\mathbb{T}^d} w_2 \cdot \mathcal{L}_\lambda(w_1).$$

For smooth w_1, w_2,

$$B_\lambda[w_1, w_2] = \int_{\mathbb{T}^d} [m_\lambda D\psi_1 \cdot D\psi_2 + f_1 Du_\lambda D\psi_2 - f_2 Du_\lambda D\psi_1 + g'(m_\lambda) f_1 f_2 + Df_1 D\psi_2 - Df_2 D\psi_1 + h_1 f_2 - h_2 f_1].$$

This last expression defines a bilinear form $B_\lambda \colon F \times F \to \mathbb{R}$.

Proof

Claim

B_λ is bounded, i.e.,

$$|B_\lambda[w_1, w_2]| \leq C\|w_1\|_F\|w_2\|_F.$$

To prove the claim, we use Holder's inequality on each summand.

Proof

Claim

There exists a linear bounded mapping, $A\colon F \to F$*, such that* $B_\lambda[w_1, w_2] = (Aw_1, w_2)_F$.

This claim follows from Claim 10 and the Riesz Representation Theorem.

Proof

Claim

There exists a positive constant, c, such that $\|Aw\|_F \geq c\|w\|_F$ for all $w \in F$.

If the previous claim were false, then there would exist a sequence, $w_n \in F$, with $\|w_n\|_F = 1$ such that $Aw_n \to 0$.

Proof

Let $w_n = (\psi_n, f_n, h_n)$. Then,

$$\int_{\mathbb{T}^d} m_\lambda |D\psi_n|^2 + g'(m_\lambda) f_n^2 = B_\lambda[w_n, w_n] \to 0. \tag{8}$$

By combining the a priori estimates on $\frac{1}{m_\lambda}$ with the fact that g is strictly increasing and smooth, we have $g'(m_\lambda) > \delta > 0$.

Proof

Then, (8) implies that $\psi_n \to 0$ in $\dot{H}^1_0$ and $f_n \to 0$ in L^2. Taking $\check{w}_n = (f_n - \int f_n, 0, 0) \in F$, we get

$$\int_{\mathbb{T}^d} [|Df_n|^2 + m_\lambda D\psi_n \cdot Df_n + f_n Du_\lambda Df_n] = B[w_n, \check{w}_n] = (Aw_n, \check{w}_n),$$

Therefore,

$$\frac{1}{2}\|Df_n\|^2_{L^2(\mathbb{T}^d)} - C\left(\|D\psi_n\|^2_{L^2(\mathbb{T}^d)} + \|f_n\|^2_{L^2(\mathbb{T}^d)}\right) \leq (Aw_n, \check{w}_n) \to 0,$$

where the constant, C, depends only on u_λ. Because $D\psi_n, f_n \to 0$ in L^2, we have $f_n \to 0$ in $H^1(\mathbb{T}^d)$.

Proof

Finally, we take $\breve{w} = (0, 1, 0)$. Accordingly, we get

$$\int_{\mathbb{T}^d} [-Du_\lambda D\psi_n + g'(m_\lambda) f_n] + h_n = B[w_n, \breve{w}] = (Aw_n, \breve{w}) \to 0.$$

Because $D\psi_n, f_n \to 0$ in L^2, we have $h_n \to 0$. Hence, $\|w_n\|_F \to 0$, which contradicts $\|w_n\|_F = 1$.

Proof

Claim

$R(A)$ is closed in F.

This claim follows from the preceding one.

Proof

Claim

$R(A) = F$.

By contradiction, suppose that $R(A) \neq F$.

- Then, because $R(A)$ is closed in F, there exists a vector, $w \neq 0$, with $w \perp R(A)$. Let $w = (\psi, f, h)$. Then,

$$0 = (Aw, w) = B_\lambda[w, w] \geq \int_{\mathbb{T}^d} \theta |D\psi|^2 + \delta |f|^2.$$

  Therefore, $\psi = 0$ and $f = 0$.

- Next, we choose $\bar{w} = (0, 1, 0)$. Similarly, we have $h = B_\lambda[\bar{w}, w] = (A\bar{w}, w) = 0$. Thus, $w = 0$, and, consequently, $R(A) = F$.

Proof I

Claim

For any $w_0 \in F^0$, there exists a unique $w \in F$ such that $B_\lambda[w, \tilde{w}] = (w_0, \tilde{w})_{F^0}$ for all $\tilde{w} \in F$. Consequently, w is the unique weak solution of the equation $\mathcal{L}_\lambda(w) = w_0$. Moreover, $w \in F^2$ and $\mathcal{L}_\lambda(w) = w_0$ in the sense of F^2.

Proof

Consider the functional $\tilde{w} \mapsto (w_0, \tilde{w})_{F^0}$ on F. By the Riesz Representation Theorem, there exists $\omega \in F$ such that $(w_0, \tilde{w})_{F^0} = (\omega, \tilde{w})_F$. Taking $w = A^{-1}\omega$, we get

$$B[w, \tilde{w}] = (Aw, \tilde{w})_F = (\omega, \tilde{w})_F = (w_0, \tilde{w})_{F^0}.$$

Therefore, f is a weak solution to

$$-\Delta f - \text{div}(m_\lambda D\psi + fDu_\lambda) = \psi_0$$

and ψ is a weak solution to

$$\Delta\psi - Du_\lambda D\psi + g'(m_\lambda)f + h = f_0.$$

Proof

- Standard results from the regularity theory for elliptic equations combined with bootstrapping arguments give $w = (\psi, f, h) \in F^2$. Thus, $\mathcal{L}_\lambda(w) = w_0$.
- Consequently, $\mathcal{L}_\lambda$ is a bijective operator from F^2 to F^0. Then, $\mathcal{L}_\lambda$ is injective as an operator from F^k to F^{k-2} for any $k \geq 2$.
- To prove that it is also surjective, take any $w_0 \in F^{k-2}$. Then, there exists $w \in F^2$ such that $\mathcal{L}_\lambda(w) = w_0$.
- Finally elliptic regularity and bootstrapping imply that $w \in F^k$. Hence, $\mathcal{L}_\lambda \colon F^k \to F^{k-2}$ is surjective and, therefore, also bijective.

Proof

Claim

$\mathcal{L}_\lambda$ is an isomorphism from F^k to F^{k-2} for any $k \geq 2$.

Because $\mathcal{L}_\lambda \colon F^k \to F^{k-2}$ is bijective, we just need to check that it is also bounded. The boundedness follows directly from bounds on u_λ and m_λ and the smoothness of V and g.

Proof

Claim

The set Λ is open.

Proof

- We choose $k > d/2 + 1$ so that $H^{k-1}(\mathbb{T}^d, \mathbb{R})$ is an algebra.
- For each $\lambda_0 \in \Lambda$, the partial derivative, $\mathcal{L} = D_2 E(\lambda_0, v_{\lambda_0})\colon F^k \to F^{k-2}$, is an isomorphism.
- By the Implicit Function Theorem, there exists a unique solution $v_\lambda \in F_+^k$ to $E(\lambda, v_\lambda) = 0$, in some neighborhood, U, of λ_0.

Finally, because $H^{k-1}(\mathbb{T}^d, \mathbb{R})$ is an algebra, bootstrapping yields that u_λ and m_λ are smooth. Therefore, v_λ is a classical solution to (6). Hence, $U \subset \Lambda$, which proves that Λ is open.

Proof

We have proven that Λ is both open and closed; hence, $\Lambda = [0, 1]$. This argument ends the proof of the theorem.