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Price formation models

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Here, we are interested in the price formation in electricity markets where:

- a large number of agents owns storage devices that can be charged and later supply the grid with electricity;
- agents seek to maximize profit by trading electricity at a price $\varpi(t)$, which is set by a supply versus demand balance condition.

Our model comprises three quantities of interest:

- a price $\varpi \in C([0, T])$
- a value function $u \in C(\mathbb{R} \times [0, T])$
- a path describing the statistical distribution of the agents, $m \in C([0, T], \mathcal{P})$.

Problem

Given $\epsilon \geq 0$, *a Hamiltonian,* $H : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $H \in C^\infty$, *an energy production rate* $\mathbf{Q} : [0, T] \to \mathbb{R}$, $\mathbf{Q} \in C^\infty([0, T])$, *a terminal cost* $\bar{u} : \mathbb{R} \to \mathbb{R}$, $\bar{u} \in C^\infty(\mathbb{R})$ *and an initial probability distribution* $\bar{m} \in \mathcal{P} \cap C_c^\infty(\mathbb{R})$, *find* $u : \mathbb{R} \times [0, T] \to \mathbb{R}$, $m \in C([0, T], \mathcal{P})$, *and* $\varpi : [0, T] \to \mathbb{R}$ *solving*

$$\begin{cases} -u_t + H(x, \varpi(t) + u_x) = \epsilon u_{xx} \\ m_t - (D_p H(x, \varpi(t) + u_x) m)_x = \epsilon m_{xx} \\ \int_\Omega D_p H(x, \varpi(t) + u_x) dm = -\mathbf{Q}(t), \end{cases} \tag{1}$$

and satisfying the initial-terminal conditions

$$\begin{cases} u(x, T) = \bar{u}(x), \\ m(x, 0) = \bar{m}(x). \end{cases}$$

- $u(x,t)$ is the value function for an agent whose charge is x at time t. u is a (continuous) viscosity of the first equation; if $\epsilon > 0$, parabolic regularity gives additional regularity.
- $m(x,t)$ determines the distribution of the energy storage of the agents at time t. m is a weak solution of the second equation.
- the spot price, $\varpi(t)$, is selected so supply $\mathbf{Q}(t)$ is balances, the condition imposed by the last equation.

Main Result

Theorem

Under the assumptions below, there exists a solution (u, m, ϖ) where u is a viscosity solution of the first equation, Lipschitz and semiconcave in x, and differentiable almost everywhere with respect to m, $m \in C([0,T], \mathcal{P})$, and ϖ is Lipschitz continuous. Moreover, if $\epsilon > 0$ this solution is unique. If $\epsilon = 0$, under additional convexity assumptions on V and the terminal cost, there is a unique solution (u, m, ϖ). Moreover, u is differentiable in x for every x, and u_{xx} and m are bounded.

Model derivation I

To simplify, we set $\epsilon = 0$.

- Each consumer has a storage device that is connected to the network, for example, an electric car battery.
- Consumers trade electricity, charging the batteries when the price is low and selling electricity to the market when the price is high.

A typical consumer has a battery with charge $\mathbf{x}(t)$. This charge changes according to

$$\dot{\mathbf{x}}(t) = \alpha(t).$$

Each consumer seeks to select α to minimize the cost.

This cost is determined by the Lagrangian

$$\ell(\alpha, x, t) = \ell_0(\alpha, x) + \varpi(t)\alpha(t).$$

For example, we often take

$$\ell_0(\alpha, x, t) = \frac{c}{2}\alpha^2(t) + V(x). \tag{2}$$

The singular case where

$$V(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1 \\ +\infty & \text{otherwise,} \end{cases}$$

corresponds to the case where the battery charges satisfies $0 \leq x \leq 1$.

Each consumer minimizes the functional

$$J(x,t,\alpha)=\int_t^T \ell(\alpha(s),\mathbf{x}(t),t)ds+\bar{u}(\mathbf{x}(T)),$$

where $\bar{u}$ is the *terminal cost* and $\alpha\in\mathcal{A}_t$, where $\mathcal{A}_t$ is the set of bounded measurable functions $\alpha:[t,T]\to A\subset\mathbb{R}$.

The *value function*, u, is the infimum of J over all controls in $\mathcal{A}_t$; that is,

$$u(x,t) = \inf_{\alpha \in \mathcal{A}_t} J(x,t,\alpha).$$

The *Hamiltonian*, H, for the preceding control problem is

$$H(x,p) = \sup_{a \in A} \left(-pa - \ell_0(x,a)\right).$$

For example, for ℓ_0 as in (2), we have

$$H(x,p) = \frac{p^2}{2c} + V(x).$$

From standard optimal control theory, u is a viscosity solution of

$$\begin{cases} -u_t + H(x, \varpi(t) + u_x) = 0 \\ u(x, T) = \bar{u}(x). \end{cases}$$

For ℓ_0 as before:

$$-u_t + \frac{1}{2c}(u_x + \varpi(t))^2 - V(x) = 0.$$

At points of differentiability of u,

$$\alpha^*(t) = -D_p H(x, \varpi(t) + u_x(\mathbf{x}(t), t)).$$

The associated *transport equation* is:

$$\begin{cases} m_t - \left(D_pH(x, u_x + \varpi(t))m\right)_x = 0, \\ m(x,0) = \bar{m}(x), \end{cases}$$

where $\bar{m}$ is the initial distribution of the agents.
Taking ℓ_0 as before, we have

$$m_t - \frac{1}{c}(m(\varpi + u_x))_x = 0.$$

Finally, we fix an *energy production function* $\mathbf{Q}(t)$ and require that the production balances demand:

$$\int_{\mathbb{R}} \alpha^*(t) m(x,t) dx = \mathbf{Q}(t);$$

that is,

$$\int_{\mathbb{R}} D_p H(x, u_x + \varpi(t)) m(x,t) dx = -\mathbf{Q}(t).$$

This constraint determines the price, $\varpi(t)$.

Assumption

The Hamiltonian H is the Legendre transform of a convex Lagrangian:

$$H(x,p) = \sup_{\alpha\in\mathbb{R}} -p\alpha - \ell_0(\alpha) - V(x), \tag{3}$$

where $\ell_0 \in C^2(\mathbb{R})$ is a uniformly convex function and $V \in C^2(\mathbb{R})$ is bounded from below.

Assumption

The potential V in (3) and the terminal data $\bar{u}$ are globally Lipschitz.

Assumption

The potential V in (3) and the terminal data $\bar{u}$ satisfy

$$|D^2_{xx}V| \leq C, \qquad |D^2_{xx}\bar{u}| \leq C$$

for some positive constant C.

Assumption

There exists a constant, $C > 0$, such that

$$|\bar{m}_{xx}|, |\bar{u}_{xx}| \leq C.$$

Assumption

There exists $\theta > 0$ such that

$$D^2_{pp}H(x,p) > \theta$$

for all $x, p \in \mathbb{R}$. In addition, there exists $C > 0$ such that

$$|D^3_{ppp}H| \leq C.$$

Existence proof main steps

- Bounds for u
- Bounds for m
- Bounds for ϖ
- Fixed point argument

Proposition

$u(x,t)$ is locally bounded and the map $x \mapsto u(x,t)$ is Lipschitz for $0 \leq t \leq T$. Moreover, the Lipschitz bound on u does not depend on ϖ nor on ϵ.

Proposition

Then, $x \mapsto u(x,t)$ is semiconcave with a semiconcavity constant that does not depend on ϵ nor on ϖ.

Proof

Fix an optimal control α^* for (x, t):

$$u(x,t) = E\left[\int_t^T \ell_0(\alpha^*) + \varpi\alpha^* + V(\mathbf{x}^*)ds + \bar{u}(\mathbf{x}(T)^*)\right].$$

Then, for any $h \in \mathbb{R}$, we have

$$u(x \pm h, t) \leq E\left[\int_t^T \ell_0(\alpha^*) + \varpi\alpha^* + V(\mathbf{x}^* \pm h)ds + \bar{u}(\mathbf{x}(T)^* \pm h)\right].$$

Therefore,

$$u(x+h,t) - 2u(x,t) + u(x-h,t) \leq Ch^2.$$

Note that C does not depend on ϖ, only on T and on the semiconcavity estimates for V and $\bar{u}$.

Proposition

Suppose that $\varpi_n \to \varpi$ uniformly on $[0, T]$, then $u^n \to u$ locally uniformly and $u^n_x \to u_x$ almost everywhere.

Proof

The local uniform convergence of u^n follows from the stability of viscosity solutions. Because u^n is semiconcave and converges uniformly to u, $u_x^n \to u_x$ almost everywhere.

Now, we examine the Fokker-Planck equation.

$$\begin{cases} m_t - \operatorname{div}(mD_pH(x, \varpi + u_x)) = \epsilon \Delta m, \\ m(x,0) = \bar{m}(x). \end{cases} \tag{4}$$

Let $\mathcal{P}$ denote the set of probability measures on $\mathbb{R}$ with finite second-moment and endowed with the 1-Wasserstein distance.

Proposition

Suppose $\epsilon > 0$. The Fokker-Planck equation has a solution $m \in C([0,T],\mathcal{P})$. Moreover,

$$d_1(m(t), m(t+h)) \leq Ch^{1/2}.$$

In addition, for any sequence $\varpi_n \to \varpi$ uniformly on $[0,T]$ and corresponing solutions u_n and m_n, we have $m_n \to m$ in $C([0,T],\mathcal{P}_1)$.

Proof

- The existence of a solution in $C([0,T],\mathcal{P}_1)$ and the estimate in the statement are well known.
- The constant C can be chosen to depend only on ϵ_0 for all $\epsilon<\epsilon_0$, on the problem data, and on $\|\varpi\|_{L^\infty}$.
- By the Ascoli-Arzela theorem, we have that $m_n \to m$ in $C([0,T],\mathcal{P}_1)$
- Because $\epsilon>0$, $m_n \to m$ in L^2. Moreover, the Fokker-Planck equation has a unique solution. Thus, it suffices to check that m is a solution. Because $u_x^n \to u_x$, almost everywhere, by semiconcavity, we have for any $\psi \in C_c^\infty$

$$\int_0^T \int_{\mathbb{R}} \psi_x D_p H(x, \varpi^n + u_x^n) m^n dxdt$$

is convergent. Hence, m is a weak solution .

Proposition

Let $\epsilon > 0$. Then

$$\int_0^T \int_{\mathbb{R}} D^2_{pp} H u^2_{xx} m dx dt \leq C.$$

Proof

We diffferentiate the Hamilton-Jacobi twice with respect to x, multiply by m, and integrate by parts using the Fokker-Planck equation.

Price-supply relation

We observe that there exists a unique ϑ_0 such that

$$\int_{\mathbb{R}} D_p H(x, \vartheta_0 + u_x(x,0))\bar{m} dx = -\mathbf{Q}(0).$$

Moreover, ϑ_0 is bounded by a constant that depends only on the problem data.

Price-supply relation

Next, we differentiate

$$\int_{\mathbb{R}} D_p H(x, \varpi + u_x) m dx = -\mathbf{Q}(t)$$

to get the identity

$$\dot{\varpi} \int_{\mathbb{R}} D^2_{pp} H m dx + \int_{\mathbb{R}} \left[D^2_{pp} H u_{xt} m + D_p H m_t \right] dx = -\dot{\mathbf{Q}}.$$

Price-supply relation

Now, we use the equations to get the identity

$$\int_{\mathbb{R}} D^2_{pp}Hu_{xt}m + D_pHm_t = \int_{\mathbb{R}} D^2_{pp}H\left(-\epsilon\Delta u_x + D_pHu_{xx} + D_xH\right)m$$
$$+ \int_{\mathbb{R}} D_pH\left(\epsilon\Delta m + (mD_pH)_x\right).$$

Because $D^2_{xp}H = 0$, we have

$$\int_{\mathbb{R}} D^2_{pp}Hu_{xt}m + D_pHm_t = \int_{\mathbb{R}} D^2_{pp}HD_xHm + \epsilon D^3_{ppp}Hu^2_{xx}m.$$

Price tracking ODE

Accordingly,

$$\dot{\varpi}\int_{\mathbb{R}} D^2_{pp}Hm = -\dot{\mathbf{Q}} - \int_{\mathbb{R}} \left(D^2_{pp}HD_xH + \epsilon D^3_{ppp}Hu^2_{xx}\right) m.$$

Given ϖ, we solve the Hamilton-Jacobi equation, then the Fokker-Planck equation, and define

$$\begin{cases} \dot{\vartheta} = \frac{-\dot{\mathbf{Q}} - \int_{\mathbb{R}} D^2_{pp} H(x, \varpi + u_x) D_x H(x, \varpi + u_x) m + \epsilon D^3_{ppp} H(x, \varpi + u_x) u^2_{xx} m}{\int_{\mathbb{R}} D^2_{pp} H(x, \varpi + u_x) m} \\ \vartheta(0) = \vartheta_0, \end{cases} \tag{5}$$

where ϑ_0 is as before.
Then, (u, m, ϖ) solves (1) if ϖ solves (5).

Proposition

Consider the setting of Problem 1 with $\epsilon > 0$. *Suppose that* $\varpi^n \to \varpi$ *uniformly in* $C([0, T])$. *Let* u^n, m^n, *and* ϑ^n *be the corresponding solutions.*
Then,

- ϑ^n *converges to* ϑ, *uniformly in* $C([0, T])$, *where* ϑ *solves* (5).
- *there exists a constant* C *that depends only on the problem data but not on* ϖ *such that* $\|\vartheta\|_{W^{1,\infty}([0,T])} \leq C$.

Proof

- The bounds in $W^{1,\infty}([0,T])$ for ϑ follow from the assumptions and the Lispchitz bounds for u.
- The uniform convergence of $\varpi_n \to \varpi$ gives the convergence of $u_x^n \to u_x$, almost everywhere and $m_n \to m$ in $C([0,T],\mathcal{P})$.
- Because $D^2_{pp}H$ is bounded from below, we have the convergence of the right-hand side of (5) as follows, for any $\psi \in C([0,T])$,

$$\int_0^T \psi \dot{\vartheta}_n ds \to \int_0^T \psi \dot{\vartheta} ds.$$

- Because the family ϑ_n is equicontinuous, any subsequence has a further convergent subsequence that must converge to ϑ. Thus, $\vartheta^n \to \vartheta$, uniformly.

Proof of Theorem 1 - existence for $\epsilon > 0$

Let $\epsilon > 0$. The map $\varpi \to \vartheta$ determined by the Hamilton-Jacobi, Fokker-Planck and (5) is continuous in $C([0, T])$, bounded, and compact due to the $W^{1,\infty}$ bound for ϖ.
Thus, by Schauder's fixed-point theorem, it has a fixed point.

Proof of Theorem 1 - existence for $\epsilon = 0$

Now, we examine the case $\epsilon = 0$.

- The key difficulty is the continuity of the map $\varpi \to m$ in the case $\epsilon = 0$.
- To overcome this difficulty, we use the vanishing viscosity method.
- The key idea is to fix $(u^\epsilon, m^\epsilon, \varpi^\epsilon)$ solve (1) with $\epsilon > 0$ and justify the limit $\epsilon \to 0$.

Proof of Theorem 1 - existence for $\epsilon = 0$

By the above, ϖ^ϵ and u^ϵ are uniformly locally bounded and Lipschitz. Therefore, as $\epsilon \to 0$, extracting a subsequence if necessary, $\varpi^\epsilon \to \varpi$ and $u^\epsilon \to u$ where u is a viscosity solution. The key difficulty is to show that m solves the transport equation.

Proof of Theorem 1 - existence for $\epsilon = 0$

We introduce a phase-space measure μ^ϵ as follows

$$\int_0^T \int_{\mathbb{R}^2} \psi(x,p,t) d\mu^\epsilon(x,p,t) = \int_0^T \int_{\mathbb{R}} \psi(x, \varpi^\epsilon + u_x^\epsilon, t) m^\epsilon dxdt$$

for all $\psi \in C_b(\mathbb{R} \times \mathbb{R} \times [0,T])$.
Because $m^\epsilon \in C([0,T], \mathcal{P})$ with a modulus of continuity that is uniform in ϵ, as $\epsilon \to 0$, we have $\mu^\epsilon \rightharpoonup \mu$; that is

$$\int_0^T \int_{\mathbb{R}^2} \psi d\mu^\epsilon \to \int_0^T \int_{\mathbb{R}^2} \psi d\mu.$$

Proof of Theorem 1 - existence for $\epsilon = 0$

Moreover,

$$\begin{aligned}&\int_0^T \int_{\mathbb{R}^2} \psi_t - D_p H(x,p) D_x \psi d\mu \\ &= \int_{\mathbb{R}} \psi(x,T) m(x,T) dx - \int_{\mathbb{R}} \psi(x,0) \bar{m}(x) dx.\end{aligned}$$

It just remains to show that $p = \varpi + u_x$, μ-almost everywhere.

Proof of Theorem 1 - existence for $\epsilon = 0$

Next, we fix $\delta > 0$ and consider a standard mollifier η_δ. We define

$$v^\delta = \eta_\delta * u.$$

We note that $|D^2 v^\delta| \leq \frac{C}{\delta^2}$. Then, using the uniform convexity of the Hamiltonian, we get

$$-v_t^\delta + \gamma \eta_\delta * |u_x - v_x^\delta|^2 + H(x, \varpi + v_x^\delta) \leq O(\delta).$$

Proof of Theorem 1 - existence for $\epsilon = 0$

Therefore, $w = v^\delta - u^\epsilon$ satisfies

$$
\begin{aligned}
&-w_t + D_pH(x, \varpi^\epsilon + u_x^\epsilon)w_x - \epsilon w_{xx} \\
&+ \gamma\eta_\delta * |u_x - v_x^\delta|^2 + \gamma|\varpi + v_x^\delta - \varpi^\epsilon - u_x^\epsilon|^2 \leq O(\delta) + O(\frac{\epsilon}{\delta^2}).
\end{aligned}
$$

Integrating with respect to m^ϵ, we conclude that

$$\int_0^T \int_{\mathbb{R}^2} \gamma\eta_\delta * |u_x - v_x^\delta|^2 + \gamma|\varpi + v_x^\delta - p|^2 d\mu^\epsilon \leq O(\delta) + O(\frac{\epsilon}{\delta^2}) + \|v^\delta - u^\epsilon\|_{L^\infty}.$$

Proof of Theorem 1 - existence for $\epsilon = 0$

Next, we let $\epsilon \to 0$, to get

$$\gamma \int_0^T \int_{\mathbb{R}^2} \eta_\delta * |u_x - v_x^\delta|^2 + |\varpi + v_x^\delta - p|^2 d\mu \leq O(\delta).$$

Finally, we let $\delta \to 0$ and conclude that m-almost every point is a point of approximate continuity of u_x. Therefore, $v_x^\delta \to u_x$ almost everywhere. Hence, $p = \varpi + u_x$ μ-almost everywhere. Therefore, we obtain

$$\begin{aligned}
&\int_0^T \int_{\mathbb{R}^2} (\psi_t - D_pH(x, \varpi + u_x) D_x\psi)\, d\mu \\
&= \int_0^T \int_{\mathbb{R}} (\psi_t - D_pH(x, \varpi + u_x) D_x\psi)\, m dx dt \\
&= \int_{\mathbb{R}} \psi(x, T) m(x, T) dx - \int_{\mathbb{R}} \psi(x, 0) \bar{m}(x) dx,
\end{aligned}$$

which gives that m solves (4) with $\epsilon = 0$.

Proposition

Suppose that $\epsilon = 0$ and that the potential, V, and the terminal cost, $\bar{u}$, are convex. Let ϖ be a Lipschitz function. Then, u is differentiable in x for every $x \in \mathbb{R}$. Moreover, u_{xx} is bounded.

Proof

By a direct inspection of the variational problem using the additional convexity assumptions, we see that $u(x, t)$ is convex in x Moreover, u is semiconcave in x. This gives the bound for u_{xx} and the differentiability of u in x.

Corollary

Suppose that $\epsilon = 0$ and that the potential, V, and the terminal cost, $\bar{u}$, are convex. Then, there exists a solution (u, m, ϖ) with u differentiable in x for every x and u_{xx} bounded. Moreover, m is also bounded.

Proof of Theorem 1 - additional regularity for $\epsilon = 0$

Additional regularity for the case $\epsilon = 0$ follows the preceding corollary.

We set

$$\Omega_T = \mathbb{R} \times [0, T],$$

and

$$\begin{aligned}
D =& (C^\infty(\Omega_T) \cap C([0, T], \mathcal{P})) \times (C^\infty(\Omega_T) \cap W^{1,\infty}(\Omega_T)) \times C^\infty([0, T]) \\
D_+ =& \{(m, u, \varpi) \in D \text{ s.t. } m > 0\}, \\
D^b =& \{(m, u, \varpi) \in D \text{ s.t. } m(x, 0) = \bar{m}(x), \; u(x, T) = \bar{u}(x)\}, \\
D^b_+ =& D^b \cap D_+,
\end{aligned}$$

Then, we define $A : D^b_+ \to D$ as

$$\begin{aligned}
A \begin{bmatrix} m \\ u \\ \varpi \end{bmatrix} =& A_1 \begin{bmatrix} m \\ u \\ \varpi \end{bmatrix} + A_2 \begin{bmatrix} m \\ u \\ \varpi \end{bmatrix} \\
=& \begin{bmatrix} u_t + \epsilon u_{xx} \\ m_t - \epsilon m_{xx} \\ 0 \end{bmatrix} + \begin{bmatrix} -H(x, Du + \varpi) \\ -\operatorname{div}(mD_pH(x, \varpi + u_x)) \\ \int_\Omega mD_pH(x, \varpi + u_x)dx + \mathbf{Q}(t) \end{bmatrix}.
\end{aligned}$$

Furthermore, for $w = (m, u, \varpi)$, $\tilde{w} = (\tilde{m}, \tilde{u}, \tilde{\varpi}) \in D$, we set

$$\langle w, \tilde{w} \rangle = \int_{\Omega_T} (m\tilde{m} + u\tilde{u})\, dxdt + \int_0^T \varpi\tilde{\varpi} dt.$$

Then, A is a *monotone operator* if

$$\langle A[w] - A[\tilde{w}], w - \tilde{w} \rangle \geq 0 \quad \text{for all} \quad w, \tilde{w} \in D_+^b.$$

Proposition

Suppose the map $p \mapsto H(x,p)$ is convex. Then A is a monotone operator.

Proof

Let $w = (m, u, \varpi)$, $\tilde{w} = (\tilde{m}, \tilde{u}, \tilde{\varpi}) \in D_+^b$. Then

$$\begin{aligned}
&\langle A_1[w] - A_1[\tilde{w}], w - \tilde{w}\rangle \\
=&\int_{\Omega_T} ((u-\tilde{u})_t + \epsilon\Delta(u-\tilde{u}))(m-\tilde{m}) \\
&+ \int_{\Omega_T} ((m-\tilde{m})_t - \epsilon\Delta(m-\tilde{m}))(u-\tilde{u}) \\
=&0,
\end{aligned}$$

because $u - \tilde{u}$ and $m - \tilde{m}$ vanish at $t = 0, T$.

Proof

Furthermore,

$$
\begin{aligned}
&\langle A_2[w] - A_2[\tilde{w}], w - \tilde{w}\rangle \\
&= \int_{\Omega_T} m\Big(H(\tilde{u}_x + \tilde{\varpi}) - H(u_x + \varpi) - (\tilde{u}_x + \tilde{\varpi} - u_x - \varpi) D_p H(u_x + \varpi)\Big) \\
&+ \int_{\Omega_T} \tilde{m}\Big(H(u_x + \varpi) - H(\tilde{u}_x + \tilde{\varpi}) - (u_x + \varpi - \tilde{u}_x - \tilde{\varpi}) D_p H(\tilde{u}_x + \tilde{\varpi})\Big) \\
\geq& 0,
\end{aligned}
$$

by the convexity of $p \mapsto H(x, p)$.

Proof

Combining the previous inequalities, we conclude that

$$\begin{aligned}&\langle A[w]-A[\tilde{w}],w-\tilde{w}\rangle\\=&\langle A_1[w]-A_1[\tilde{w}],w-\tilde{w}\rangle+\langle A_2[w]-A_2[\tilde{w}],w-\tilde{w}\rangle\geq 0.\end{aligned}$$

Proof of main result, uniqueness

- Let (m, u, ϖ) and $(\tilde{m}, \tilde{u}, \tilde{\varpi})$ solve Problem 1. Note that m and $\tilde{m}$ are absolutely continuous and strictly positive.
- The monotonicity gives

$$\int_0^T \int_{\mathbb{R}} |\varpi + u_x - \tilde{\varpi} - \tilde{u}_x|^2 (\tilde{m} + m) = 0.$$

  Therefore, $\varpi + u_x = \tilde{\varpi} + \tilde{u}_x$ a.e..
- Hence,

$$u_t = \tilde{u}_t,$$

  almost everywhere and, thus, $u = \tilde{u}$.
- Finally, the uniqueness of the Fokker-Planck equation, for $\epsilon > 0$ or for the transport equation, when $\epsilon = 0$ give $m = \tilde{m}$.

Linear-quadratic model I

We set

$$\ell(t,\alpha) = \frac{c}{2}\alpha^2 + \alpha\varpi(t),$$

where c is a constant that accounts for the usage-depreciation of the battery. The corresponding MFG is

$$\begin{cases} -u_t + \frac{(\varpi(t)+u_x)^2}{2c} = 0 \\ m_t - \frac{1}{c}(m(\varpi(t)+u_x))_x = 0 \\ \frac{1}{c}\int_{\mathbb{R}}(\varpi(t)+u_x)m dx = -\mathbf{Q}(t). \end{cases}$$

The stored energy by each agent follows optimal trajectories that solve the Euler Lagrange equation:

$$c\ddot{\mathbf{x}} + \dot{\varpi} = 0.$$

Integrating the previous equation in time, we get

$$\dot{\mathbf{x}}(t) = \frac{1}{c}\left(\theta - \varpi(t)\right), \tag{6}$$

where θ is time independent.

Next, differentiating the Hamilton-Jacobi equation:

$$-(u_x)_t + (u_x + \varpi)\frac{u_{xx}}{c} = 0.$$

Using the previous equation, taking into account the transport equation, give

$$\frac{d}{dt}\int_{\mathbb{R}} u_x m dx = \int_{\mathbb{R}} u_{xt} m + u_x m_t = \int_{\mathbb{R}} u_{xt} m + \frac{1}{c} u_x \left(m(\varpi + u_x)\right)_x$$
$$= \frac{1}{c}\int_{\mathbb{R}} (\varpi + u_x) u_{xx} m - u_{xx} m(\varpi + u_x) dx = 0.$$

Thus, the supply vs demand balance condition becomes

$$\mathbf{Q}(t) = -\frac{1}{c}\int_{\mathbb{R}} (u_x + \varpi) m dx = \frac{1}{c}\left(\Theta - \varpi\right),$$

where

$$\Theta = -\int_{\mathbb{R}} u_x m dx \tag{7}$$

is constant. From the above, we obtain the following linear price-supply relation

$$\varpi = \Theta - c\mathbf{Q}(t). \tag{8}$$

Integrating (6) in time and taking into account the linear price-supply relation (8):

$$\mathbf{x}(T)=\mathbf{x}(t)+\frac{1}{c}\int_t^T(\theta-\varpi(s))ds=x+\frac{T-t}{c}(\theta-\Theta)+\int_t^T\mathbf{Q}(s)ds.$$

Accordingly, u is given by the optimization problem

$$u(x,t)=\inf_\theta\int_t^T\left[\frac{(\theta-\Theta+c\mathbf{Q}(s))^2}{2c}+\frac{1}{c}(\theta-\Theta+c\mathbf{Q}(s))(\Theta-c\mathbf{Q}(s))\right]$$
$$+\bar{u}\left(x+\frac{(\theta-\Theta)}{c}(T-t)+K(t)\right),$$

where

$$K(t)=\int_t^T Q(s)ds.$$

By setting $\mu = \theta - \Theta$, we get

$$u(x,t) = \inf_{\mu} \int_t^T \left[\frac{(\mu + c\mathbf{Q}(s))^2}{2c} + \frac{1}{c}(\mu + c\mathbf{Q}(s))(\Theta - c\mathbf{Q}(s)) \right] ds$$
$$+ \bar{u}\left(x + \frac{\mu}{c}(T-t) + K(t) \right).$$

Thus, given Θ, we determine a function, u^{Θ}, solving the preceding minimization problem:

$$u^{\Theta}(x,t) = \inf_{\mu} \left[\frac{T-t}{2c}\mu^2 + \frac{1}{c}(T-t)\Theta\mu + \int_t^T \left(\Theta - c\frac{\mathbf{Q}(s)}{2} \right) \mathbf{Q}(s) ds \right.$$
$$\left. + \bar{u}\left(x + \frac{\mu}{c}(T-t) + K(t) \right) \right].$$

- The optimality condition in the preceding minimization problem gives
$$\mu + \bar{u}_x\left(\mathbf{x}(T)\right) = -\Theta. \tag{9}$$
Thus, if $\bar{u}$ is a convex function, there is a unique solution, $\mu(\Theta)$ for each given Θ.
- Given Θ, we obtain a solution, u^Θ for the Hamilton-Jacobi equation.
- We use the resulting expression for u^Θ in (7) at $t = 0$ to get
$$\Theta = -\int_{\mathbb{R}} u_x^\Theta(x, 0) m_0(x) dx. \tag{10}$$
- Solving the preceding equation, we obtain Θ and hence ϖ using the price-supply relation, (8).

As an example, we consider the terminal cost

$$\bar{u}(y) = \frac{\gamma}{2}(y - \zeta)^2 .$$

Solving(9), we obtain

$$\mu = -\frac{\gamma(K(t) + x - \zeta) + \Theta}{1 + \gamma\frac{T-t}{c}}.$$

Accordingly, we have

$$u^{\Theta}(x,t) = \frac{\gamma(K(t)+x-\zeta)^2 + \frac{(t-T)}{c}\Theta(2\gamma(K(t)+x-\zeta)+\Theta)}{2\left(1+\gamma\frac{T-t}{c}\right)} + \Theta K(t) - c\int_t^T \frac{\mathbf{Q}^2(s)}{2}ds.$$

Therefore,

$$u_x(x,t) = \gamma \frac{K(t) + x - \zeta - \frac{(T-t)}{c}\Theta}{1 + \gamma \frac{T-t}{c}}$$

Using the previous expression for $t = 0$ in (10), we obtain the following equation for Θ

$$\Theta = -\gamma \frac{K(0) + \bar{x} - \zeta - \frac{T}{c}\Theta}{1 + \gamma \frac{T}{c}}$$

where

$$\bar{x} = \int_{\mathbb{R}} x m_0 dx.$$

Thus,

$$\Theta = -\gamma(K(0) + \bar{x} - \zeta).$$

Therefore, using (8), we obtain

$$\varpi = -\gamma(K(0) + \bar{x} - \zeta) - c\mathbf{Q}.$$

Finally, we use the above results and conclude that each agent dynamics is

$$\begin{cases} \dot{\mathbf{x}} = \frac{(\bar{x}-x)\gamma}{1+\frac{T}{c}\gamma} + \mathbf{Q} \\ \mathbf{x}(0) = x. \end{cases}$$

In alternative, using

$$\dot{\mathbf{x}}(t) = -\frac{\varpi + u_x(\mathbf{x}(t), t)}{c}$$

we have

$$\begin{cases} \dot{\mathbf{x}}(t) = \frac{(\bar{\mathbf{x}}(t) - \mathbf{x}(t))\gamma}{1 + \frac{T-t}{c}\gamma} + \mathbf{Q} \\ \mathbf{x}(0) = x, \end{cases} \tag{11}$$

where

$$\bar{\mathbf{x}}(t) = \int_{\mathbb{R}} x m(x,t) dx.$$

Averaging (11) with respect to m, we obtain

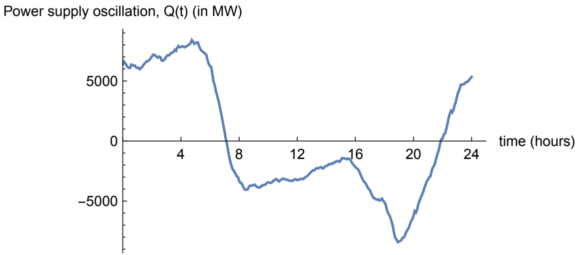
$$\dot{\bar{\mathbf{x}}}(t) = \mathbf{Q}(t),$$

which is simply the conservation of energy. Thus, the trajectory of an individual agent can be computed by solving

$$\begin{cases} \dot{\mathbf{x}}(t) = \frac{(\bar{\mathbf{x}}(t) - \mathbf{x}(t))\gamma}{1 + \frac{T-t}{c}\gamma} + \mathbf{Q}(t) \\ \dot{\bar{\mathbf{x}}}(t) = \mathbf{Q}. \end{cases}$$

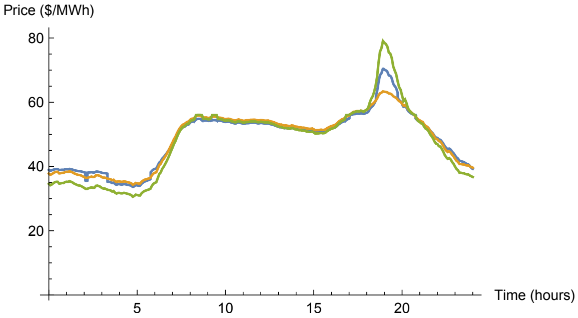
The previous system is a closed system of ordinary differential equations that only involves $\mathbf{Q}$ and the parameters of the problem.

Daily energy consumption in the UK, during a twenty-four hour period.



Normalized electricity production **Q**.

- We compare our price-formation model with the MFG model by De Paola et al, where the energy price is a function of the aggregate consumption.Thus, there may be an energy imbalance.
- we consider the state-independent quadratic cost model; thus, the price depends only on the constant that accounts for battery's wear and tear.
- We calibrate our model against the model by De Paola et al. using a least squares approach.



24-hour electricity price. In green, energy's price when no batteries are connected to the grid. In blue, price with batteries connected as predicted by the model by De Paola et al. In yellow, the price corresponding to our model.