Linear-Quadratic Mean-Field-Type Games

common noise, jump-diffusion, regime switching

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Lecture 3: Social cost, Robustness, Bargaining, Empathy, Coopetition

Agenda

1. Full Cooperation Problem (with Common Noise)
2. Nash Bargaining Problem
3. Robustness and Zero-Sum case
4. Other-regarding payoffs: empathy, antipathy, self-abnegation
5. MFT-LQJD with regime switching

Global Optimization Problem (with Common Noise)

$$\begin{array}{l} \text{Team} \; : \quad \inf_{(u_1,\ldots,u_n)} \mathbb{E} \sum_i L_i, \; \text{subject to:} \\ ds = \{a(s-\hat{s}) + (a+\bar{a})\hat{s} \\ \qquad + \sum_{i=1}^{n} b_i(u_i - \hat{u}_i) + \sum_{i=1}^{n} (b_i + \bar{b}_i)\hat{u}_i \Big\} dt \\ \qquad + \sigma dB + \int_{\Theta} \mu(t,\theta)\tilde{N}(dt, d\theta) + \sigma_o dB_o, \\ s(0) = s_0, \quad \perp\!\!\!\perp \{B, N\} \end{array} \tag{1}$$

Cooperative Solution (with Common Noise)

$H_0: \ q_0(t) := \sum_i q_i(t) > 0, \bar{q}_0(t) := \sum_i q_i(t) + \bar{q}_i(t) > 0.$

Theorem

Mean-Field-Type Cooperative Solution

$$\begin{array}{l}
\inf_{(u_1,\ldots,u_n)} \mathbb{E} \sum_i [L_i] = \\
\mathbb{E}\{\frac{1}{2}\alpha_0(0) var(s(0)) + \frac{1}{2}\beta_0(0)[\hat{s}(0)]^2 + \gamma_0(0)\hat{s}(0) + \delta_0(0)\}, \\
u_i^* = -\frac{b_i}{r_i}\alpha_0(s - \hat{s}) - \frac{(b_i+\bar{b}_i)}{r_i+\bar{r}_i}(\beta_0\hat{s} + \gamma_0), \\
d\hat{s} = \{[a + \bar{a} - \beta_0 \sum_{i=1}^n \frac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i}]\hat{s} + \gamma_0 \sum_{i=1}^n \frac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i}\}dt + \sigma_o dB_o, \\
\hat{s}(0) = \hat{s}_0.
\end{array} \tag{2}$$

Stochastic Riccati System for Cooperative Solution

$$
\begin{aligned}
& d\alpha_0 = -\left\{2a\alpha_0 - \alpha_0^2 \sum_i \frac{b_i^2}{r_i} + q_i\right\} dt + [\alpha_{0,o}]dB_o, \\
& \alpha_0(T) = q_0(T), \\
& d\beta_0 = -\left\{2\beta_0(a+\bar{a}) - \beta_0^2 \sum_i \frac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i} + q_0 + \bar{q}_0\right\} dt + \beta_{0,o}dB_o, \\
& \beta_0(T) = q_0(T) + \bar{q}_0(T), \\
& d\gamma_0 = -\left\{(a+\bar{a})\gamma_0 + \beta_{0,o}\sigma_o - \beta_0\gamma_0 \sum_i \frac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i}\right\} dt - \sigma_o\beta_0 dB_o, \\
& \gamma(T) = 0, \\
& d\delta_0 = -\left\{[\sigma^2 + \int_\Theta \mu^2(t,\theta)\nu(d\theta)]\frac{\alpha_0}{2} + \sigma_o^2\frac{\beta_0}{2} + \gamma_{0,o}\sigma_o - \frac{1}{2}\gamma_0^2 \sum_i \frac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i}\right\} dt \\
& \qquad -\sigma_o\gamma_0 dB_o,
\end{aligned}
\tag{3}
$$

Summary: Multi-agent fully-cooperative approach

- Full cooperation reduces the Riccati system into a simpler one.
- the common noise brings Stochastic Riccati system
- $\hat{s}$ is an $\mathcal{F}^{B_o}$-adapted process
- Regularize the processes γ_0 and δ_0

Nash Bargaining solution

$$NBS(\mathcal{V}, L^{NE}) = \arg\max_{v\in\mathcal{V}}\{\prod_{i\in\mathcal{N}}[L_i^{NE} - v_i]\}$$

- $$\begin{array}{l} \inf_{(u_1,\ldots,u_n)} \mathbb{E}\sum_{i=1}^n w_i L_i = \\ \mathbb{E}\{\frac{1}{2}\alpha_0(0)var(s(0)) + \frac{1}{2}\beta_0(0)[\hat{s}(0)]^2 + \gamma_0(0)\hat{s}(0) + \delta_0(0)\}, \\ \hat{u}(w) \in \arg\min_u \langle w, L(u)\rangle \end{array} \quad (4)$$
- Optimal Bargaining strategy: $\hat{u}_i(w) = -\frac{b_i}{w_i r_i}\alpha_0(s-\hat{s}) - \frac{(b_i+\bar{b}_i)}{w_i(r_i+\bar{r}_i)}(\beta_0\hat{s}+\gamma_0)$,
- Set the weight $w_i^* = \frac{\prod_{j\neq i}[L_j^{NE} - L_j(\hat{u}(w^*))]}{\sum_{k=1}^n \prod_{j\neq k}[L_j^{NE} - L_j(\hat{u}(w^*))]}$.

Nash bargaining

$$
\begin{aligned}
d\alpha_0 &= -\left\{2a\alpha_0 - \alpha_0^2 \sum_i \frac{b_i^2}{w_i r_i} + \sum_i w_i q_i\right\} dt + \alpha_{0,o} dB_o,\\
\alpha_0(T) &= \sum_i w_i q_i(T),\\
d\beta_0 &= -\left\{2\beta_0(a+\bar{a}) - \beta_0^2 \sum_i \frac{(b_i+\bar{b}_i)^2}{w_i(r_i+\bar{r}_i)} + \sum_i w_i(q_i+\bar{q}_i)\right\} dt + \beta_{0,o} dB_o,\\
\beta_0(T) &= \sum_i w_i(q_i+\bar{q}_i(T)),\\
d\gamma_0 &= -\left\{(a+\bar{a})\gamma_0 + \beta_{0,o}\sigma_o - \beta_0\gamma_0 \sum_i \frac{(b_i+\bar{b}_i)^2}{w_i(r_i+\bar{r}_i)}\right\} dt - \sigma_o \beta_0 dB_o,\\
\gamma_0(T) &= 0,\\
d\delta_0 &= -\left\{[\sigma^2 + \int_\Theta \mu^2(t,\theta)\nu(d\theta)]\frac{\alpha_0}{2} + \sigma_o^2\frac{\beta_0}{2} + \gamma_{0,o}\sigma_o - \frac{1}{2}\gamma_0^2 \sum_i \frac{(b_i+\bar{b}_i)^2}{w_i(r_i+\bar{r}_i)}\right\} dt\\
&\quad -\sigma_o\gamma_0 dB_o,\\
d\hat{s} &= \{[a+\bar{a} - \beta_0 \sum_{i=1}^n \frac{(b_i+\bar{b}_i)^2}{w_i(r_i+\bar{r}_i)}]\hat{s} - \gamma_0 \sum_{i=1}^n \frac{(b_i+\bar{b}_i)^2}{w_i(r_i+\bar{r}_i)}\} dt + \sigma_o dB_o,\\
\hat{s}(0) &= \hat{s}_0.
\end{aligned}
\tag{5}
$$

Summary: Multi-agent bargaining solution

- Non-convex (in u)
- Solve a weighted full cooperation problem
- choose carefully the weight to match the bargaining power

Robust Optimization Problem (with Common Noise)

$$
\begin{aligned}
L(u_1,u_2) = \tfrac{1}{2}[&q(T)(s(T)-\hat{s}(T))^2 + [q(T)+\bar{q}(T)]\hat{s}^2(T)\\
&+\int_0^T q(t)(s(t)-\hat{s}(t))^2 + [q(t)+\bar{q}(t)]\hat{s}^2(t)dt\\
+\int_0^T & r_1(t)(u_1(t)-\hat{u}_1(t))^2 + (r_1(t)+\bar{r}_1(t))[\hat{u}_1(t)]^2dt\\
+\int_0^T & r_2(t)(u_2(t)-\hat{u}_2(t))^2 + (r_2(t)+\bar{r}_2(t))[\hat{u}_2(t)]^2dt],
\end{aligned}
$$

$$
\begin{aligned}
&\textbf{Problem}: \ \inf_{u_1} \sup_{u_2} \mathbb{E}[L(u_1,u_2)],\\
&ds = \{a(t)(s(t)-\hat{s}(t)) + (a(t)+\bar{a}(t))\hat{s}(t)\\
&+\textstyle\sum_{i=1}^2 b_i(t)(u_i(t)-\hat{u}_i(t)) + \sum_{i=1}^2 (b_i(t)+\bar{b}_i(t))\hat{u}_i(t)\Big\}\, dt\\
&+\sigma(t)dB(t) + \int_\Theta \mu(t,\theta)\tilde{N}(t,d\theta) + \sigma_o(t)dB_o(t),\\
&s(0)=s_0 \in L^2(\Omega,\mathbb{R}), \quad s(0) \perp\!\!\!\perp \{B,N\}
\end{aligned}
\tag{6}
$$

For large population, see:

D. Bauso, H. Tembine, T. Başar: Robust Mean Field Games, Dynamic Games and Applications Sept. 2016, Vol. 6, Issue 3, pp 277-303

MinMax Solution and Saddle Point with Common Noise

A: $r_1 > 0,\ r_1 + \bar{r}_1 > 0, r_2 < 0,\ r_2 + \bar{r}_2 < 0$

Theorem

MinMax Solution and Saddle Point with Common Noise

$$
\begin{array}{l}
\inf_{u_1} \sup_{u_2} \mathbb{E}[L(u_1, u_2)] = \\
\mathbb{E}\{\frac{1}{2}\alpha(0) var(s(0)) + \frac{1}{2}\beta(0)[\hat{s}(0)]^2 + \gamma(0)\hat{s}(0) + \delta(0)\}, \\
u_1^* = -\frac{b_1}{r_1}\alpha(s-\hat{s}) - \frac{(b_1+\bar{b}_1)}{r_1+\bar{r}_1}(\beta\hat{s}+\gamma), \\
u_2^* = -\frac{b_2}{r_2}\alpha(s-\hat{s}) - \frac{(b_2+\bar{b}_2)}{r_2+\bar{r}_2}(\beta\hat{s}+\gamma), \\
d\hat{s} = \{[a + \bar{a} - \beta\sum_{i=1}^2 \frac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i}]\hat{s} + \gamma\sum_{i=1}^2 \frac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i}\}dt + \sigma_o dB_o, \\
\hat{s}(0) = \hat{s}_0.
\end{array}
\tag{7}
$$

Stochastic Riccati System for MinMax

$$
\begin{aligned}
& d\alpha = -\left\{2a\alpha - \alpha^2 \sum_{i=1}^2 \frac{b_i^2}{r_i} + q\right\} dt + \alpha_o dB_o, \\
& \alpha(T) = q(T), \\
& d\beta = -\left\{2\beta(a+\bar{a}) - \beta^2 \sum_{i=1}^2 \frac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i} + q + \bar{q}\right\} dt + \beta_o dB_o, \\
& \beta(T) = q(T) + \bar{q}(T), \\
& d\gamma = -\left\{(a+\bar{a})\gamma + \beta_o\sigma_o - \beta\gamma \sum_{i=1}^2 \frac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i}\right\} dt - \sigma_o\beta dB_o, \\
& \gamma(T) = 0, \\
& d\delta = -\left\{[\sigma^2 + \int_\Theta \mu^2(t,\theta)\nu(d\theta)]\frac{\alpha}{2} + \sigma_o^2\frac{\beta}{2} \right. \\
& \left. + \gamma_o\sigma_o - \frac{1}{2}\gamma^2 \sum_{i=1}^2 \frac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i}\right\} dt - \sigma_o\gamma dB_o
\end{aligned}
\tag{8}
$$

Summary: Maxmin robust solution

- Saddle point solution obtained from a simple stochastic Riccati system
- One single equation α instead of two equations (α_1, α_2).

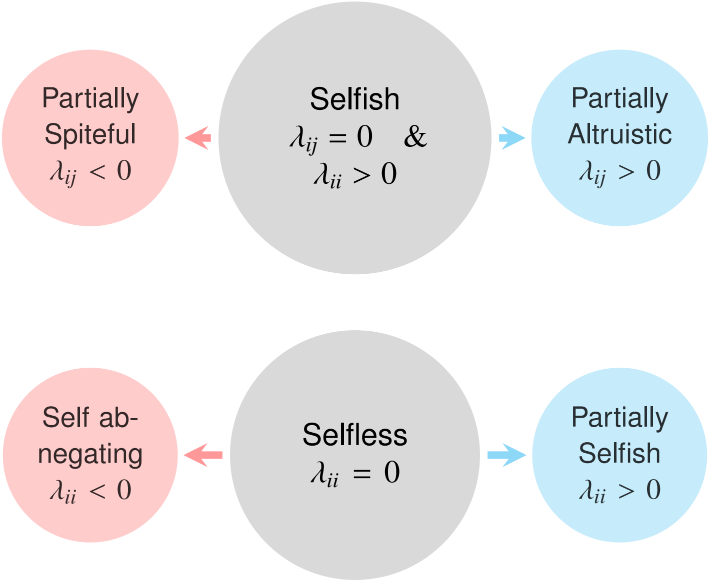
Other-regarding payoffs

- Empathy structure: $\Lambda \in \mathbb{R}^{n\times n}$
- Empathetic objective function: l_i^{Λ}
- $$l_i^{\Lambda} := \lambda_{ii} l_i + \sum_{j\in\mathcal{N}_i} \lambda_{ij} l_j,$$
- One-hop neighbors of i :

$$\mathcal{N}_i = \{j \in \mathcal{N} \mid j \neq i,\ \lambda_{ij} \neq 0\}.$$

T. E. Duncan, H. Tembine: Other-regarding payoffs in linear-quadratic mean-field-type games with common noise: A direct method, Preprint, 2018

Empathy and other-regarding payoffs



Other-regarding payoff

$$
\begin{aligned}
&\textbf{Problem}: \ \inf_{u_i} \ \mathbb{E}L_i^{\Lambda}(u_1,\ldots,u_n)\\
&= \frac{1}{2}(\lambda_{ii}q_i(T) + \sum_{j\in\mathcal{N}_i}\lambda_{ij}q_j(T))\mathbb{E}(s(T)-\hat{s}(T))^2\\
&+\frac{1}{2}(\lambda_{ii}(q_i(T)+\bar{q}_i(T)) + \sum_{j\in\mathcal{N}_i}\lambda_{ij}(q_j(T)+\bar{q}_j(T)))E\hat{s}^2(T)\\
&+\frac{1}{2}\mathbb{E}\int_0^T \Big\{(\lambda_{ii}q_i + \sum_{j\in\mathcal{N}_i}\lambda_{ij}q_j)(s-\hat{s})^2\\
&\qquad +(\lambda_{ii}(q_i+\bar{q}_i) + \sum_{j\in\mathcal{N}_i}\lambda_{ij}(q_j+\bar{q}_j))\hat{s}^2\\
&\qquad +\lambda_{ii}r_i(u_i-\hat{u}_i)^2 + \sum_{j\in\mathcal{N}_i}\lambda_{ij}r_j(u_j-\hat{u}_j)^2\\
&\qquad +\lambda_{ii}(r_i+\bar{r}_i)\hat{u}_i^2 + \sum_{j\in\mathcal{N}_i}\lambda_{ij}(r_j+\bar{r}_j)\hat{u}_j^2\Big\}\,dt,
\end{aligned} \tag{9}
$$

subject to

$$
\begin{aligned}
ds &= \Big\{a(s-\hat{s}) + (a+\bar{a})\hat{s} + \sum_{i=1}^n b_i(u_i-\hat{u}_i) + \sum_{i=1}^n (b_i+\bar{b}_i)\hat{u}_i\Big\}\,dt\\
&\quad +\sigma dB + \int_\Theta \mu(t,\theta)\tilde{N}(dt,d\theta) + \sigma_o dB_o.\\
&s(0) \perp\!\!\!\perp \{B, N\}
\end{aligned}
$$

T. E. Duncan, H. Tembine: Other-regarding payoffs in linear-quadratic mean-field-type games with common noise: A direct method, Preprint, 2018

Other-regarding payoff: solution

Theorem

Empathetic equilibria

$$\begin{aligned}
&\inf_{u_i\in\mathcal{U}_i} \mathbb{E}[L_i^{\Lambda}] \\
&= \mathbb{E}\{\tfrac{1}{2}\alpha_i(0)(s(0)-\hat{s}(0))^2 + \tfrac{1}{2}\beta_i(0)[\hat{s}(0)]^2 + \gamma_i(0)\hat{s}(0) + \delta_i(0)\} \\
&u_i^* = -\frac{b_i}{\lambda_{ii} r_i}\alpha_i(s-\hat{s}) - \frac{(b_i+\bar{b}_i)}{\lambda_{ii}(r_i+\bar{r}_i)}(\beta_i\hat{s}+\gamma_i), \\
&d\hat{s} = \{[a+\bar{a} - \textstyle\sum_{i=1}^n \frac{(b_i+\bar{b}_i)^2}{\lambda_{ii}(r_i+\bar{r}_i)}\beta_i]\hat{s} - \sum_{i=1}^n \frac{(b_i+\bar{b}_i)^2}{\lambda_{ii}(r_i+\bar{r}_i)}\gamma_i\}dt \\
&+\sigma_o dB_o, \\
&\hat{s}(0) = \hat{s}_0, \quad s(0) \perp\!\!\!\perp \{B, N\}
\end{aligned} \tag{10}$$

Other-regarding payoff: solution

$$
\begin{aligned}
d\alpha_i &= -\Big\{2a\alpha_i - \frac{b_i^2}{\lambda_{ii} r_i}\alpha_i^2 - 2\alpha_i \sum_{j\neq i} \frac{b_j^2}{\lambda_{jj} r_j}\alpha_j + \lambda_{ii} q_i + \sum_{j\in\mathcal{N}_i} \lambda_{ij} q_j \\
&+ \sum_{j\in\mathcal{N}_i} \frac{b_j^2 \lambda_{ij}}{\lambda_{jj}^2 r_j}\alpha_j^2\Big\} dt + \alpha_{i,o} dB_o, \\
\alpha_i(T) &= \lambda_{ii} q_i(T) + \sum_{j\in\mathcal{N}_i} \lambda_{ij} q_j(T), \\
d\beta_i &= -\Big\{2(a+\bar{a})\beta_i - \frac{(b_i+\bar{b}_i)^2}{\lambda_{ii}(r_i+\bar{r}_i)}\beta_i^2 - 2\beta_i \sum_{j\neq i} \frac{(b_j+\bar{b}_j)^2}{\lambda_{jj}(r_j+\bar{r}_j)}\beta_j \\
&+ \lambda_{ii}(q_i+\bar{q}_i) + \sum_{j\in\mathcal{N}_i} \lambda_{ij}(q_j+\bar{q}_j) \\
&+ \sum_{j\in\mathcal{N}_i} \lambda_{ij} \frac{(b_j+\bar{b}_j)^2}{\lambda_{jj}^2 (r_j+\bar{r}_j)}\beta_j^2\Big\} dt + \beta_{i,o} dB_o, \\
\beta_i(T) &= \lambda_{ii}(q_i(T)+\bar{q}_i(T)) + \sum_{j\in\mathcal{N}_i} \lambda_{ij}(q_j(T)+\bar{q}_j(T)),
\end{aligned} \tag{11}
$$

Other-regarding payoff: solution

$$
\begin{aligned}
&d\gamma_i = -\Big\{(a+\bar{a})\gamma_i + \beta_{i,o}\sigma_o - \tfrac{(b_i+\bar{b}_i)^2}{\lambda_{ii}(r_i+\bar{r}_i)}\beta_i\gamma_i \\
&-\beta_i \textstyle\sum_{j\neq i} \tfrac{(b_j+\bar{b}_j)^2}{\lambda_{jj}(r_j+\bar{r}_j)}\gamma_j - \gamma_i \textstyle\sum_{j\neq i} \tfrac{(b_j+\bar{b}_j)^2}{\lambda_{jj}(r_j+\bar{r}_j)}\beta_j \\
&+\textstyle\sum_{j\in\mathcal{N}_i} \lambda_{ij}\tfrac{(b_j+\bar{b}_j)^2}{\lambda_{jj}^2(r_j+\bar{r}_j)}\beta_j\gamma_j\Big\}dt - \sigma_o\beta_i dB_o, \\
&\gamma_i(T) = 0, \\
&d\delta_i = -\Big\{[\sigma^2 + \textstyle\int_\Theta \mu^2(t,\theta)\nu(d\theta)]\tfrac{1}{2}\alpha_i + \tfrac{1}{2}\sigma_o^2\beta_i + \gamma_{i,o}\sigma_o \\
&-\gamma_i \textstyle\sum_{j\neq i} \tfrac{(b_j+\bar{b}_j)^2}{\lambda_{jj}(r_j+\bar{r}_j)}\gamma_j - \tfrac{(b_i+\bar{b}_i)^2}{\lambda_{ii}(r_i+\bar{r}_i)}\tfrac{1}{2}\gamma_i^2 \\
&+\tfrac{1}{2}\textstyle\sum_{j\in\mathcal{N}_i} \lambda_{ij}\tfrac{(b_j+\bar{b}_j)^2}{\lambda_{jj}^2(r_j+\bar{r}_j)}\gamma_j^2\Big\}dt - \sigma_o\gamma_i dB_o, \\
&\delta_i(T) = 0,
\end{aligned}
\tag{12}
$$

Other-regarding payoff: new behaviors

- Empathy structure can be inconsistent
- Pure altruism can be inconsistent
- Partial altruism may help in reducing payoff inequality
- Empathy spite may lead to non-admissible strategies
- Empathy spite may lead to non-existence of equilibria

Regime Switching

MFT-LQJD with regime switching

$$
\begin{cases}
\inf_{u_i(\cdot)\in\mathcal{U}_i} \mathbb{E}\left[L_i(u_1,\ldots,u_n)\right], \\
\text{ subject to} \\
ds(t) = \Big\{a(t,\tilde{s}(t))s(t) + \bar{a}(t,\tilde{s}(t))\mathbb{E}[s(t) \mid \mathcal{F}_t^{B_o}] \\
+ \sum_{i=1}^{n} b_i(t,\tilde{s}(t))u_i(t) + \sum_{i=1}^{n} \bar{b}_i(t,\tilde{s}(t))\mathbb{E}[u_i(t) \mid \mathcal{F}_t^{B_o}]\Big\} dt \\
+\sigma(t,\tilde{s}(t))dB(t) + \int_{\Theta} \mu(t,\theta,\tilde{s}(t))\tilde{N}(dt,d\theta) + \kappa(t,.).\tilde{M}(dt) \\
+\sigma_o(t,\tilde{s}(t))dB_o(t), \\
s(0) \in L^2(\Omega,\mathbb{R}),\ s(0) \perp\!\!\!\perp \{B,N\}
\end{cases}
\tag{13}
$$

$\tilde{s}(t)$: switching between regimes $1,\ldots,S$, with rate $q_{\tilde{s}k}$

MFT-LQJD with regime switching: solution

$$
\begin{aligned}
&\inf_{u_i\in\mathcal{U}_i} \mathbb{E}[L_i] \\
&= \mathbb{E}\{ \tfrac{1}{2}\alpha_i(0,\tilde{s}(0))(s(0)-\hat{s}(0))^2 + \tfrac{1}{2}\beta_i(0,\tilde{s}(0))\hat{s}(0)^2 \quad +\gamma_i(0,\tilde{s}(0))\hat{s} + \delta_i(0,\tilde{s}(0))\}, \\
&u_i^* = -\frac{b_i}{r_i}\alpha_i(s-\hat{s}) - \frac{(b_i+\bar{b}_i)}{r_i+\bar{r}_i}(\beta_i\hat{s}+\gamma_i), \\
&d\hat{s} = \{[a+\bar{a} - \textstyle\sum_{i=1}^n \frac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i}\beta_i]\hat{s} - \sum_{i=1}^n \frac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i}\gamma_i\}dt + \sigma_o dB_o, \\
&\hat{s}(0) = \hat{s}_0, \\
&d\alpha_i = -\left\{2a\alpha_i - \frac{b_i^2}{r_i}\alpha_i^2 - 2\alpha_i \textstyle\sum_{j\neq i}\frac{b_j^2}{r_j}\alpha_j + q_i + \sum_{k\neq\tilde{s}}[\alpha_i(t,k)-\alpha_i(t,\tilde{s})]q_{\tilde{s}k}\right\}dt + [\alpha_{i,o}]dB_o, \\
&\alpha_i(T,\tilde{s}) = q_i(T,\tilde{s}), \\
&d\beta_i = -\Big\{2(a+\bar{a})\beta_i - \frac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i}\beta_i^2 - 2\beta_i\textstyle\sum_{j\neq i}\frac{(b_j+\bar{b}_j)^2}{r_j+\bar{r}_j}\beta_j + q_i + \bar{q}_i \\
&\qquad + \textstyle\sum_{k\neq\tilde{s}}[\beta_i(t,k)-\beta_i(t,\tilde{s})]q_{\tilde{s}k}\Big\}dt + [\beta_{i,o}]dB_o, \\
&\beta_i(T,\tilde{s}) = q_i(T,\tilde{s}) + \bar{q}_i(T,\tilde{s}), \\
&d\gamma_i = -\Big\{(a+\bar{a})\gamma_i - \frac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i}\beta_i\gamma_i - \beta_i\textstyle\sum_{j\neq i}\frac{(b_j+\bar{b}_j)^2}{r_j+\bar{r}_j}\gamma_j - \gamma_i\sum_{j\neq i}\frac{(b_j+\bar{b}_j)^2}{r_j+\bar{r}_j}\beta_j \\
&\qquad +\beta_{i,o}\sigma_o + \textstyle\sum_{k\neq\tilde{s}}[\gamma_i(t,k)-\gamma_i(t,\tilde{s})]q_{\tilde{s}k}\Big\}dt - \sigma_o\beta_i dB_o, \\
&\gamma_i(T,\tilde{s}) = 0, \\
&d\delta_i = -\Big\{\tfrac{1}{2}\alpha_i[\sigma^2 + \textstyle\int_\Theta \mu^2(t_-,.,\theta)\nu(d\theta)] + \tfrac{1}{2}\sigma_o^2\beta_i \\
&\qquad + \tfrac{1}{2}\textstyle\sum_{k\neq\tilde{s}}\alpha_i(t,k)\kappa^2(t_-,k)q_{\tilde{s}k} + \sum_{k\neq\tilde{s}}[\delta_i(t,k)-\delta_i(t,\tilde{s})]q_{\tilde{s}k} + \gamma_{i,o}\sigma_o \\
&\qquad - \frac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i}\tfrac{1}{2}\gamma_i^2 - \gamma_i\textstyle\sum_{j\neq i}\frac{(b_j+\bar{b}_j)^2}{r_j+\bar{r}_j}\gamma_j\Big\}dt - \sigma_o\gamma_i dB_o, \\
&\delta_i(T,\tilde{s}) = 0.
\end{aligned} \tag{14}
$$

MFT-LQJD with regime switching: weighted global optimization

$$
\begin{aligned}
&\inf_{(u_1,\ldots,u_n)} \mathbb{E} \sum_i w_i L_i, \\
&ds = \left\{ a(s-\hat{s}) + (a+\bar{a})\hat{s} + \sum_{i=1}^n b_i(u_i - \hat{u}_i) + \sum_{i=1}^n (b_i + \bar{b}_i)\hat{u}_i \right\} dt \\
&\qquad +\sigma dB + \int_\Theta \mu(t,\theta)\tilde{N}(dt,d\theta) + \sigma_o dB_o + \kappa(t).\tilde{M}(dt), \\
&s(0) = s_0, \quad s(0) \perp\!\!\!\perp \{B, N\}
\end{aligned}
\tag{15}
$$

T. E. Duncan, H. Tembine: Nash bargaining solution in linear-quadratic mean-field-type games: A direct method, Preprint, 2018

Weighted global optimization solution

$$
\begin{aligned}
&\inf_{(u_1,\ldots,u_n)} \mathbb{E}\sum_i w_i L_i = \\
&E\{\tfrac{1}{2}\alpha_0(0)var(s(0)) + \tfrac{1}{2}\beta_0(0)[\hat{s}(0)]^2 + \gamma_0(0)\hat{s}(0) + \delta_0(0)\}, \\
&\hat{u}_i(w) = -\frac{b_i}{w_i r_i}\alpha_0(s-\hat{s}) - \frac{(b_i+\bar{b}_i)}{w_i(r_i+\bar{r}_i)}(\beta_0\hat{s}+\gamma_0), \\
&d\hat{s} = \{[a+\bar{a}-\beta_0\sum_{i=1}^n \frac{(b_i+\bar{b}_i)^2}{w_i(r_i+\bar{r}_i)}]\hat{s} - \gamma_0\sum_{i=1}^n \frac{(b_i+\bar{b}_i)^2}{w_i(r_i+\bar{r}_i)}\}dt + \sigma_o dB_o, \\
&\hat{s}(0) = \hat{s}_0, \\
&d\alpha_0 = -\left\{2a\alpha_0 - \alpha_0^2\sum_i \frac{b_i^2}{w_i r_i} + \sum_i w_i q_i + \sum_{k\neq\tilde{s}}[\alpha_0(t,k)-\alpha_0(t,\tilde{s})]q_{\tilde{s}k}\right\}dt + [\alpha_{0,o}]dB_o, \\
&\alpha_0(T) = \sum_i w_i q_i(T), \\
&d\beta_0 = -\left\{2\beta_0(a+\bar{a}) - \beta_0^2\sum_i \frac{(b_i+\bar{b}_i)^2}{w_i(r_i+\bar{r}_i)} + \sum_i w_i(q_i+\bar{q}_i) + \sum_{k\neq\tilde{s}}[\beta_0(t,k)-\beta_0(t,\tilde{s})]q_{\tilde{s}k}\right\}dt + [\beta_{0,o}]dB_o, \\
&\beta_0(T) = \sum_i w_i(q_i+\bar{q}_i(T)), \\
&d\gamma_0 = -\left\{(a+\bar{a})\gamma_0 + \beta_{0,o}\sigma_o - \beta_0\gamma_0\sum_i \frac{(b_i+\bar{b}_i)^2}{w_i(r_i+\bar{r}_i)} + \sum_{k\neq\tilde{s}}[\gamma_0(t,k)-\gamma_0(t,\tilde{s})]q_{\tilde{s}k}\right\}dt - \sigma_o\beta_0 dB_o, \\
&\gamma_0(T) = 0, \\
&d\delta_0 = -\Bigg\{[\sigma^2 + \int_\Theta \mu^2(t,\theta)\nu(d\theta)]\frac{\alpha_0}{2} + \sigma_o^2\frac{\beta_0}{2} + \gamma_{0,o}\sigma_o - \frac{1}{2}\gamma_0^2\sum_i \frac{(b_i+\bar{b}_i)^2}{w_i(r_i+\bar{r}_i)} \\
&\qquad + \frac{1}{2}\sum_{k\neq\tilde{s}}\alpha_0(t,k)\kappa^2(t_-,k)q_{\tilde{s}k} + \sum_{k\neq\tilde{s}}[\delta_0(t,k)-\delta_0(t,\tilde{s})]q_{\tilde{s}k}\Bigg\}dt - \sigma_o\gamma_0 dB_o, \\
&\delta_0(T) = 0
\end{aligned}
\tag{16}
$$

How useful is this framework?

- from numerical view point, the explicitly solvable games help to build up a reference trajectory
- The numerical plot (computer simulations) will be compared to these solutions to check whether the numerical output makes sense.
- Verification/Validation procedure

Agenda: next step

- Recitation 3: Blockchain-based technologies

THANK YOU

Linear-Quadratic Mean-Field-Type Games

common noise, jump-diffusion, regime switching

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IPAM Graduate Summer School

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Recitation 3: Blockchain-based technology

Agenda

1. How to design a regulated price dynamics?
2. a more stable and regulated price

Blockchain-based technology: example of unstable price

Figure: Coindesk database: the price of bitcoin went from 10K USD to 20 K USD and back to below 7 K USD within 2-3 months in 2017-2018.

Blockchain-based technology: example of a more stable price

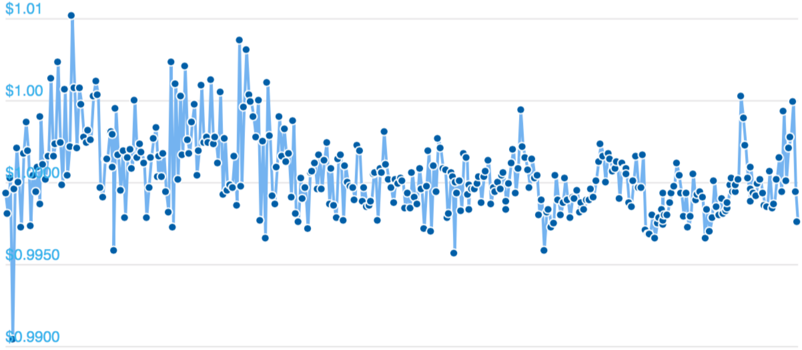


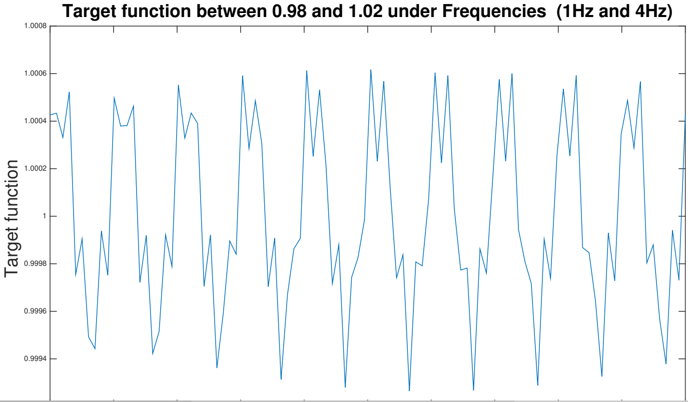
Figure: Coindesk database: the price of tether USD went from 0.99 USD to 1.01 USD

Construction of a regulated price

choose the coefficients $c, \hat{c}$ such that target price $p_{tp,i}(t) \in [\underline{p}_i, \bar{p}_i]$

$$p_{tp,i}(t) = c_{i0} + \sum_{k=1}^{2} c_{ik} \cos(2\pi kt) + \hat{c}_{ik} \sin(2\pi kt),$$

Let $c_{i0} := \frac{\underline{p}_i + \bar{p}_i}{2}$, $c_{i1} := \frac{\bar{p}_i - \underline{p}_i}{100}$, $\hat{c}_{i1} := \frac{\bar{p}_i - \underline{p}_i}{150}$, $c_{12} := \frac{\bar{p}_i - \underline{p}_i}{200}$, $\hat{c}_{12} := \frac{\bar{p}_i - \underline{p}_i}{250}$.



A more stable and regulated price

$$\begin{aligned} dp_i &= \eta[D_i - p_i - (S_i + u_i)]dt \\ &+ \left(\sigma_i dB_i + \int_{\theta\in\Theta} \mu_i(\theta)\tilde{N}_i(dt, d\theta)\right) + \sigma_o dB_o, \end{aligned} \tag{1}$$

$$\begin{cases} L_{mftg} = q_i(t_1)var(p_i(t_1) - p_{tp,i}(t_1)) \\ +[q_i(t_1) + \bar{q}_i(t_1)][Ep_i(t_1) - p_{tp,i}(t_1)]^2 \\ + \int_{t_0}^{t_1} q_i(t)var(p_i(t) - p_{tp,i}(t))dt \\ + \int_{t_0}^{t_1} (q_i(t) + \bar{q}_i(t))[Ep_i(t) - p_{tp,i}(t)]^2 \, dt. \end{cases} \tag{2}$$

A more stable and regulated price

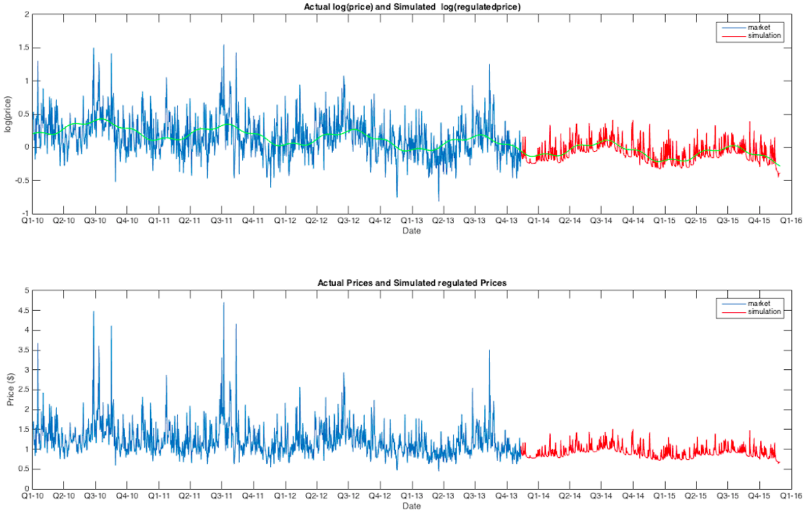


Figure: Real market price and simulation of the regulated price dynamics as a continuation price under MFTG strategy.

Consumption-Investment-Insurance over several assets

Some few prior works

Mossin, J. (1968), Aspects of rational insurance purchasing, Journal of Political Economy, 79: 553-568.

van Heerwaarden, A. (1991), Ordering of Risks, Thesis, Tinbergen Institute, Amsterdam.

Kristen S. Moore and Virginia R. Young, Optimal Insurance in a Continuous-Time Model, Insurance: Mathematics and Economics 39, 47-68, 2006.

with a small non-quadratic part:

MFTG: Consumption-Investment-Insurance

$$
\begin{aligned}
&\sup_{(u^c,u^I,u^{ins})} \ -qe^{-\lambda t_1}[s(t_1)-\hat{s}(t_1)]^2 + \int_{t_0}^{t_1} e^{-\lambda t}\log u^c \ dt \\
&ds = \kappa_0(r_0(\tilde{s})+\hat{\mu}_0(\tilde{s}))sdt \\
&+\sum_{k=1}^d[\hat{\mu}_k-(r_0(\tilde{s})+\hat{\mu}_0(\tilde{s}))\kappa_0+\text{Drift}_k(\tilde{s})]u_k^I dt \\
&-u^c dt - \bar{\lambda}(\tilde{s})(1+\bar{\theta}(\tilde{s}))E[u^{ins}]dt \\
&+\sum_{k=1}^d u_k^I \text{Diffusion}_k(\tilde{s}) + \sum_{k=1}^d u_k^I \text{Jump}_k(\tilde{s}), \\
&-(L-u^{ins})dN,
\end{aligned} \tag{3}
$$

$$
\begin{aligned}
&L = l(\tilde{s})s, \\
&Drift_k = \eta[D_k - p_k - (S_k + u_{mftg,k})]dt \\
&+\frac{1}{2}(\sigma_i^2+\sigma_o^2)dt + \int_\Theta [e^{\gamma_k}-1-\gamma_k]\nu(d\theta)dt, \\
&Diffusion_k = (\sigma_k dB_k + \sigma_o dB_o), \\
&Jump_k = \int_\Theta [e^{\gamma_k}-1]\tilde{N}_k(dt,d\theta),
\end{aligned} \tag{4}
$$

MFTG: Consumption-Investment-Insurance

Investment over several asset

MFTG strategy

Consumption

the optimal consumption strategy process is proportional to the wealth process: $u^c = \frac{e^{-\lambda t}}{\alpha_1} x$

Insurance

the optimal insurance is a decreasing and convex function of $\bar{\theta}$.
$u^{ins} := \left[l(\tilde{s}) - \frac{1+\theta(\tilde{s})}{2+\theta(\tilde{s})} \right]_+ x$

Network Security as a Public Good

$$
\begin{aligned}
&\sup_{u_i}\ \mathbb{E}\{-\tfrac{q}{2}e^{-\lambda_i t_1}\,(s(t_1)-\hat{s}(t_1))^2+\int_{t_0}^{t_1}e^{-\lambda_i t}\,[s(1-\epsilon s)-C_i(u_i)]\,dt\},\\
&ds=[-as+\textstyle\sum_{i=1}^{n}u_i]dt+s[\sigma dB+\int_{\Theta}\mu(\theta)\tilde{N}(dt,d\theta)+\sigma_o dB_o],\\
&s_0,\\
&q>0,\ \lambda_i>0, t_0<t_1,\\
&\hat{s}(t)=\mathbb{E}[s(t)\mid \mathcal{F}_t^{B_o}]
\end{aligned}
\tag{5}
$$

Agenda: next step

- Lecture 4: How far can we go without PDEs without SMPs?

THANK YOU