Part II: Boltzmann mean field games

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- *Kinetic theory: was originally developed to describe the statistical evolution of a non-equilibrium many-particle system in phase space.*
- *Ludwig Boltzmann made significant contributions in kinetic theory by investigating the properties of dilute gases.*

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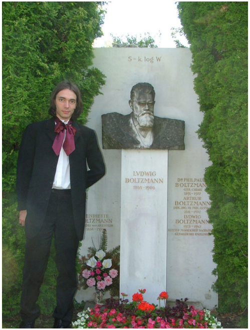

The Boltzmann equation

The classic Boltzmann equation describes the evolution of the one-particle distribution function of a rarefied monatomic gas.
Let $f = f(x, v, t)$ denote the probability to find a particle at position $x \in \mathbb{R}^3$ with velocity $v \in \mathbb{R}^3$ at time $t > 0$. Then the equation reads as:

$$\frac{\partial f}{\partial t}(x, v, t) = - \underbrace{v \cdot \nabla_x f(x, v, t)}_{\text{free particle transport}} + \underbrace{\mathcal{Q}(f, f)(x, v, t)}_{\text{effects of binary collisions}}$$

The original Boltzmann equation was derived under the following assumptions:

- *Binary interactions: such as in dilute gases, where interactions of more than two particles can be neglected.*
- *Elastic collisions $\Rightarrow$ conservation of mass and momentum.*
- *Collisions involve only uncorrelated particles.*

Collisions

Elastic binary collision: Given two particles with velocity v and w the post-collisional velocities v^* and w^* we have

$$v^* = \frac{1}{2}(v + w + |v - w|n)$$
$$w^* = \frac{1}{2}(v + w - |v - w|n),$$

where n is the unit normal vector.
Conservation of momentum and kinetic energy:

$$v + w = v^* + w^*$$
$$|v|^2 + |w|^2 = |v^*|^2 + |w^*|^2$$

Collision operator in the case of hard spheres:

$$\mathcal{Q}(f, g)(v) = \int_{\mathbb{R}^3 \times S^2} B((v - w) \cdot n)(f(v^*)g(w^*) - f(v)g(v))dwdn.$$

where B is the collision kernel.

Fundamental properties of the collision operator

- Conservation of mass, momentum and energy

$$\int_{\mathbb{R}^3} \mathcal{Q}(f,f)\psi(v)dv = 0 \text{ for } \psi = 1, v, |v|^2.$$

- H-Theorem: The entropy $-\int_{\mathbb{R}^3} f \log f dv$ is non-decreasing in time. That is

$$-\frac{d}{dt}\int_{\mathbb{R}^3} f \log f dv = -\int_{\mathbb{R}^3} \mathcal{Q}(f,f)\log(f)dv \geq 0.$$

Any equilibrium distribution, which is a maximum of the entropy, has to be of Maxwellian form

$$M(\rho, u, T)(v) = \frac{\rho}{(2\pi T)^{\frac{d}{2}}} \exp(-\frac{|u-v|^2}{2T}),$$

where ρ, u and T are the density, mean velocity and temperature of the gas

$$\rho = \int_{\mathbb{R}^3} f(v)dv, \quad u = \frac{1}{\rho}\int_{\mathbb{R}^3} vf(v)dv, \quad T = \frac{1}{3\rho}\int_{\mathbb{R}^3} |u-v|^2 f(v)dv.$$

From molecules to agents

Classic kinetic theory

- *large number of molecules*
- *described by their position and velocity*
- *velocity is changed in collision*

Wealth distribution in simple markets

- *large number of trading agents*
- *each characterised by its wealth*
- *goods are exchanged in 'collisions'*

Opinion formation in a society

- *large networks or people or a society*
- *each person has a opinion on a certain topic*
- *opinion is changed due to interactions*

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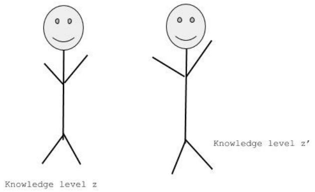
Knowledge diffusion and growth[1]

Lucas and Moll's model setup:

- *Consider a continuum of individuals, which are characterised by their knowledge level $z \in \mathbb{R}^+$.*
- *Let $s = s(z, t)$ denote the time that an individual with knowledge level z spends on learning.*
- *Each individual has one unit of time, which he/she can split between producing goods with the knowledge already obtained or meeting others to enhance their knowledge level.*



If two individuals with knowledge level z and z' meet, they exchange ideas.

[1] R. E. Lucas Jr and B. Moll. Knowledge growth and the allocation of time. *Journal of Political Economics*, 2014

Knowledge diffusion and growth[1]

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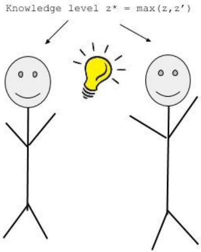


$\Rightarrow$ their post-collision knowledge corresponds to

$$z^* = \max(z, z').$$

[1] R. E. Lucas Jr and B. Moll. Knowledge growth and the allocation of time. *Journal of Political Economics*, 2014

Knowledge diffusion and growth

Evolution of the distribution of agents $f = f(z,t)$ with respect to their knowledge level z:

$$\partial_t f(z,t) = -\alpha(s(z,t))f(z,t)\int_z^\infty f(y,t)dy + f(z,t)\int_0^z \alpha(s(y,t))f(y,t)dy.$$

- *The function $\alpha = \alpha(s)$ is the interaction probability of an individual, which spends an s-th fraction of its time on learning (also called the learning function).* *Possible choices:*

$$\alpha(s) = \alpha_0 s^n, \quad n \in (0,1).$$

- *Individual productivity:*

$$y(t) = (1 - s(z,t))z$$

- *Total earnings in an economy:*

$$Y(t) = \int_0^\infty (1 - s(z,t))zf(z,t)\,dz.$$

How much time should one spend on learning ?

Each individual wants to maximise its earnings by choosing the optimal fraction of learning time $s = s(z,t)$:

$$V(x,t') = \max_{s\in\mathcal{S}}\Big[\int_{t'}^{T}\int_0^\infty e^{-r(t-t')}(1-s(z,t))z\rho_x(z,t)dzdt\Big],$$

subject to

$$\partial_t\rho_x(z,t) = -\alpha(s)\rho_x(z,t)\int_z^\infty f(y,t)\,dy + f(z,t)\int_0^z \alpha(s)\rho_x(y,t)\,dy$$

with $\rho_x(z,t') = \delta_x$.

Hamilton-Jacobi Bellman (HJB) equation for the value function $V = V(z,t)$:

$$\begin{aligned}&\partial_t V(z,t) - rV(z,t)\\ &\qquad + \max_{s\in\mathcal{S}}\Big((1-s(z,t))z + \alpha(s)\int_z^\infty [V(y,t)-V(z,t)]f(y,t)dy\Big) = 0,\end{aligned}$$

where $\mathcal{S}$ denotes the set of admissible controls $\mathcal{S} = \{s : \mathcal{I}\times[0,T]\to[0,1]\}$ and $\mathcal{I} = \mathbb{R}^+$ or $\mathcal{I} = [0,\bar{z}]$.

The BMFG system

$$\partial_t f(z,t) = -\alpha(S(z,t))f(z,t)\int_z^\infty f(y,t)dy + f(z,t)\int_0^z \alpha(S(y,t))f(y,t)dy.$$

$$\partial_t V(z,t) - rV(z,t) =$$

$$-\max_{s\in\mathcal{S}}\left[(1-s(z,t))z - \alpha(s(z,t))\int_z^\infty [V(y,t)-V(z,t)]f(y,t)dy\right]$$

$$S(z,t) = \arg\max_{s\in\mathcal{S}}\left[(1-s(z,t))z + \alpha(s(z,t))\int_z^\infty [V(y,t)-V(z,t)]f(y,t)dy\right],$$

$$f(z,0) = f_0(z),$$

$$V(z,T) = 0.$$

Highly nonlinear problem: Boltzmann type equation describing the evolution of individuals forward in time and a HJB equation for their optimal strategy backward in time.

Special case $\alpha = \alpha_0$

In this case the equations decouple and the maximum of

$$(1 - s(z,t))z + \alpha(s) \int_z^\infty [V(y,t) - V(z,t)] f(y,t) dy$$

is $S(z,t) = 0$.

The Boltzmann equation can be written in terms of the cdf $F(z,t) = \int_0^z f(y,t)dy$:

$$\partial_t F(z,t) = -\alpha_0 (1 - F(z,t)) F(z,t).$$

Then the function $G(z,t) = 1 - F(z,t)$ *satisfies the Fisher KPP equation.*

Analysis of the Boltzmann equation [2]

First we consider the Boltzmann type equation for a given learning function $\alpha = \alpha(z,t)$*:*

$$\partial_t f(z,t) = -\alpha(z,t) f(z,t) \int_z^{\bar{z}} f(y,t)\, dy + f(z,t) \int_0^z \alpha(y,t) f(y,t) dy,$$
$$f(z,0) = f_0(z),$$

on the interval $\mathcal{I} = [0, \bar{z}]$*, where* $f_0 \in L^\infty(\mathcal{I})$ *is a given probability density.*

Theorem

Let $\alpha = \alpha(z,t) \in L^1(\mathcal{I}) \times L^\infty([0,T])$*. Then the Boltzmann equation has a global in time solution* $f = f(z,t) \in L^1(\mathcal{I}) \times L^\infty([0,T])$*.*

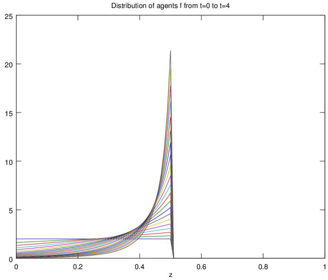
[2] M. Burger, A. Lorz and MTW, On a Boltzmann mean-field model for knowledge growth, SIAM Appl Math 76(5), 2016

But if f_0 has compact support....

Theorem

Let $\alpha(z,t) \geq \underline{\alpha} > 0$ *and* $\bar{z} \in supp(f)$*, then*

$$f(\cdot, t) \rightharpoonup^* \delta_{\bar{z}}.$$



The Hamilton-Jacobi-Bellman equation

Consider the HJB equation for a given $f \in C(0,T,L^1)$ *on* $\mathcal{I} = \mathbb{R}^+$*:*

$$\begin{aligned}\partial_t V(z,t) - rV(z,t) = -\max_{s\in\mathcal{S}} \big[&(1-s(z,t))z - \alpha(s(z,t))V(z,t)((1-H)*f) \\ &+\alpha(s(z,t))((1-H)*(Vf))\big] \\ V(z,T) = 0.&\end{aligned}$$

Assumptions:

(A1) Let the final data $V(\cdot,T)$ be non-negative and non-decreasing.

(A2) Let the interaction function satisfy:

$$\alpha : [0,1] \to \mathbb{R}^+,\ \alpha \in C^\infty([0,1]),\ \alpha(0)=0,\ \alpha'(0)=\infty,\ \alpha'' < 0 \text{ and } \alpha \text{ monotone.}$$

Theorem

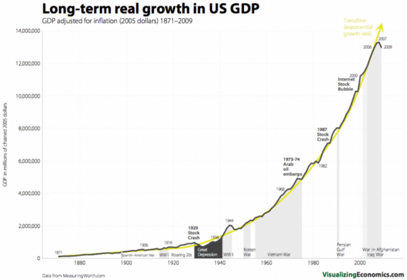
Let $f \in C(0, T, L^1)$ be given and α satisfies assumption (A2). Then there exists a unique solution $V \in C(0, T, L^\infty)$ of the HJB equation with $V(z, T) = 0$. Moreover, let $\tilde{V}$ be a solution of the HJB equation with $\tilde{f}$. Then there exist constants m and D (independent of $\tilde{V}$ and $\tilde{f}$) such that

$$\|V - \tilde{V}\|_\infty \leq De^{mt}\|f - \tilde{f}\|_{C(0,T,L^1)}\|\tilde{V}\|_\infty.$$

Theorem

Let $f_0(z) \in L^\infty(\mathcal{I})$ be a probability density and (A1) and (A2) be satisfied. If $\lim_{s\to 0} \frac{(\alpha')^3}{\alpha''} < \infty$, then the fully coupled Boltzmann mean field game system on $\mathcal{I} = \mathbb{R}^+$ has a unique local in time solution.

- *Economic growth describes the increase of the inflation-adjusted market value of the goods and services produced in an economy over time - commonly measured in the gross domestic product (GDP).*
- *The GDP of most developed countries has grown about two percent since World War II.*



- *Economists are interested in solutions which correspond to sustained growth - so called balanced growth path (BGP) solutions.*

Endogenous growth theory

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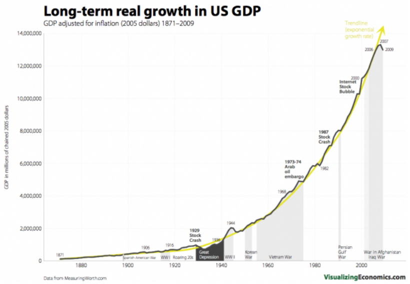


- *Economists are interested in solutions which correspond to sustained growth - so called balanced growth path (BGP) solutions.*

Can we find BGP solutions for the BMFG system ?

Balanced growth path solutions

Let us assume there exists a growth parameter $\gamma \in \mathbb{R}^+$ *and consider the re-scaling:*

$$f(z,t) = e^{-\gamma t}\phi(ze^{-\gamma t}), \quad V(z,t) = e^{\gamma t}v(ze^{-\gamma t}) \text{ and } s(z,t) = \sigma(ze^{-\gamma t})$$

Rescaled BMFG system in $(v,\phi,\sigma) = (v(x),\phi(x),\sigma(x))$ *with* $x = e^{-\gamma t}z$ *reads as:*

$$-\gamma\phi(x) - \gamma\phi'(x)x = \phi(x)\int_0^x \alpha(\sigma(y))\phi(y)\,dy - \alpha(\sigma(x))\phi(x)\int_x^\infty \phi(y)\,dy$$

$$(r-\gamma)v(x) + \gamma v'(x)x = \max_{\sigma\in\Xi}\left\{(1-\sigma)x + \alpha(\sigma)\int_x^\infty [v(y)-v(x)]\phi(y)\,dy\right\}$$

where $\Xi = \{\sigma : \mathbb{R}^+ \to [0,1]\}$ *denotes the set of admissible controls.*

Re-scaling results in exponential growth of the overall production:

$$Y(t) = e^{\gamma t}\int_0^\infty [1-\sigma(x)]x\phi(x)dx.$$

Existence of BGP solutions

Does such a growth parameter γ exist ?

Existence of BGP solutions

The initial commutative distribution function $F(z,0) = \int_0^z f_0(z)dz$ has a Pareto tail, if there exist constants $k, \theta \in \mathbb{R}^+$ such that

$$\lim_{z \to \infty} \frac{1 - F(z,0)}{z^{-1/\theta}} = k. \tag{P}$$

Lemma

Let (P) be satisfied. Then $F = F(z,t)$ has a Pareto tail with the same decay rate θ for all times $t \in [0, T]$.

Theorem

Let (P) be satisfied and $\alpha = \alpha_0$. then there exists a unique BGP solution $(\Phi, v, 0)$ and a scaling constant γ given by

$$\gamma = \alpha_0 \theta \int_{\mathcal{I}} f_0(z)\, dz, \quad \Phi(x) = \frac{1}{1 + kx^{-1/\theta}} \text{ with } \Phi(x) = \int_0^x \phi(y) dy.$$

Existence of BGP solutions

Degenerate solution:

$$\gamma = 0,\ v = \frac{x}{r} \text{ and } S \equiv 0 \Rightarrow \Phi(x) = 1 \text{ for } x > 0$$
$$\Rightarrow \phi(x) = \delta_0$$

Challenge for the analysis and numerics: construct a solution Φ with a strictly positive Pareto tail $k > 0$.

Variable transformation:

$$\zeta := x^{-1/\theta} \text{ and } K(\zeta) := \frac{1 - \Phi(x)}{\gamma\zeta},$$

where θ and k denote the Pareto indices. We solve the correspondingly transformed equation with an initial condition at $\zeta = 0$ (determined by the Pareto tail condition).

Existence of BGP solutions

Theorem

Let $r > \theta\alpha(1)$ and $\tilde{k} > 0$, then the BGP system has a non-trivial solution satisfying the Pareto-tail condition with $k = \frac{\gamma}{\theta}\tilde{k}$.

Idea of proof: Fixed point argument

- Solve equations for (Φ, γ) given (v, S).
- Solve equations for (v, S) and given (Φ, γ).

Diffusion and knowledge growth[3]

Achdou et al. postulate that diffusion

- *enhances growth in the case of a Pareto tail*
- *and leads to exponential growth also for compactly supported initial values.*

Special case $\alpha = \alpha_0$:

- *The Fisher KPP equation (with diffusion) admits travelling wave solutions*

$$G(z,t) = \Phi(z - \gamma t)$$

  with a minimal wave speed $\gamma = 2\sqrt{\nu\alpha_0}$.
- *Travelling waves correspond to BGP solutions (in logarithmic variables).*

[3] Y. Achdou, F.J. Buera, J.-M. Lasry, P.-L. Lions and B. Moll, PDE models in macroeconomics, *Phil. Trans. Roy. Soc. A*, 372, 2014.

Diffusion and knowledge growth

Let the knowledge of each agent evolve by a geometric Brownian motion (independent of the time spent on learning), that is

$$Z_t = \exp(\sqrt{2\nu} W_t)$$

where W_t is a Wiener process, independently for each agent.

Then the corresponding Boltzmann mean field game system with diffusion reads as:

$$\begin{aligned}
\partial_t f(z,t) - \nu \partial_{zz}(z^2 f(z,t)) + \nu \partial_z (z f(z,t)) &= f(z,t) \int_0^z \alpha(S(y,t)) f(y,t) dy \\
&\quad - \alpha(S(z,t)) f(z,t) \int_z^\infty f(y,t) dy, \\
\partial_t V(z,t) + \nu z^2 \partial_{zz} V(z,t) + \nu z \partial_z V(z,t) - rV(z,t) &= \\
&- \max_{s \in \mathcal{S}} \left[(1-s)z + \alpha(s) \int_z^\infty [V(y,t) - V(z,t)] f(y,t) dy \right].
\end{aligned}$$

Assuming the existence of the scaling parameter γ for a balanced growth path we rewrite the system in the known BGP variables (ϕ, σ, v)

$$
\begin{aligned}
&-\gamma\phi(x) - \gamma x\phi'(x) - \nu(x^2\phi(x))'' + \nu(x\phi(x))' = \\
&\qquad \phi(x)\int_0^x \alpha(\sigma(y))\phi(y)dy - \alpha(\sigma(x))\phi(x)\int_x^\infty \phi(y)dy \\
&(r-\gamma)v(x) + \gamma x v'(x) - \nu x^2 v''(x) - \nu x v'(x) = \\
&\qquad -\max_{\sigma\in\Sigma}\left[(1-\sigma)x + \alpha(\sigma)\int_x^\infty [v(y)-v(x)]\phi(y)dy\right].
\end{aligned}
$$

Achdou et al. [4] postulated the existence of BGP solutions to this system with a rescaling parameter γ given by

$$
\gamma = 2\sqrt{\nu\int_0^\infty \alpha(\sigma(y))\phi(y)dy}.
$$

[4] Y. Achdou, F.J. Buera, J.-M. Lasry, P.-L. Lions and B. Moll, PDE models in macroeconomics, *Phil. Trans. Roy. Soc. A*, 372, 2014.

This model is quite simplistic....

... since meetings between individuals are completely asymmetric. Individuals can only increase their knowledge through active search, the 'smarter' individual gains nothing in the meeting.

Symmetric meetings: if an individual with knowledge level y initiated the meeting, the one with the higher knowledge level z may learn with a probability β. This gives:

$$\begin{aligned}\frac{\partial f}{\partial t} = & - f(z,t)\int_z^\infty [\alpha(s(z,t)) + \beta\alpha s(y,t)] f(y,t)dy \\ & + f(z,t)\int_0^z [\alpha(s(y,t)) + \beta\alpha(s(z,t))] f(y,t)dy.\end{aligned}$$

Limits to learning

If two individuals meet, the one with the lower knowledge level z adopts the higher knowledge level y with a certain probability $k(\frac{y}{z})$. Then

$$\begin{aligned}\partial_t f(z,t) =& f(z,t)\int_0^z \alpha(s(y,t))f(y,t)k(\frac{z}{y})dy \\ &- \alpha(s(z,t))f(z,t)\int_z^\infty f(y,t)k(\frac{y}{z})dy.\end{aligned}$$

Possible choice for k:

$$k(x) = \delta + (1-\delta)x^{-\kappa} \quad \textit{where } \kappa > 0.$$

Alternative interpretation of k: *interaction probability depends on the distance between knowledge levels.*

In the case of a constant interaction rate $\alpha = \alpha_0$ the CDF $F = F(z,t)$ evolves according to

$$\partial_t F(z,t) = -\alpha(1 - F(z,t)F(z,t).$$

Then

$$\lim_{t \to \infty} F(z,t) = \frac{1}{1 + kx^{-\frac{1}{\theta}}}$$

Exogenous knowledge shock: undiscovered ideas modelled by a CDF $G = G(z)$

$$\partial_t F(z,t) = -\alpha(1 - F(z,t))F(z,t) - \beta(1 - G(z))F(z,t)$$

Asymptotic behaviour

Depends on the 'tails' of F and G:

- *If neither $F(z,0)$ nor $G(z)$ has a Pareto tail there will be no growth in the long run.*
- *If $F(z,0)$ has a fatter tail than $G(z)$ then the BGP has a growth rate of $\gamma = \alpha\theta$ (external ideas do not influence the asymptotic behaviour).*
- *If $G(z)$ has a fatter tail (denoted by $\frac{1}{\xi}$) the BGP path grows at a rate $\gamma = \alpha\xi$ and the asymptotic distribution satisfies*

$$\lim_{t\to\infty} F(z,t) = \frac{1}{1 + \frac{\beta}{\alpha} m x^{-\frac{1}{\xi}}}$$

  where $m > 0$.

- *If they have the same Pareto tail then the asymptotic distribution satisfies*

$$\lim_{t\to\infty} F(z,t) = \frac{1}{1 + [k + \frac{\beta}{\alpha} m] x^{-\frac{1}{\theta}}}$$

  with $m > 0$.

The time-dependent solver

The solver is based on a fixed point scheme:

1. *Given f_0 and S^k solve*

$$\frac{1}{\tau}(f_i^{k+1} - f_i^k) - \frac{\nu}{h^2}(z_{i+\frac{1}{2}}^2 f_{i+1}^{k+1} - (z_{i+\frac{1}{2}}^2 + z_{i-\frac{1}{2}}^2) f_i^{k+1} + z_{i-\frac{1}{2}}^2 f_{i+1}^{k+1}) + \frac{\nu}{h}(z_{i+\frac{1}{2}} f_i^{k+1} - z_{i-\frac{1}{2}} f_{i-1}^{k+1}) = g_1(f^k, S^k),$$

   for every time $t^k = k\tau,\ k > 1$, using a trapezoidal rule to approximate the integrals in g_1.

2. *Update the maximizer S^k.*
3. *Given the evolution of the density f^k and the maximizer S^k solve the HJB equation*

$$\frac{1}{\tau}(V_i^{k+1} - V^k) + \frac{\nu}{h^2} z_i^2 (V_{i+1}^k - 2V_i^k + V_{i-1}^k) + \frac{\nu}{h} z_i (V_i^k - V_{i-1}^k) - rV_i^k = g_2(S^{k+1}, f^{k+1}, V^{k+1}),$$

   backward in time using a trapezoidal rule to approximate g_2.

4. *Go to step (1) until convergence.*

The BGP solver

The BGP solver is also based on a fixed point scheme:

1. *Given* ϕ^{n+1}, γ^n *and* σ^n *solve*

$$(r-\gamma^n)v_i^{n+1} + \frac{(\gamma^n-\nu)}{h}x_i(v_i^{n+1}-v_{i-1}^{n+1}) - \frac{\nu x_i^2}{h^2}(v_{i+1}^{n+1}-2v_i^{n+1}+v_{i-1}^{n+1}) = -q_2(\phi_{n+1}, v^n, \sigma^n)$$

   using the trapezoidal rule to approximate the right hand side q_2.

2. *Compute the maximum* σ^{n+1} *and update the growth parameter* γ^{n+1} *via*

$$\gamma^{n+1} = 2\Big(\nu\int_{\mathcal{I}}\alpha(\sigma^{n+1}(y))\phi^{n+1}(y)dy\Big)^{\frac{1}{2}}.$$

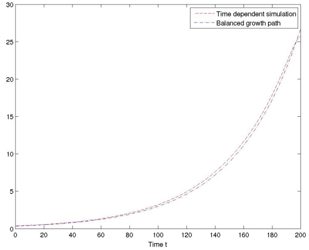
3. *Given* v^n, σ^n *and* γ^n *solve*

$$-(\gamma^n-\nu)\phi_i^{n+1} - \frac{(\gamma^n-\nu)}{h}x_i(\phi_{i+1}^{n+1}-\phi_i^{n+1}) - (\Xi_i - \alpha(\sigma_i^n)(1-\Phi_i))\phi^{n+1} - \frac{\nu}{h^2}(x_{i+\frac{1}{2}}^2\phi_{i+1}^{n+1} - (x_{i+\frac{1}{2}}^2 + x_{i-\frac{1}{2}}^2)\phi_i^{n+1} + x_{i-\frac{1}{2}}^2\phi_{i-1}^{n+1}) = 0,$$

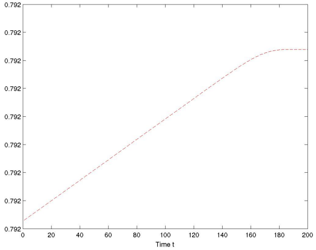
   subject to the constraint $(\phi_1^{n+1}+\phi_2^{n+1}+\ldots\frac{1}{2}\phi_N^{n+1})h = 1$ *(normalisation* $\int_0^{\bar{z}}\phi(y)dy = 1$ *plus* $\phi_0^{n+1} = 0$*).*

4. *Go to (1) until convergence.*

Simulations



(a) Transient vs. BGP



(b) Linear growth

Figure: Evolution of the production function $Y = Y(t)$ in time for different choices of n and θ

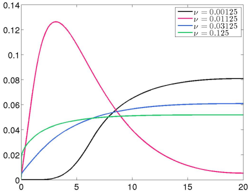
Numerical simulations

- *Simulations of the time-dependent problem as well as the BGP system are performed iteratively.*
- *We solve the systems on a bounded domain with no-flux boundary conditions.*
- *To exclude degenerate BGP solutions we set*

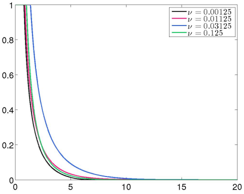
$$\phi_0 = 0.$$

- *We use a finite difference discretization in space and approximate the integrals using the trapezoidal rule.*

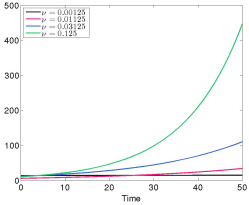
BGP simulations with knowledge diffusion



(a) Agent distribution for different values of ν.



(b) Fraction of time σ devoted to learning for different values of ν.



(c) Production function Y.

This model is quite simplistic....

... since meetings between individuals are completely asymmetric. Individuals can only increase their knowledge through active search, the 'smarter' individual gains nothing in the meeting.

Symmetric meetings: if an individual with knowledge level y initiated the meeting, the one with the higher knowledge level z may learn with a probability β. This gives:

$$\begin{aligned}\frac{\partial f}{\partial t} = &- f(z,t)\int_z^{\infty}[\alpha(s(z,t)) + \beta\alpha s(y,t)]f(y,t)dy \\ &+ f(z,t)\int_0^{z}[\alpha(s(y,t)) + \beta\alpha(s(z,t))]f(y,t)dy.\end{aligned}$$

Limits to learning

If two individuals meet, the one with the lower knowledge level z adopts the higher knowledge level y with a certain probability $k(\frac{y}{z})$. Then

$$\partial_t f(z,t) = f(z,t)\int_0^z \alpha(s(y,t)) f(y,t) k(\frac{z}{y}) dy \\ - \alpha(s(z,t)) f(z,t) \int_z^\infty f(y,t) k(\frac{y}{z}) dy.$$

Possible choice for k:

$$k(x) = \delta + (1-\delta)x^{-\kappa} \quad \textit{where } \kappa > 0.$$

Alternative interpretation of k: *interaction probability depends on the distance between knowledge levels.*

To finish....

References:

- Y. Achdou, F.J. Buera, J.-M. Lasry, P.-L. Lions and B. Moll, PDE models in macroeconomics, *Phil. Trans. Roy. Soc. A*, 372, 2014.
- R. E. Lucas Jr and B. Moll. Knowledge growth and the allocation of time. *Journal of Political Economics*, 2014.
- M. Burger, A. Lorz and M.T. Wolfram, On a Boltzmann model for knowledge growth. *SIAM Applied Math* 76(5), 2016.
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To finish....

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Thanks for the attention and have a great time at rest of the school !