Part II: Boltzmann mean field games

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• **Kinetic theory**: was originally developed to describe the statistical evolution of a non-equilibrium many-particle system in phase space.

• **Ludwig Boltzmann** made significant contributions in kinetic theory by investigating the properties of dilute gases.
Kinetic theory

- **Kinetic theory**: was originally developed to describe the statistical evolution of a non-equilibrium many-particle system in phase space.
- Ludwig Boltzmann made significant contributions in kinetic theory by investigating the properties of dilute gases.
The classic Boltzmann equation describes the evolution of the one-particle distribution function of a rarefied monatomic gas. Let $f = f(x, v, t)$ denote the probability to find a particle at position $x \in \mathbb{R}^3$ with velocity $v \in \mathbb{R}^3$ at time $t > 0$. Then the equation reads as:

$$\frac{\partial f}{\partial t}(x, v, t) = - v \cdot \nabla_x f(x, v, t) + Q(f, f)(x, v, t)$$

The original Boltzmann equation was derived under the following assumptions:

- **Binary interactions**: such as in dilute gases, where interactions of more than two particles can be neglected.
- **Elastic collisions** ⇒ conservation of mass and momentum.
- **Collisions** involve only **uncorrelated particles**.
**Elastic binary collision:** Given two particles with velocity \( v \) and \( w \) the post-collisional velocities \( v^* \) and \( w^* \) we have

\[
v^* = \frac{1}{2} (v + w + |v - w|n)
\]
\[
w^* = \frac{1}{2} (v + w - |v - w|n),
\]

where \( n \) is the unit normal vector.

**Conservation of momentum and kinetic energy:**

\[
v + w = v^* + w^*
\]
\[
|v|^2 + |w|^2 = |v^*|^2 + |w^*|^2
\]

Collision operator in the case of hard spheres:

\[
Q(f, g)(v) = \int_{\mathbb{R}^3 \times S^2} B((v - w) \cdot n)(f(v^*)g(w^*) - f(v)g(v))dwdn.
\]

where \( B \) is the collision kernel.
Fundamental properties of the collision operator

- Conservation of mass, momentum and energy
  \[ \int_{\mathbb{R}^3} Q(f, f) \psi(v) dv = 0 \text{ for } \psi = 1, v, |v|^2. \]

- H-Theorem: The entropy \(- \int_{\mathbb{R}^3} f \log fdv\) is non-decreasing in time. That is
  \[ -\frac{d}{dt} \int_{\mathbb{R}^3} f \log fdv = -\int_{\mathbb{R}^3} Q(f, f) \log(f) dv \geq 0. \]

Any equilibrium distribution, which is a maximum of the entropy, has to be of Maxwellian form
\[ M(\rho, u, T)(v) = \frac{\rho}{(2\pi T)^{d/2}} \exp\left( -\frac{|u - v|^2}{2T} \right), \]
where \(\rho, u\) and \(T\) are the density, mean velocity and temperature of the gas
\[ \rho = \int_{\mathbb{R}^3} f(v) dv, \quad u = \frac{1}{\rho} \int_{\mathbb{R}^3} vf(v) dv, \quad T = \frac{1}{3\rho} \int_{\mathbb{R}^3} |u - v|^2 f(v) dv. \]
From molecules to agents

Classic kinetic theory
- large number of molecules
- described by their position and velocity
- velocity is changed in collision

Wealth distribution in simple markets
- large number of trading agents
- each characterised by its wealth
- goods are exchanged in 'collisions'

Opinion formation in a society
- large networks or people or a society
- each person has an opinion on a certain topic
- opinion is changed due to interactions


Knowledge diffusion and growth

Lucas and Moll’s model setup:

- Consider a continuum of individuals, which are characterised by their knowledge level $z \in \mathbb{R}^+$.  
- Let $s = s(z, t)$ denote the time that an individual with knowledge level $z$ spends on learning.  
- Each individual has one unit of time, which he/she can split between producing goods with the knowledge already obtained or meeting others to enhance their knowledge level.

If two individuals with knowledge level $z$ and $z'$ meet, they exchange ideas.

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- **Let** $s = s(z, t)$ **denote the time that an individual with knowledge level** $z$ **spends on learning.**
- **Each individual has one unit of time, which he/she can split between producing goods with the knowledge already obtained or meeting others to enhance their knowledge level.**

\[ \Rightarrow \text{their post-collision knowledge corresponds to} \]

\[ z^* = \max(z, z'). \]

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Knowledge diffusion and growth

Evolution of the distribution of agents $f = f(z, t)$ with respect to their knowledge level $z$:

$$\partial_t f(z, t) = -\alpha(s(z, t)) f(z, t) \int_z^\infty f(y, t) dy + f(z, t) \int_0^z \alpha(s(y, t)) f(y, t) dy.$$

- The function $\alpha = \alpha(s)$ is the interaction probability of an individual, which spends an $s$-th fraction of its time on learning (also called the learning function). Possible choices:

  $$\alpha(s) = \alpha_0 s^n, \quad n \in (0, 1).$$

- Individual productivity:

  $$y(t) = (1 - s(z, t))z$$

- Total earnings in an economy:

  $$Y(t) = \int_0^\infty (1 - s(z, t))zf(z, t) \, dz.$$
How much time should one spend on learning?

Each individual wants to maximise its earnings by choosing the optimal fraction of learning time \( s = s(z, t) \):

\[
V(x, t') = \max_{s \in S} \left[ \int_{t'}^{T} \int_{0}^{\infty} e^{-r(t-t')} (1 - s(z, t)) z \rho_x(z, t) dz dt \right],
\]

subject to

\[
\partial_t \rho_x(z, t) = -\alpha(s) \rho_x(z, t) \int_{z}^{\infty} f(y, t) dy + f(z, t) \int_{0}^{z} \alpha(s) \rho_x(y, t) dy
\]

with \( \rho_x(z, t') = \delta_x \).

Hamilton-Jacobi Bellman (HJB) equation for the value function \( V = V(z, t) \):

\[
\partial_t V(z, t) - rV(z, t)
+ \max_{s \in S} \left( (1 - s(z, t)) z + \alpha(s) \int_{z}^{\infty} [V(y, t) - V(z, t)] f(y, t) dy \right) = 0,
\]

where \( S \) denotes the set of admissible controls \( S = \{ s : \mathcal{I} \times [0, T] \to [0, 1] \} \) and \( \mathcal{I} = \mathbb{R}^+ \) or \( \mathcal{I} = [0, \bar{z}] \).
The BMFG system

\[ \partial_t f(z, t) = -\alpha(S(z, t)) f(z, t) \int_{\infty}^{\infty} f(y, t) dy + f(z, t) \int_{0}^{z} \alpha(S(y, t)) f(y, t) dy. \]

\[ \partial_t V(z, t) - rV(z, t) = \]

\[ - \max_{s \in S} \left[ (1 - s(z, t)) z - \alpha(s(z, t)) \int_{z}^{\infty} [V(y, t) - V(z, t)] f(y, t) dy \right] \]

\[ S(z, t) = \arg \max_{s \in S} \left[ (1 - s(z, t)) z + \alpha(s(z, t)) \int_{z}^{\infty} [V(y, t) - V(z, t)] f(y, t) dy \right], \]

\[ f(z, 0) = f_0(z), \]

\[ V(z, T) = 0. \]

Highly nonlinear problem: Boltzmann type equation describing the evolution of individuals forward in time and a HJB equation for their optimal strategy backward in time.
Special case $\alpha = \alpha_0$

In this case the equations decouple and the maximum of

$$(1 - s(z, t))z + \alpha(s) \int_z^\infty [V(y, t) - V(z, t)]f(y, t)dy$$

is $S(z, t) = 0$.

The Boltzmann equation can be written in terms of the cdf $F(z, t) = \int_0^z f(y, t)dy$:

$$\partial_t F(z, t) = -\alpha_0 (1 - F(z, t))F(z, t).$$

Then the function $G(z, t) = 1 - F(z, t)$ satisfies the Fisher KPP equation.
Analysis of the Boltzmann equation

First we consider the Boltzmann type equation for a given learning function $\alpha = \alpha(z, t)$:

$$\partial_t f(z, t) = -\alpha(z, t)f(z, t) \int_z^{\bar{z}} f(y, t) \, dy + f(z, t) \int_0^z \alpha(y, t)f(y, t) \, dy,$$

$$f(z, 0) = f_0(z),$$

on the interval $\mathcal{I} = [0, \bar{z}]$, where $f_0 \in L^\infty(\mathcal{I})$ is a given probability density.

Theorem

Let $\alpha = \alpha(z, t) \in L^1(\mathcal{I}) \times L^\infty([0, T])$. Then the Boltzmann equation has a global in time solution $f = f(z, t) \in L^1(\mathcal{I}) \times L^\infty([0, T])$.

\[\text{M. Burger, A. Lorz and MTW, On a Boltzmann mean-field model for knowledge growth, SIAM Appl Math 76(5), 2016}\]
But if $f_0$ has compact support....

**Theorem**

Let $\alpha(z, t) \geq \alpha > 0$ and $\bar{z} \in \text{supp}(f)$, then

$$f(\cdot, t) \rightharpoonup^* \delta_{\bar{z}}.$$
Consider the HJB equation for a given \( f \in C(0, T, L^1) \) on \( I = \mathbb{R}^+ \):

\[
\partial_t V(z, t) - rV(z, t) = -\max_{s \in S} [(1 - s(z, t))z - \alpha(s(z, t))V(z, t)((1 - H) * f) \\
+ \alpha(s(z, t))((1 - H) * (Vf))]
\]

\( V(z, T) = 0. \)

Assumptions:

(A1) Let the final data \( V(\cdot, T) \) be non-negative and non-decreasing.

(A2) Let the interaction function satisfy:

\[
\alpha : [0, 1] \rightarrow \mathbb{R}^+, \alpha \in C^\infty([0, 1]), \alpha(0) = 0, \alpha'(0) = \infty, \alpha'' < 0 \text{ and } \alpha \text{ monotone.}
\]
The full BMFG system

Theorem

Let $f \in C(0, T, L^1)$ be given and $\alpha$ satisfies assumption (A2). Then there exists a unique solution $V \in C(0, T, L^\infty)$ of the HJB equation with $V(z, T) = 0$. Moreover, let $\tilde{V}$ be a solution of the HJB equation with $\tilde{f}$. Then there exist constants $m$ and $D$ (independent of $\tilde{V}$ and $\tilde{f}$) such that

$$\|V - \tilde{V}\|_\infty \leq De^{mt}\|f - \tilde{f}\|_{C(0, T, L^1)}\|\tilde{V}\|_\infty.$$ 

Theorem

Let $f_0(z) \in L^\infty(\mathcal{I})$ be a probability density and (A1) and (A2) be satisfied. If $
\lim_{s \to 0} \frac{(\alpha')^3}{\alpha'} < \infty$, then the fully coupled Boltzmann mean field game system on $\mathcal{I} = \mathbb{R}^+$ has a unique local in time solution.
Endogenous growth theory

- Economic growth describes the increase of the inflation-adjusted market value of the goods and services produced in an economy over time - commonly measured in the gross domestic product (GDP).
- The GDP of most developed countries has grown about two percent since World War II.

Economists are interested in solutions which correspond to sustained growth - so called balanced growth path (BGP) solutions.
Endogenous growth theory

- Economic growth describes the increase of the inflation-adjusted market value of the goods and services produced in an economy over time - commonly measured in the gross domestic product (GDP).

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Can we find BGP solutions for the BMFG system?
Balanced growth path solutions

Let us assume there exists a growth parameter $\gamma \in \mathbb{R}^+$ and consider the re-scaling:

$$f(z, t) = e^{-\gamma t}\phi(ze^{-\gamma t}), \quad V(z, t) = e^\gamma t v(ze^{-\gamma t}) \text{ and } s(z, t) = \sigma(ze^{-\gamma t})$$

Rescaled BMFG system in $(v, \phi, \sigma) = (v(x), \phi(x), \sigma(x))$ with $x = e^{-\gamma t}z$ reads as:

$$-\gamma \phi(x) - \gamma \phi'(x)x = \phi(x) \int_0^x \alpha(\sigma(y))\phi(y) \, dy - \alpha(\sigma(x))\phi(x) \int_x^\infty \phi(y) \, dy$$

$$(r - \gamma)v(x) + \gamma v'(x)x = \max_{\sigma \in \Xi} \left\{ (1 - \sigma)x + \alpha(\sigma) \int_x^\infty [v(y) - v(x)]\phi(y) \, dy \right\}$$

where $\Xi = \{ \sigma : \mathbb{R}^+ \to [0, 1] \}$ denotes the set of admissible controls.

Re-scaling results in exponential growth of the overall production:

$$Y(t) = e^{\gamma t} \int_0^\infty [1 - \sigma(x)]x\phi(x) \, dx.$$
Existence of BGP solutions

Does such a growth parameter $\gamma$ exist?
Existence of BGP solutions

The initial commutative distribution function $F(z, 0) = \int_0^z f_0(z)dz$ has a Pareto tail, if there exist constants $k, \theta \in \mathbb{R}^+$ such that

$$\lim_{z \to \infty} \frac{1 - F(z, 0)}{z^{-1/\theta}} = k. \quad (P)$$

Lemma

Let $(P)$ be satisfied. Then $F = F(z, t)$ has a Pareto tail with the same decay rate $\theta$ for all times $t \in [0, T]$.

Theorem

Let $(P)$ be satisfied and $\alpha = \alpha_0$. then there exists a unique BGP solution $(\Phi, v, 0)$ and a scaling constant $\gamma$ given by

$$\gamma = \alpha_0 \theta \int_\mathbb{I} f_0(z)dz, \quad \Phi(x) = \frac{1}{1 + kx^{-1/\theta}} \quad \text{with} \quad \Phi(x) = \int_0^x \phi(y)dy.$$
Existence of BGP solutions

Degenerate solution:

\[ \gamma = 0, \ v = \frac{x}{r} \text{ and } S \equiv 0 \Rightarrow \Phi(x) = 1 \text{ for } x > 0 \]

\[ \Rightarrow \phi(x) = \delta_0 \]

Challenge for the analysis and numerics: construct a solution \( \Phi \) with a strictly positive Pareto tail \( k > 0 \).

Variable transformation:

\[ \zeta := x^{-1/\theta} \text{ and } K(\zeta) := \frac{1 - \Phi(x)}{\gamma \zeta}, \]

where \( \theta \) and \( k \) denote the Pareto indices. We solve the correspondingly transformed equation with an initial condition at \( \zeta = 0 \) (determined by the Pareto tail condition).
Existence of BGP solutions

Theorem

Let $r > \theta \alpha(1)$ and $\tilde{k} > 0$, then the BGP system has a non-trivial solution satisfying the Pareto-tail condition with $k = \frac{\gamma}{\theta} \tilde{k}$.

Idea of proof: Fixed point argument

- Solve equations for $(\Phi, \gamma)$ given $(\nu, S)$.
- Solve equations for $(\nu, S)$ and given $(\Phi, \gamma)$.
Achdou et al. postulate that diffusion

• enhances growth in the case of a Pareto tail
• and leads to exponential growth also for compactly supported initial values.

Special case $\alpha = \alpha_0$:  

• The Fisher KPP equation (with diffusion) admits travelling wave solutions
  $$G(z, t) = \Phi(z - \gamma t)$$
  with a minimal wave speed $\gamma = 2\sqrt{\nu \alpha_0}$.
• Travelling waves correspond to BGP solutions (in logarithmic variables).

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Diffusion and knowledge growth

Let the knowledge of each agent evolve by a geometric Brownian motion (independent of the time spent on learning), that is

$$Z_t = \exp(\sqrt{2\nu} W_t)$$

where $W_t$ is a Wiener process, independently for each agent.

Then the corresponding Boltzmann mean field game system with diffusion reads as:

$$\partial_t f(z, t) - \nu \partial_{zz}(z^2 f(z, t)) + \nu \partial_z (zf(z, t)) = f(z, t) \int_0^z \alpha(S(y, t))f(y, t)dy$$

$$- \alpha(S(z, t))f(z, t) \int_z^\infty f(y, t)dy,$$

$$\partial_t V(z, t) + \nu z^2 \partial_{zz} V(z, t) + \nu z \partial_z V(z, t) - rV(z, t) =$$

$$- \max_{s \in S} \left[(1 - s)z + \alpha(s) \int_z^\infty [V(y, t) - V(z, t)]f(y, t)dy \right].$$
Assuming the existence of the scaling parameter $\gamma$ for a balanced growth path we rewrite the system in the known BGP variables $(\phi, \sigma, \nu)$

$$
- \gamma \phi(x) - \gamma x \phi'(x) - \nu (x^2 \phi(x))'' + \nu (x \phi(x))' = \\
\phi(x) \int_0^x \alpha(\sigma(y))\phi(y)dy - \alpha(\sigma(x))\phi(x) \int_x^\infty \phi(y)dy
$$

$$(r - \gamma)\nu(x) + \gamma x \nu'(x) - \nu x^2 \nu''(x) - \nu x \nu'(x) = \\
- \max_{\sigma \in \Sigma} \left[ (1 - \sigma)x + \alpha(\sigma) \int_x^\infty [\nu(y) - \nu(x)]\phi(y)dy \right].$$

Achdou et al.\textsuperscript{4} postulated the existence of BGP solutions to this system with a rescaling parameter $\gamma$ given by

$$
\gamma = 2 \sqrt{\nu \int_0^\infty \alpha(\sigma(y))\phi(y)dy}.
$$

This model is quite simplistic....

... since meetings between individuals are completely asymmetric. Individuals can only increase their knowledge through active search, the 'smarter' individual gains nothing in the meeting.

**Symmetric meetings:** if an individual with knowledge level $y$ initiated the meeting, the one with the higher knowledge level $z$ may learn with a probability $\beta$. This gives:

$$
\frac{\partial f}{\partial t} = -f(z, t) \int_{\infty}^{\infty} \left[ \alpha(s(z, t)) + \beta\alpha(s(y, t)) \right] f(y, t) dy
$$

$$
+ f(z, t) \int_{0}^{z} \left[ \alpha(s(y, t)) + \beta\alpha(s(z, t)) \right] f(y, t) dy.
$$
If two individuals meet, the one with the lower knowledge level \( z \) adopts the higher knowledge level \( y \) with a certain probability \( k\left(\frac{y}{z}\right) \). Then

\[
\partial_t f(z, t) = f(z, t) \int_0^z \alpha(s(y, t)) f(y, t) k\left(\frac{z}{y}\right) dy \\
- \alpha(s(z, t)) f(z, t) \int_z^\infty f(y, t) k\left(\frac{y}{z}\right) dy.
\]

Possible choice for \( k \):

\[
k(x) = \delta + (1 - \delta) x^{-\kappa} \quad \text{where } \kappa > 0.
\]

Alternative interpretation of \( k \): interaction probability depends on the distance between knowledge levels.
In the case of a constant interaction rate $\alpha = \alpha_0$ the CDF $F = F(z, t)$ evolves according to

$$\partial_t F(z, t) = -\alpha(1 - F(z, t))F(z, t).$$

Then

$$\lim_{t \to \infty} F(z, t) = \frac{1}{1 + kx^{-\frac{1}{\theta}}},$$

**Exogenous knowledge shock:** undiscovered ideas modelled by a CDF $G = G(z)$

$$\partial_t F(z, t) = -\alpha(1 - F(z, t))F(z, t) - \beta(1 - G(z))F(z, t)$$
Asymptotic behaviour

Depends on the 'tails' of $F$ and $G$:

- If neither $F(z, 0)$ nor $G(z)$ has a Pareto tail there will be no growth in the long run.
- If $F(z, 0)$ has a fatter tail than $G(z)$ then the BGP has a growth rate of $\gamma = \alpha \theta$ (external ideas do not influence the asymptotic behaviour).
- If $G(z)$ has a fatter tail (denoted by $\frac{1}{\xi}$) the BGP path grows at a rate $\gamma = \alpha \xi$ and the asymptotic distribution satisfies

$$\lim_{t \to \infty} F(z, t) = \frac{1}{1 + \frac{\beta}{\alpha} m x^{-\frac{1}{\xi}}}$$

where $m > 0$.
- If they have the same Pareto tail then the asymptotic distribution satisfies

$$\lim_{t \to \infty} F(z, t) = \frac{1}{1 + \left[k + \frac{\beta}{\alpha} m \right] x^{-\frac{1}{\theta}}}$$

with $m > 0$. 
The time-dependent solver

The solver is based on a fixed point scheme:

1. **Given** $f_0$ and $S^k$ solve

   \[
   \frac{1}{\tau} (f_{i}^{k+1} - f_{i}^{k}) - \frac{\nu}{h^2} (z_{i+\frac{1}{2}} f_{i+1}^{k+1} - (z_{i+\frac{1}{2}} + z_{i-\frac{1}{2}}) f_{i}^{k+1} + z_{i-\frac{1}{2}} f_{i+1}^{k+1}) \\
   + \frac{\nu}{h} (z_{i+\frac{1}{2}} f_{i+1}^{k+1} - z_{i-\frac{1}{2}} f_{i-1}^{k+1}) = g_1(f^k, S^k),
   \]

   for every time $t^k = k\tau$, $k > 1$, using a trapezoidal rule to approximate the integrals in $g_1$.

2. **Update** the maximizer $S^k$.

3. **Given** the evolution of the density $f^k$ and the maximizer $S^k$ solve the HJB equation

   \[
   \frac{1}{\tau} (V_{i}^{k+1} - V_{i}^{k}) + \frac{\nu}{h^2} z_{i}^2 (V_{i+1}^{k} - 2V_{i}^{k} + V_{i-1}^{k}) + \frac{\nu}{h} z_{i} (V_{i}^{k} - V_{i-1}^{k}) \\
   - rV_{i}^{k} = g_2(S^{k+1}, f^{k+1}, V^{k+1}),
   \]

   backward in time using a trapezoidal rule to approximate $g_2$.

4. **Go to step (1) until convergence.**
The BGP solver

The BGP solver is also based on a fixed point scheme:

1. Given $\phi^{n+1}$, $\gamma^n$ and $\sigma^n$ solve

\[
(r - \gamma^n)v_{i}^{n+1} + \frac{(\gamma^n - \nu)}{h}x_i(\nu_{i}^{n+1} - \nu_{i-1}^{n+1}) - \frac{\nu x_i^2}{h^2}(\nu_{i+1}^{n+1} - 2\nu_{i}^{n+1} + \nu_{i-1}^{n+1}) = -q_2(\phi_{n+1}, \nu^n, \sigma^n)
\]

using the trapezoidal rule to approximate the right hand side $q_2$.

2. Compute the maximum $\sigma^{n+1}$ and update the growth parameter $\gamma^{n+1}$ via

\[
\gamma^{n+1} = 2(\nu \int_{I} \alpha(\sigma^{n+1}(y))\phi^{n+1}(y) dy) \frac{1}{2}.
\]

3. Given $\nu^n$, $\sigma^n$ and $\gamma^n$ solve

\[
-(\gamma^n - \nu)\phi_{i}^{n+1} - \frac{(\gamma^n - \nu)}{h}x_i(\phi_{i+1}^{n+1} - \phi_{i}^{n+1}) - (\Xi_i - \alpha(\sigma^n)(1 - \Phi_i))\phi_{i}^{n+1}
\]

\[
- \frac{\nu}{h^2}(x_{i+\frac{1}{2}}^2\phi_{i+1}^{n+1} - (x_{i+\frac{1}{2}}^2 + x_{i-\frac{1}{2}}^2)\phi_{i}^{n+1} + x_{i-\frac{1}{2}}^2\phi_{i-1}^{n+1}) = 0,
\]

subject to the constraint $(\phi_{1}^{n+1} + \phi_{2}^{n+1} + \ldots + \frac{1}{2} \phi_{N}^{n+1})h = 1$ (normalisation $\int_{0}^{\bar{z}} \phi(y) dy = 1$ plus $\phi_{0}^{n+1} = 0$).

4. Go to (1) until convergence.
Simulations

(a) Transient vs. BGP

(b) Linear growth

Figure: Evolution of the production function $Y = Y(t)$ in time for different choices of $n$ and $\theta$
Numerical simulations

- *Simulations of the time-dependent problem as well as the BGP system are performed iteratively.*
- *We solve the systems on a bounded domain with no-flux boundary conditions.*
- *To exclude degenerate BGP solutions we set*
  \[ \phi_0 = 0. \]
- *We use a finite difference discretization in space and approximate the integrals using the trapezoidal rule.*
BGP simulations with knowledge diffusion

(a) Agent distribution for different values of $\nu$.

(b) Fraction of time $\sigma$ devoted to learning for different values of $\nu$.

(c) Production function $Y$. 
This model is quite simplistic....

... since meetings between individuals are completely asymmetric. Individuals can only increase their knowledge through active search, the 'smarter' individual gains nothing in the meeting.

**Symmetric meetings:** if an individual with knowledge level $y$ initiated the meeting, the one with the higher knowledge level $z$ may learn with a probability $\beta$. This gives:

$$\frac{\partial f}{\partial t} = - f(z, t) \int_{z}^{\infty} [\alpha(s(z, t)) + \beta \alpha s(y, t)] f(y, t) \, dy$$

$$+ f(z, t) \int_{0}^{z} [\alpha(s(y, t)) + \beta \alpha(s(z, t))] f(y, t) \, dy.$$
Limits to learning

If two individuals meet, the one with the lower knowledge level $z$ adopts the higher knowledge level $y$ with a certain probability $k(\frac{y}{z})$. Then

$$\partial_t f(z, t) = f(z, t) \int_0^z \alpha(s(y, t)) f(y, t) k\left(\frac{y}{z}\right) dy - \alpha(s(z, t)) f(z, t) \int_z^\infty f(y, t) k\left(\frac{y}{z}\right) dy.$$ 

Possible choice for $k$:

$$k(x) = \delta + (1 - \delta)x^{-\kappa} \text{ where } \kappa > 0.$$ 

Alternative interpretation of $k$: interaction probability depends on the distance between knowledge levels.
References:

References:


*Thanks for the attention and have a great time at rest of the school!*