Linear-Quadratic Mean-Field-Type Games

common noise, jump-diffusion, regime switching

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Acknowledgments

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Lecture 4: How far can we go with noPDEs–noSMPs ?

Agenda

1. Infinite-dimensional sPDE approach
2. Master System
3. Variance, Covariance, Mean-Variance, Mean-Covariance
4. Discrete Time Case
5. From Discrete Step to Continuous Time

Part II: Bellman approach

Finding feedback strategies $u_i(t, s, \hat{s}, \tilde{s}, B_0, s_0, \hat{s}_0, \tilde{s}_0)$ that are $\mathcal{F}_t^{s,\tilde{s},B_o}$–measurable

Still want the infinite-dimensional sPIDE approach?

Stochastic Fokker-Plank-Kolmogorov :

$m_0(ds, \tilde{s}),$

$dm = \{-[mD]_s + \frac{\sigma^2+\sigma_o^2}{2} m_{ss} + J^*[m] + \tilde{S}^*[m]\}dt - \sigma_o m_s dB_o,$

Stochastic Bellman system :

$$\hat{V}_i(T, m, \tilde{s}) = \frac{1}{2} q_i(T, \tilde{s}) \int_y (y - \bar{m})^2\, m(dy) + \frac{1}{2}[q_i(T, \tilde{s}) + \bar{q}_i(T, \tilde{s})]\bar{m}^2, \tag{1}$$

$$-d\hat{V}_i + \Big\{ \hat{V}_{i,t}(t, m, \tilde{s}) + \int_s H_i(s, m, \tilde{s}, \hat{V}_{i,m}, \hat{V}_{i,sm}, \hat{V}_{i,ssm}) m(ds)$$
$$+\sigma_o \int m_s \sigma_o \partial_s [\int_y m(dy) \hat{V}_{i,sm}] ds + \frac{\sigma_o^2}{2} \int_{s,y} \hat{V}_{i,mm}(s, y) m_s(ds) m_s(dy) \Big\}\, dt$$
$$+[\sigma_o \int_s m(ds) \hat{V}_{i,sm}] dB_o = 0,$$

$i \in \{1, 2, \ldots\}$

$$
\begin{aligned}
&H_i(s, m, \tilde{s}, \hat{V}_{i,m}, \hat{V}_{i,sm}, \hat{V}_{i,ssm}) \\
&= \inf_{u_i} \left\{ \tfrac{1}{2} q_i (s - \hat{s})^2 + \tfrac{1}{2}[q_i + \bar{q}_i]\hat{s}^2 + \tfrac{1}{2} r_i (u_i - \hat{u}_i)^2 + \tfrac{1}{2}(r_i + \bar{r}_i)[\hat{u}_i]^2 \right. \\
&\left. + [a(s - \hat{s}) + (a + \bar{a})\hat{s} + \textstyle\sum_{i=1}^n b_i (u_i - \hat{u}_i) + \textstyle\sum_{i=1}^n (b_i + \bar{b}_i)\hat{u}_i] \hat{V}_{i,sm} \right\} \\
&+ \tfrac{\sigma^2 + \sigma_o^2}{2} \hat{V}_{i,ssm} + \textstyle\int_\Theta [\hat{V}_{i,m}(t, s + \mu) - \hat{V}_{i,m} - \mu \hat{V}_{i,sm}] \nu(d\theta) \\
&+ \textstyle\sum_{k \neq \tilde{s}} [\hat{V}_{i,m}(t, s + \kappa(k)) - \hat{V}_{i,m} - \kappa \hat{V}_{i,sm}] q_{\tilde{s}k},
\end{aligned}
\tag{2}
$$

$$
\begin{aligned}
&J[\phi] := \textstyle\int_\Theta [\phi(s + \mu) - \phi(s) - \mu \phi_s] \nu(d\theta), \\
&\tilde{S}[\phi] := \textstyle\sum_{k \neq \tilde{s}} [\phi(s + \kappa(t, k), k) - \phi(s, \tilde{s}) - \kappa(t, \tilde{s}) \phi_s] q_{\tilde{s}k}.
\end{aligned}
$$

Master System: $\mathcal{V}_i(s, m) := \partial_m \hat{V}_i(m),\ i \in \{1, 2, \ldots\}$

No common noise case

$$
\begin{aligned}
&\hat{V}_{i,t}(t,m) + \int_s H_i(s,m,\hat{V}_{i,m},\hat{V}_{i,sm},\hat{V}_{i,ssm})m(ds) = 0,\\
&\hat{V}_i(T,m) = \frac{1}{2}q(T)\int_y (y-\bar{m})^2\, m(dy) + \frac{1}{2}[q(T)+\bar{q}(T)]\left[\int_y y\, m(dy)\right]^2,\\
&H_i(s,m,\hat{V}_{i,m},\hat{V}_{i,sm},\hat{V}_{i,smm})\\
&= \inf_{u_i}\Big\{\frac{1}{2}q_i(s-\bar{s})^2 + \frac{1}{2}[q_i+\bar{q}_i]\bar{s}^2 + \frac{1}{2}r_i(u_i-\bar{u}_i)^2 + \frac{1}{2}(r_i+\bar{r}_i)[\bar{u}_i]^2\\
&+[a(s-\bar{s}) + (a+\bar{a})\bar{s} + \sum_{i=1}^n b_i(u_i-\bar{u}_i) + \sum_{i=1}^n (b_i+\bar{b}_i)\bar{u}_i]\hat{V}_{i,sm}\Big\}\\
&+\frac{\sigma^2}{2}\hat{V}_{i,ssm} + \int_\Theta [\hat{V}_{i,m}(t_-, s+\mu) - \hat{V}_{i,m} - \mu\hat{V}_{i,sm}]\nu(d\theta).
\end{aligned}
\tag{3}
$$

Part III: Pontryagin approach

Finding open-loop strategies $u_i(t, B_0, s_0, \tilde{s}_0)$ that are $\mathcal{F}_t^{B_0, s_0, \tilde{s}_0}$–measurable

- $p_i = \partial_{sm} V_i(m)[(t, s(t), \tilde{s}(t))]$
- $dp_i = -\alpha_i^* dt + q_i^* dB + \int r_i^*(t,\theta)\tilde{N}(t, d\theta) + \sum_{k\neq\tilde{s}} s_i^*(t,k)\tilde{M}(t,k) + q_{i,o}^* dB_o$
- Dynamics of $s(t)$ and $\mathbb{E}[s(t)|\mathcal{F}_t^{s_0,B_o}]$
- $d\hat{s} = \left\{(a+\bar{a})\hat{s} + \sum_{i=1}^n (b_i + \bar{b}_i)\hat{u}_i\right\} dt + \sigma_o dB_o.$
- Best response strategy expressed in terms of $\mathbb{E}[p_i|\mathcal{F}_t^{s_0,B_o}]$

Variance, Covariance, Mean-Variance, Mean-Covariance

Variance and Covariance

$$
\begin{aligned}
&\textbf{Problem}: \ \inf_{u_i} \ \mathbb{E}L_i \\
&L_i = \frac{1}{2}q_{iT}s^2(T) + \frac{1}{2}\bar{q}_{iT}\bar{s}^2(T) + \epsilon_{i3}(T)\bar{s}(T) + \frac{1}{2}\int_0^T \left(q_i s^2 + \bar{q}_i\bar{s}^2 + r_i u_i^2 + \bar{r}_i\bar{u}_i^2\right)dt \\
&+ \int_0^T \left(\epsilon_{i1} s u_i + \bar{\epsilon}_{i1}\bar{s}\bar{u}_i + \epsilon_{i2}\bar{u}_i + \epsilon_{i3}\bar{s}\right)dt, \\
&ds = [b_0 + b_1 s + \bar{b}_1\bar{s} + \sum_{j=1}^n b_{j2}u_j + \sum_{j=1}^n \bar{b}_{j2}\bar{u}_j]dt \\
&+[\sigma_0 + \sigma_1 s + \bar{\sigma}_1\bar{s} + \sum_{j=1}^n \sigma_{j2}u_j + \sum_{j=1}^n \bar{\sigma}_{j2}\bar{u}_j]dB \\
&+ \int_\Theta[\mu_0 + \mu_1 s + \bar{\mu}_1\bar{s} + \sum_{j=1}^n \mu_{j2}u_j + \sum_{j=1}^n \bar{\mu}_{j2}\bar{u}_j]\tilde{N}(dt, d\theta), \\
&\ s(0) \perp\!\!\!\perp \{B, N\}, \\
&\tilde{s}(t) \sim \tilde{q}_{\tilde{s}\tilde{s}'}
\end{aligned}
\tag{4}
$$

Equivalent problem

$$
\begin{aligned}
&\textbf{Problem}: \inf_{u_i} \ \mathbb{E}\hat{L}_i \\
&\hat{L}_i = \tfrac{1}{2}q_{iT}var(s(T)) + \tfrac{1}{2}(q_{iT}+\bar{q}_{iT})\bar{s}^2(T) + \varepsilon_{i3}(T)\bar{s}(T) \\
&+\tfrac{1}{2}\int_0^T \left(q_i var(s) + (q_i+\bar{q}_i)\bar{s}^2 + r_i var(u_i) + (r_i+\bar{r}_i)\bar{u}_i^2\right)dt \\
&+\int_0^T \left(\epsilon_{i1}cov(s,u_i) + (\epsilon_{i1}+\bar{\epsilon}_{i1})\bar{s}\bar{u}_i + \varepsilon_{i2}\bar{u}_i + \varepsilon_{i3}\bar{s}\right)dt, \\
&ds = [b_0 + b_1(s-\bar{s}) + (b_1+\bar{b}_1)\bar{s} + \textstyle\sum_{j=1}^n b_{j2}(u_j-\bar{u}_j) + \sum_{j=1}^n (b_{j2}+\bar{b}_{j2})\bar{u}_j]dt \\
&+[\sigma_0 + \sigma_1(s-\bar{s}) + (\sigma_1+\bar{\sigma}_1)\bar{s} + \textstyle\sum_{j=1}^n \sigma_{j2}(u_j-\bar{u}_j) + \sum_{j=1}^n (\sigma_{j2}+\bar{\sigma}_{j2})\bar{u}_j]dB \\
&+\int_\Theta [\mu_0 + \mu_1(s-\bar{s}) + (\mu_1+\bar{\mu}_1)\bar{s} + \textstyle\sum_{j=1}^n \mu_{j2}(u_j-\bar{u}_j) + \sum_{j=1}^n (\mu_{j2}+\bar{\mu}_{j2})\bar{u}_j)]\tilde{N}(dt,d\theta \\
&\quad s(0) \perp\!\!\!\perp \{B,N\}, \\
&\tilde{s}(t) \sim \tilde{q}_{\tilde{s}\tilde{s}'}.
\end{aligned} \tag{5}
$$

Result

Mean-Field Equilibria

$$\begin{array}{l} \inf_{u_i \in \mathcal{U}_i} \mathbb{E}[L_i] \\ = \mathbb{E}\{\frac{1}{2}\alpha_i(0,\tilde{s}_0)(s(0)-\hat{s}(0))^2 + \frac{1}{2}\beta_i(0,\tilde{s}_0)[\hat{s}(0)]^2 + \gamma_i(0,\tilde{s}_0)\hat{s}(0) + \delta_i(0,\tilde{s}_0)\} \\ u_i^* = -\frac{\tilde{\tau}_i}{c_i}(s-\bar{s}) - \frac{\bar{\lambda}_{i2}\bar{s}}{\bar{c}_i} - \frac{\bar{\lambda}_{i3}}{\bar{c}_i}, \end{array} \tag{6}$$

Coefficients of the solution: α

$$
\begin{aligned}
&\dot{\alpha}_i = -2\alpha_i b_1 - \alpha_i \left(\sigma_1^2 + \int_\Theta \mu_1^2 \nu(d\theta)\right) - q_i - \sum_{\tilde{s}'\in\tilde{\mathcal{S}}}[\alpha_i(.,\tilde{s}') - \alpha_i(.,\tilde{s})]\tilde{q}_{\tilde{s}\tilde{s}'} \\
&-\alpha_i\left(\sum_{j\neq i} \sigma_{j2}\frac{\tilde{\tau}_j}{c_j}\right)^2 - \int_\Theta \alpha_i\left(\sum_{j\neq i}^n \mu_{j2}\frac{\tilde{\tau}_j}{c_j}\right)^2 \nu(d\theta) \\
&+2\alpha_i \sum_{j\neq i}\left(\sigma_1\sigma_{j2} + \int_\Theta \mu_1\mu_{j2}\nu(d\theta)\right)\frac{\tilde{\tau}_j}{c_j} + 2\alpha_i \sum_{j\neq i} b_{j2}\frac{\tilde{\tau}_j}{c_j} + \frac{\tilde{\tau}_i^2}{c_i}, \\
&\alpha_i(T,\tilde{s}) = q_i(T,\tilde{s}),
\end{aligned}
\tag{7}
$$

Coefficients of the solution: β

$$\begin{aligned}
&\dot{\beta}_i = -2\beta_i(b_1 + \bar{b}_1) - \alpha_i \left((\sigma_1 + \bar{\sigma}_1)^2 + \int_\Theta (\mu_1 + \bar{\mu}_1)^2 \nu(d\theta)\right) \\
&-(q_i + \bar{q}_i) - \sum_{\tilde{s}' \in \tilde{\mathcal{S}}} [\beta_i(., \tilde{s}') - \beta_i(., \tilde{s})] \tilde{q}_{\tilde{s}\tilde{s}'} \\
&-\alpha_i \left(\sum_{j \neq i} (\sigma_{j2} + \bar{\sigma}_{j2}) \frac{\bar{\lambda}_{j2}}{\bar{c}_j} \right)^2 - \alpha_i \int_\Theta \left(\sum_{j \neq i} (\mu_{j2} + \bar{\mu}_{j2}) \frac{\bar{\lambda}_{j2}}{\bar{c}_j} \right)^2 \nu(d\theta) + \frac{\bar{\lambda}_{i2}^2}{\bar{c}_i} \\
&+2 \sum_{j \neq i} \alpha_i \left((\sigma_1 + \bar{\sigma}_1)(\sigma_{j2} + \bar{\sigma}_{j2}) + \int_\Theta (\mu_1 + \bar{\mu}_1)(\mu_{j2} + \bar{\mu}_{j2}) \nu(d\theta) \right) \frac{\bar{\lambda}_{j2}}{\bar{c}_j} \\
&+2 \sum_{j \neq i} \beta_i (b_{j2} + \bar{b}_{j2}) \frac{\bar{\lambda}_{j2}}{\bar{c}_j}, \\
&\beta_i(T, \tilde{s}) = q_i(T, \tilde{s}) + \bar{q}_i(T, \tilde{s}),
\end{aligned} \tag{8}$$

Coefficients of the solution: γ

$$
\begin{aligned}
&\dot{\gamma}_i = -\beta_i b_0 - \gamma_i(b_1 + \bar{b}_1) - \alpha_i\left(\sigma_0(\sigma_1 + \bar{\sigma}_1) + \int_\Theta \mu_0(\mu_1 + \bar{\mu}_1)\nu(d\theta)\right)\\
&-\varepsilon_{i3} - \sum_{\tilde{s}' \in \tilde{\mathcal{S}}}[\gamma_i(.,\tilde{s}') - \gamma_i(.,\tilde{s})]\tilde{q}_{\tilde{s}\tilde{s}'}\\
&-\alpha_i\left(\sum_{j\neq i}(\sigma_{j2} + \bar{\sigma}_{j2})\frac{\bar{\lambda}_{j2}}{\bar{c}_j}\right)\left(\sum_{j\neq i}(\sigma_{j2} + \bar{\sigma}_{j2})\frac{\bar{\lambda}_{j3}}{\bar{c}_j}\right)\\
&-\alpha_i\int_\Theta\left(\sum_{j\neq i}(\mu_{j2} + \bar{\mu}_{j2})\frac{\bar{\lambda}_{j2}}{\bar{c}_j}\right)\left(\sum_{j\neq i}(\mu_{j2} + \bar{\mu}_{j2})\frac{\bar{\lambda}_{j3}}{\bar{c}_j}\right)\nu(d\theta)\\
&+\sum_{j\neq i}\left[\alpha_i\left(\sigma_0(\sigma_{j2} + \bar{\sigma}_{j2}) + \int_\Theta \mu_0(\mu_{j2} + \bar{\mu}_{j2})\nu(d\theta)\right) + \gamma_i(b_{j2} + \bar{b}_{j2})\right]\frac{\bar{\lambda}_{j2}}{\bar{c}_j}\\
&+\sum_{j\neq i}\left[\alpha_i\left((\sigma_1 + \bar{\sigma}_1)(\sigma_{j2} + \bar{\sigma}_{j2}) + \int_\Theta(\mu_1 + \bar{\mu}_1)(\mu_{j2} + \bar{\mu}_{j2})\nu(d\theta)\right)\right]\frac{\bar{\lambda}_{j3}}{\bar{c}_j}\\
&+\sum_{j\neq i}\beta_i(b_{j2} + \bar{b}_{j2})\frac{\bar{\lambda}_{j3}}{\bar{c}_j} + \frac{\bar{\lambda}_{i2}\bar{\lambda}_{i3}}{\bar{c}_i},\\
&\gamma_i(T,\tilde{s}) = \epsilon_{i3}(T,\tilde{s}),
\end{aligned}
\tag{9}
$$

Coefficients of the solution: δ

$$\begin{aligned}
&\dot{\delta}_i = -\gamma_i b_0 - \frac{\alpha_i}{2}\left(\sigma_0^2 + \int_\Theta \mu_0^2 \nu(d\theta)\right) - \sum_{\tilde{s}'\in\tilde{\mathcal{S}}}[\delta_i(.,\tilde{s}') - \delta_i(.,\tilde{s})]\tilde{q}_{\tilde{s}\tilde{s}'} \\
&-\alpha_i \frac{1}{2}\left(\sum_{j\neq i}(\sigma_{j2} + \bar{\sigma}_{j2})\frac{\bar{\lambda}_{j3}}{\bar{c}_j}\right)^2 - \alpha_i \int_\Theta \frac{1}{2}\left(\sum_{j\neq i}(\mu_{j2} + \bar{\mu}_{j2})\frac{\bar{\lambda}_{j3}}{\bar{c}_j}\right)^2 \nu(d\theta) \\
&+\sum_{j\neq i}\left[\alpha_i\left(\sigma_0(\sigma_{j2} + \bar{\sigma}_{j2}) + \int_\Theta \mu_0(\mu_{j2} + \bar{\mu}_{j2})\nu(d\theta)\right)\right]\frac{\bar{\lambda}_{j3}}{\bar{c}_j} \\
&+\sum_{j\neq i}\gamma_i(b_{j2} + \bar{b}_{j2})\frac{\bar{\lambda}_{j3}}{\bar{c}_j} + \frac{\bar{\lambda}_{i3}^2}{2\bar{c}_i}, \\
&\delta_i(T,\tilde{s}) = 0,
\end{aligned} \tag{10}$$

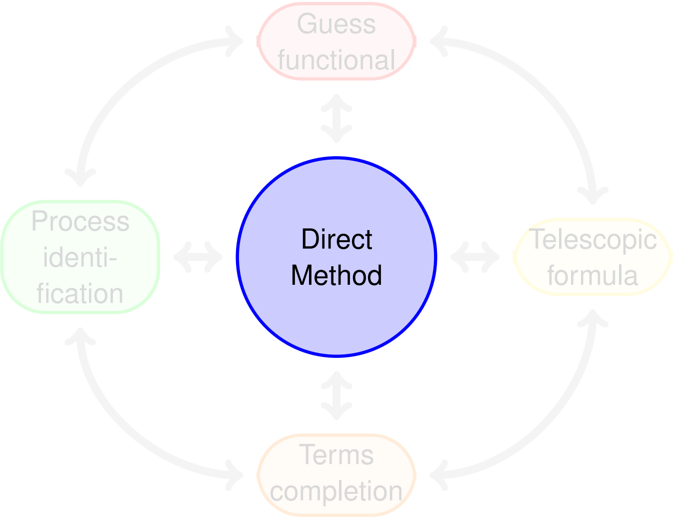
Coefficients of the solution: $\tau, \bar{\tau}, c, \bar{c}$

$$
\begin{aligned}
& A\tilde{\tau} = v, \quad \bar{A}\bar{\tau} = \bar{v}, \\
& A_{ii} = \bar{A}_{ii} = 1, \\
& A_{ij} = \frac{\alpha_i}{c_j}\{\sigma_{i2}\sigma_{j2} + \int_\Theta \mu_{i2}\, \mu_{j2}\nu(d\theta)\}, \ \forall j \neq i \\
& v_i = \left[\epsilon_{i1} + \alpha_i \left\{b_{i2} + \sigma_1\sigma_{i2} + \int_\Theta \mu_1\mu_{i2}\nu(d\theta)\right\}\right], \\
& \bar{A}_{ij} = \frac{\alpha_i}{\bar{c}_j}\left[(\sigma_{i2} + \bar{\sigma}_{i2})(\sigma_{j2} + \bar{\sigma}_{j2}) + \int_\Theta (\mu_{i2} + \bar{\mu}_{i2})(\mu_{j2} + \bar{\mu}_{j2})\nu(d\theta)\right], \ \forall j \neq i \\
& \bar{v}_i = \varepsilon_{i2} + \alpha_i\left(\sigma_0(\sigma_{i2} + \bar{\sigma}_{i2}) + \int_\Theta \mu_0(\mu_{i2} + \bar{\mu}_{i2})\nu(d\theta)\right) + \gamma_i(b_{i2} + \bar{b}_{i2}) \\
& + \left[\varepsilon_{i1} + \bar{\varepsilon}_{i1} + \alpha_i\left((\sigma_1 + \bar{\sigma}_1)(\sigma_{i2} + \bar{\sigma}_{i2}) + \int_\Theta (\mu_1 + \bar{\mu}_1)(\mu_{i2} + \bar{\mu}_{i2})\nu(d\theta)\right)\right]\bar{s} \\
& + \beta_i(b_{i2} + \bar{b}_{i2})\bar{s}, \\
& \bar{A}\bar{\tau} = \bar{A}\left(\bar{\lambda}_2\bar{s} + \bar{\lambda}_3\right) = \bar{v}, \\
& c_i = r_i + \alpha_i\sigma_{i2}^2 + \int_\Theta \alpha_i\mu_{i2}^2\nu(d\theta), \\
& \bar{c}_i = (r_i + \bar{r}_i) + \alpha_i\left[(\sigma_{i2} + \bar{\sigma}_{i2})^2 + \int_\Theta (\mu_{i2} + \bar{\mu}_{i2})^2\nu(d\theta)\right],
\end{aligned}
\tag{11}
$$

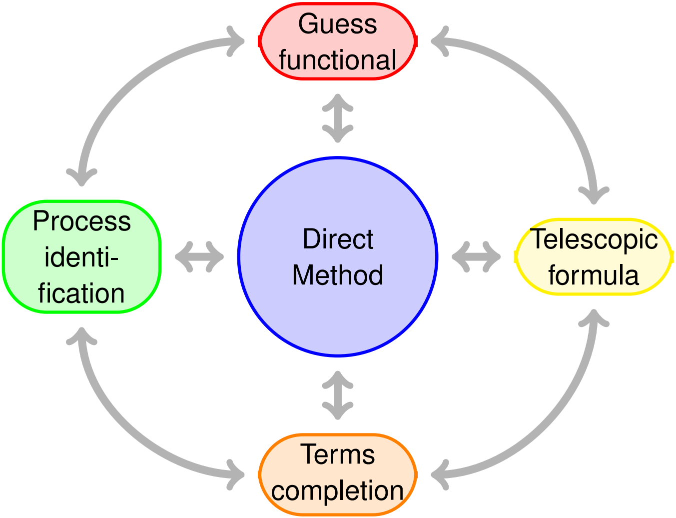
How to solve the discrete time game problem?

- No Bellman-Shapley operator,
- No Lagrange multipliers

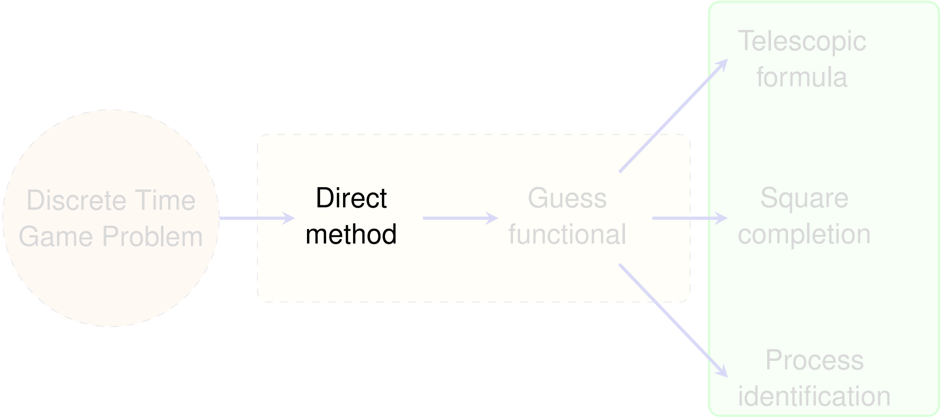
Direct Method in Discrete Time



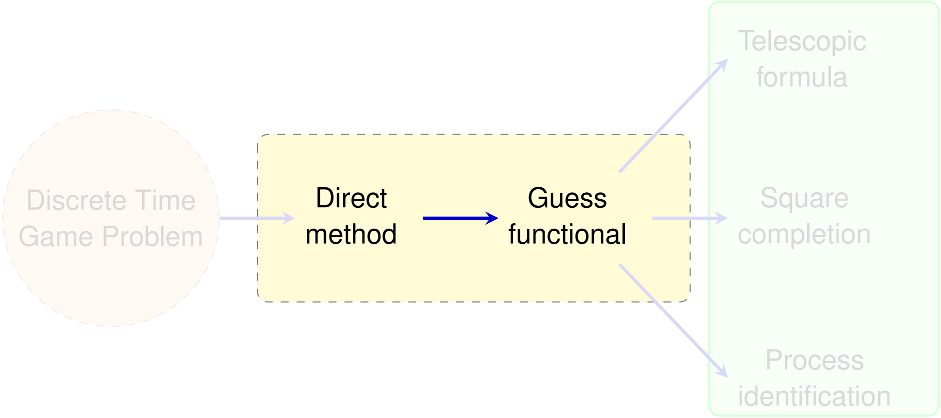
Direct Method in Discrete Time



Procedure


Discrete Time Game Problem
Direct method
Guess functional
Telescopic formula
Square completion
Process identification

Procedure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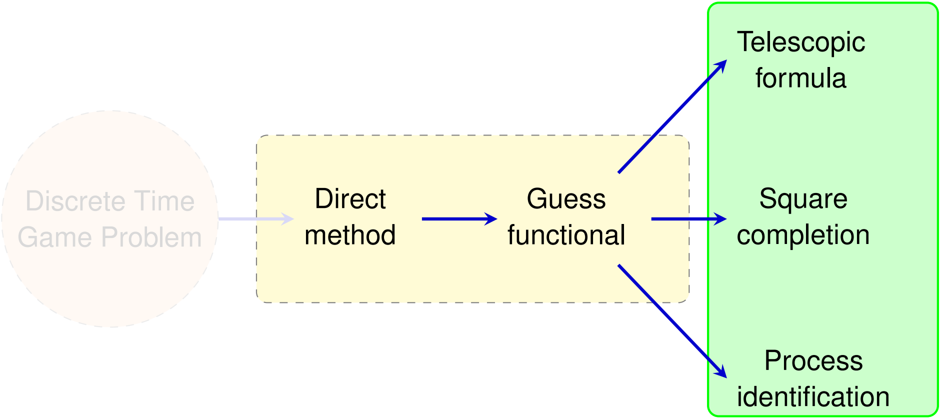

Discrete Time Game Problem
Direct method
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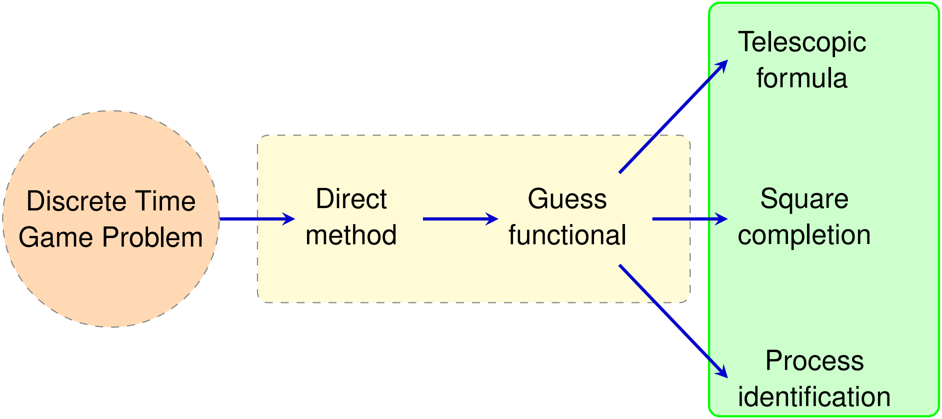


Discrete Time Game Problem
Direct method
Guess functional
Telescopic formula
Square completion
Process identification

Telescopic formula

Telescopic

$$f_{iK} = f_{i0} + \sum_{k=0}^{K-1} [f_{i,k+1} - f_{ik}],$$

Discrete time game problem

$$
\begin{array}{l}
\textbf{Problem}: \inf_{u_i} \ \mathbb{E}L_i \\
L_i = q_{iK}s_K^2 + \bar{q}_{iK}(E[s_K])^2 + \epsilon_{i3K}E[s_K] \\
+ \sum_{k=0}^{K-1} q_{ik}s_k^2 + \bar{q}_{ik}(E[s_k])^2 + r_{ik}u_{ik}^2 + \bar{r}_{ik}(E[u_{ik}])^2 \\
+ \sum_{k=0}^{K-1} 2\epsilon_{i1k}s_k u_{ik} + 2\bar{\epsilon}_{i1k}(E[s_k])E[u_{ik}] + \epsilon_{i2k}E[u_{ik}] + \epsilon_{i3k}E[s_k], \\
\text{subject to} \\
s_{k+1} = \left(b_{0k} + b_{1k}s_k + \bar{b}_{1k}E[s_k] + \sum_{j=1}^n b_{j2k}u_{jk} + \sum_{j=1}^n \bar{b}_{j2k}E[u_{jk}]\right) \\
+ \left(\sigma_{0k} + \sigma_{1k}s_k + \bar{\sigma}_{1k}E[s_k] + \sum_{j=1}^n \sigma_{j2k}u_{jk} + \sum_{j=1}^n \bar{\sigma}_{j2k}E[u_{jk}]\right) w_{k+1}, \\
s(0) = s_0,
\end{array}
\tag{12}
$$

Discrete time: solution

$$\begin{aligned} u_{ik} &= -\eta_{ik}(s_k - Es_k) - \bar{\eta}_{ik} Es_k - \hat{\eta}_{ik}, \\ L_i^* &= \alpha_{i0} var(s_0) + \beta_{i0}(Es_0)^2 + \gamma_{i0}(Es_0) + \delta_{i0}, \end{aligned} \tag{13}$$

Idea of the Proof

Telescopic formula:

$$f_{iK} = f_{i0} + \sum_{k=0}^{K-1} [f_{i,k+1} - f_{ik}],$$

with the guess

$$f_{ik} = \alpha_{ik}(s_k - Es_k)^2 + \beta_{ik}(Es_k)^2 + \gamma_{ik}(Es_k) + \delta_{ik}$$

Coefficients η_{ik}, c_{ik}

$$\begin{aligned}
&\eta_{ik} = -\frac{1}{c_{ik}}\alpha_{i,k+1}\sum_{j\neq i} b_{j2k}\eta_{jk} - \frac{1}{c_{ik}}\alpha_{i,k+1}\sigma_{i2k}\sum_{j\neq i}\sigma_{j2k}\eta_{jk} \\
&+\frac{1}{c_{ik}}\alpha_{i,k+1}b_{1k}b_{i2k} + \frac{1}{c_{ik}}\alpha_{i,k+1}\sigma_{1k}\sigma_{i2k} + \frac{\epsilon_{i1k}}{c_{ik}}, \\
&c_{ik} = r_{ik} + \alpha_{i,k+1}(b_{i2k}^2 + \sigma_{i2k}^2),
\end{aligned} \tag{14}$$

Coefficients $\bar{\eta}_{ik}, \bar{c}_{ik}, \hat{\eta}_{ik}$

$$
\begin{aligned}
&\bar{\eta}_{ik} = -\frac{1}{\bar{c}_{ik}}\alpha_{i,k+1}(\sigma_{i2k} + \bar{\sigma}_{i2k}) \sum_{j\neq i}(\sigma_{j2k} + \bar{\sigma}_{j2k})\bar{\eta}_{jk} \\
&-\frac{1}{\bar{c}_{ik}}\beta_{i,k+1}(b_{i2k} + \bar{b}_{i2k}) \sum_{j\neq i}(b_{j2k} + \bar{b}_{j2k})\bar{\eta}_{jk} \\
&+\frac{1}{\bar{c}_{ik}}\alpha_{i,k+1}(\sigma_{1k} + \bar{\sigma}_{1k})(\sigma_{i2k} + \bar{\sigma}_{i2k}) \\
&+\frac{1}{\bar{c}_{ik}}\beta_{i,k+1}(b_{1k} + \bar{b}_{1k})(b_{i2k} + \bar{b}_{i2k}) + \frac{1}{\bar{c}_{ik}}(\epsilon_{i1k} + \bar{\epsilon}_{i1k}), \\
&\bar{c}_{ik} = (r_{ik} + \bar{r}_{ik}) + \alpha_{i,k+1}(\sigma_{i2k} + \bar{\sigma}_{i2k})^2 + \beta_{i,k+1}(b_{i2k} + \bar{b}_{i2k})^2, \\
&\hat{\eta}_{ik} = -\frac{1}{\bar{c}_{ik}}\alpha_{i,k+1}(\sigma_{i2k} + \bar{\sigma}_{i2k}) \sum_{j\neq i}(\sigma_{j2k} + \bar{\sigma}_{j2k})\hat{\eta}_{jk} \\
&-\frac{1}{\bar{c}_{ik}}\beta_{i,k+1}(b_{i2k} + \bar{b}_{i2k}) \sum_{j\neq i}(b_{j2k} + \bar{b}_{j2k})\hat{\eta}_{jk} \\
&+\frac{1}{\bar{c}_{ik}}\alpha_{i,k+1}\sigma_{0k}(\sigma_{i2k} + \bar{\sigma}_{i2k}) \\
&+\frac{1}{\bar{c}_{ik}}\beta_{i,k+1}b_{0k}(b_{i2k} + \bar{b}_{i2k}) + \frac{1}{2\bar{c}_{ik}}(\gamma_{i,k+1}(b_{i2k} + \bar{b}_{i2k}) + \epsilon_{i2k}),
\end{aligned}
\tag{15}
$$

Coefficients α_{ik}, β_{ik}

$$\begin{aligned}
&\alpha_{ik} = q_{ik} - \eta_{ik}^2 c_{ik} \\
&+\alpha_{i,k+1}[b_{1k} - \textstyle\sum_{j\neq i} b_{j2k}\eta_{jk}]^2 + \alpha_{i,k+1}[\sigma_{1k} - \textstyle\sum_{j\neq i} \sigma_{j2k}\eta_{jk}]^2, \\
&\alpha_{iK} = q_{iK}, \\
&\beta_{ik} = (q_{ik} + \bar{q}_{ik}) - \bar{c}_{ik}\bar{\eta}_{ik}^2 \\
&+\beta_{i,k+1}[(b_{1k} + \bar{b}_{1k}) - \textstyle\sum_{j\neq i}(b_{j2k} + \bar{b}_{j2k})\bar{\eta}_{jk}]^2 \\
&+\alpha_{i,k+1}[(\sigma_{1k} + \bar{\sigma}_{1k}) - \textstyle\sum_{j\neq i}(\sigma_{j2k} + \bar{\sigma}_{j2k})\bar{\eta}_{jk}]^2, \\
&\beta_{iK} = q_{iK} + \bar{q}_{iK},
\end{aligned} \tag{16}$$

Coefficients γ_{ik}, δ_{ik}

$$
\begin{aligned}
&\gamma_{ik} = \gamma_{i,k+1}(b_{1k} + \bar{b}_{1k}) + \epsilon_{i3k} - 2\bar{c}_{ik}\bar{\eta}_{ik}\hat{\eta}_{ik} \\
&+2\alpha_{i,k+1}\sigma_{0k}(\sigma_{1k} + \bar{\sigma}_{1k}) + 2\beta_{i,k+1}b_{0k}(b_{1k} + \bar{b}_{1k}) \\
&+2\alpha_{i,k+1}[\sum_{j\neq i}(\sigma_{j2k} + \bar{\sigma}_{j2k})\bar{\eta}_{jk}][\sum_{j\neq i}(\sigma_{j2k} + \bar{\sigma}_{j2k})\hat{\eta}_{jk}] \\
&+2\beta_{i,k+1}[\sum_{j\neq i}(b_{j2k} + \bar{b}_{j2k})\bar{\eta}_{jk}][\sum_{j\neq i}(b_{j2k} + \bar{b}_{j2k})\hat{\eta}_{jk}] \\
&-2\alpha_{i,k+1}(\sigma_{1k} + \bar{\sigma}_{1k})\sum_{j\neq i}(\sigma_{j2k} + \bar{\sigma}_{j2k})\hat{\eta}_{jk} \\
&-2\beta_{i,k+1}\sum_{j\neq i}(b_{1k} + \bar{b}_{1k})(b_{j2k} + \bar{b}_{j2k})\hat{\eta}_{jk} \\
&-2\alpha_{i,k+1}\sigma_{0k}\sum_{j\neq i}(\sigma_{j2k} + \bar{\sigma}_{j2k})\bar{\eta}_{jk} \\
&-2\beta_{i,k+1}\sum_{\neq i}^{n} b_{0k}(b_{j2k} + \bar{b}_{j2k})\bar{\eta}_{jk} \\
&-\gamma_{i,k+1}\sum_{j\neq i}(b_{j2k} + \bar{b}_{j2k})\bar{\eta}_{jk}, \\
&\gamma_{iK} = \epsilon_{i3K}, \\
&\delta_{ik} = \delta_{i,k+1} - \bar{c}_{ik}\hat{\eta}_{ik}^2 + \alpha_{i,k+1}\sigma_{0k}^2 + \beta_{i,k+1}b_{0k}^2 + \gamma_{i,k+1}b_{0k} \\
&+\alpha_{i,k+1}\{\sum_{j\neq i}(\sigma_{j2k} + \bar{\sigma}_{j2k})\hat{\eta}_{jk}\}^2 + \beta_{i,k+1}[\sum_{j\neq i}(b_{j2k} + \bar{b}_{j2k})\hat{\eta}_{jk}]^2 \\
&-2\alpha_{i,k+1}\sigma_{0k}\sum_{j\neq i}(\sigma_{j2k} + \bar{\sigma}_{j2k})\hat{\eta}_{jk} \\
&-2\beta_{i,k+1}\sum_{\neq i}^{n} b_{0k}(b_{j2k} + \bar{b}_{j2k})\hat{\eta}_{jk} - \gamma_{i,k+1}\sum_{j\neq i}(b_{j2k} + \bar{b}_{j2k})\hat{\eta}_{jk}, \\
&\delta_{iK} = 0.
\end{aligned}
\tag{17}
$$

From Discrete Step to Continuous Time

$$
\begin{aligned}
&\Delta > 0,\\
&\tilde{b}_{0k} = b_{0k}\Delta,\ \tilde{b}_{1k} = 1 + b_{1k}\Delta,\\
&\tilde{b}_{j2k} = b_{j2k}\Delta, \bar{\tilde{b}}_{j2k} = \bar{b}_{j2k}\Delta,\\
&\tilde{\sigma} = \sigma\sqrt{\Delta},\ \bar{\tilde{\sigma}} = \bar{\sigma}\sqrt{\Delta},\\
&\tilde{q}_{ik} = q_{ik}\Delta,\ \bar{\tilde{q}}_{ik} = \bar{q}_{ik}\Delta,\\
&\tilde{r}_{ik} = r_{ik}\Delta,\ \bar{\tilde{r}}_{ik} = \bar{r}_{ik}\Delta,\\
&\tilde{\epsilon}_{i2k} = \epsilon_{i2k}\Delta\ ,\ \tilde{\epsilon}_{i3k} = \epsilon_{i3k}\Delta,\\
&w_{k+1} = \frac{1}{\sqrt{\Delta}}(B_{(k+1)\Delta} - B_{k\Delta}),\ K = [\frac{T}{\Delta}].
\end{aligned}
\tag{18}
$$

$$
\begin{cases}
\frac{\tilde{b}_{0k}\tilde{b}_{j2k}}{\Delta} \to 0, \\
\frac{(b_{i2}+\bar{b}_{i2})(b_{j2}+\bar{b}_{j2})}{\Delta} \to 0, \\
[\tilde{b}_{1k} - \sum_{j\neq i}\tilde{b}_{j2k}\eta_{jk}]^2 = 1 + \Delta[2b_1 - 2\sum_{j\neq i} b_{j2k}\eta_{jk}] + O(\Delta^2), \\
w_{k+1} \to dB(t),
\end{cases}
\tag{19}
$$

as the step size Δ goes to zero

$$\begin{cases} u_i = -\eta_i(x - \mathbb{E}x) - \bar{\eta}_i\mathbb{E}x - \hat{\eta}_i, \\ \eta_i = -\frac{1}{c_i}\alpha_i\sigma_{i2}\sum_{j\neq i}\sigma_{j2}\eta_j \\ +\frac{1}{c_i}\alpha_i b_{i2} + \frac{1}{c_i}\alpha_i\sigma_1\sigma_{i2} + \frac{\epsilon_{i1}}{c_i}, \\ \\ \bar{\eta}_i = -\frac{1}{\bar{c}_i}\alpha_i(\sigma_{i2} + \bar{\sigma}_{i2})\sum_{j\neq i}(\sigma_{j2} + \bar{\sigma}_{j2})\bar{\eta}_j \\ +\frac{1}{\bar{c}_i}\alpha_i(\sigma_1 + \bar{\sigma}_1)(\sigma_{i2} + \bar{\sigma}_{i2}) \\ +\frac{1}{\bar{c}_i}(\epsilon_{i1} + \bar{\epsilon}_{i1}), \\ \hat{\eta}_i = -\frac{1}{\bar{c}_i}\alpha_i(\sigma_{i2} + \bar{\sigma}_{i2})\sum_{j\neq i}(\sigma_{j2} + \bar{\sigma}_{j2})\hat{\eta}_j \\ +\frac{1}{\bar{c}_i}\alpha_i\sigma_0(\sigma_{i2} + \bar{\sigma}_{i2}) + \frac{1}{2\bar{c}_i}(\gamma_i(b_{i2} + \bar{b}_{i2}) + \epsilon_{i2}), \end{cases} \tag{20}$$

and the equilibrium cost of decision-maker i is

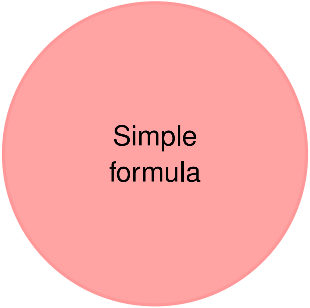
$$\mathbb{E}[\hat{L}_i] = \alpha_i(0)(x_0 - \mathbb{E}x_0)^2 + \beta_i(0)(\mathbb{E}x_0)^2 + \gamma_i(0)(\mathbb{E}x_0) + \delta_i(0).$$

The scaling $\frac{\alpha_{i,k+1}-\alpha_{ik}}{\Delta}$ converges to the time-derivative of α_i as Δ goes to zero.

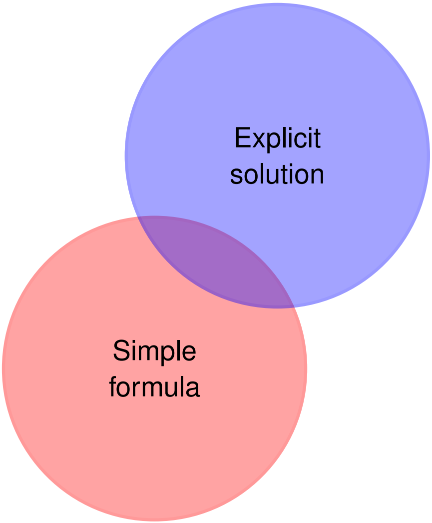
$$\frac{\alpha_{i,k+1}-\alpha_{ik}}{\Delta} \to \dot{\alpha}_i.$$

We retrieve the continuous time solution !

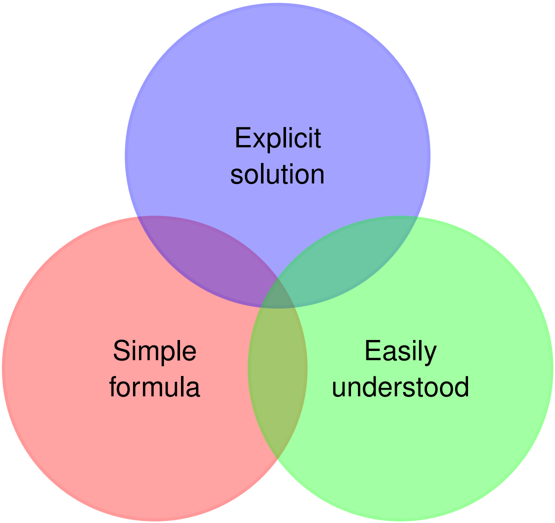
Direct method is key



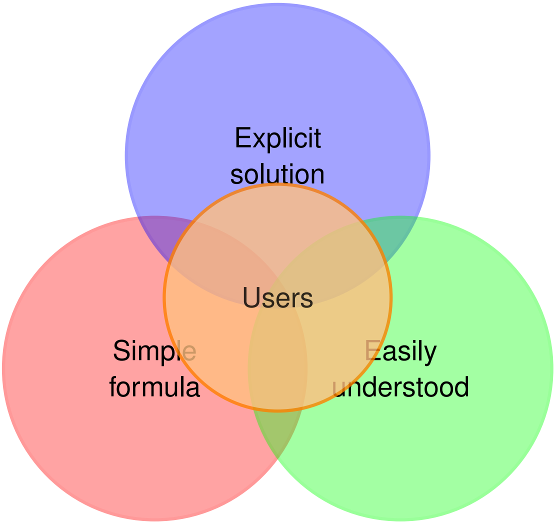
Direct method is key



Direct method is key



Direct method is key



THANK YOU