Linear-Quadratic Mean-Field-Type Games

common noise, jump-diffusion, regime switching

Tembine Hamidou

Learning & Game Theory Laboratory, NYUAD
Wuchen Li, UCLA
IPAM Graduate Summer School

June 18-29, 2018

Acknowledgments

- Collaborators:
  - Jian Gao, Yida Xu, Michail Smyrnakis, Massa Ndong, Julian Barreiro-Gomez,
  - Eitan Altman (INRIA), Tamer Başar (UIUC), Jean-Yves LeBoudec (EPFL), Alain Bensoussan (UT), Boualem Djehiche (KTH), Tyrone E. Duncan (Kansas)
- We appreciate support from U.S. Air Force Office of Scientific Research under grant number FA9550-17-1-0259.

Lecture 2: LQ games (with mean-field dependence)

Agenda

1. Jovanovic & Rosenthal 1982-1988's Model
2. Mean-Field Games (with Jump-Diffusion)
3. LQ mean-field games: Direct method
4. LQ mean-field-type games: Direct method
5. Games with common noise

Mean-Field Games

What tools are useful for users studying dynamical systems with infinitely many interacting agents?

- Mean-field game theory

Jovanovic & Rosenthal 1982-1988's Model

- Time $t \in \mathcal{T} := \{1, 2, \ldots\}$
- At each time t, each agent chooses a feasible control action $u(t) \in \mathcal{U}(t, s(t), m_1(t)) \subset \mathbb{U}, \ m_1 = \mathbb{P}_s$
- State change: $\mathbb{P}(s(t+1) = s' \mid s(t) = s, u(t) = u, m(t) = m) = F(s'; t, s, u, m),$ $m(t, .) = \mathbb{P}_{(s(t), u(t))},$
- The representative agent receives an Instant payoff: $r(t, s(t), u(t), m(t)) \in \mathbb{R}.$
- The evolution of m is given by Kolmogorov equation: $m(t+1) = \Phi(t, m(t)) = \int_{s,u} Fm(t, dsdu).$
- Long-term discounted payoff: $\frac{\sum_{t=1}^{+\infty} r(t,s(t),u(t),m(t)) \prod_{1\leq k\leq t} \beta(k)}{\sum_{t=1}^{+\infty} \prod_{1\leq k\leq t} \beta(k)}$

Mean-Field Games: some references

- Infinite number of agents: Borel 1921, Volterra'26, Hotelling'29,von Neumann'44, Nash'51, Wardrop'52, Aumann'64, Selten'70, Schmeidler'73, Dubey et al.'80-, etc
- Discrete-time/state mean-field games:
  - Jovanovic'82, Jovanovic & Rosenthal'88, Bergins & Bernhardt'92, Weibull& Benaïm'03-, Weintraub, Benkard, Van Roy'05-, Sandholm '06-, Adlaska, Johari, Goldsmith'08-, Benaïm & Le Boudec'08-, Gast & Gaujal'09, Bardenave'09-, Gomes, Mohr & Souza'10-, Borkar & Sundaresan'12, Elliott'12-,Bayraktar, Budhiraja, Cohen'17- etc
- Continuous-time mean-field games
  - Krusell & Smith'98, Benamou & Brenier'00-, Huang, Caines, Malhame'03-, Lasry & Lions'06-, Kotelenez & Kurtz'07-, Li & Zhang'08-, Buckdahn, Djehiche, Li and Peng'09-, Gueant'09-, Gomes et al.'09-, Yin, Mehta, Meyn, and Shanbhag'10, Djehiche et al' 10, Feng et al.'10-, Dogbe'10-, Achdou et al.'10-, LaChapelle'10-, Zhu, Başar'11, Bardi'12, Bensoussan,Sung,Yam,Yung'12-, Kolokoltsov'12-, Carmona & Delarue'12-,Yong'13-,Gangbo & Swiech'14-, Pham'16-,Fischer'17-,Nuno'17-,etc

common assumptions: **Indistinguishability per class+large number of agents+regularity**

A Class of Mean-Field Games (with Jump-Diffusion)

- **Infinite** number of agents
- individual state:
  $ds_i = D(t, s_i(t), u_i(t), m(t))dt + \sigma dB_i + \int_{\Theta} \mu(t,\theta)\tilde{N}_i(dt, d\theta),\ s_0 \sim m_0.$
- payoff: $\mathbb{E}[h(s_i(T), m(T)) + \int_0^T l(t, s_i(t), u_i(t), m(t))dt]$
- m : "mean-field" generated by the infinite population.

Best response to mean-field: $\arg\min_{u_i}\{H(t, s_i, m, p) = l + D.p\}$
Dynamic Nash Equilibrium : No agent has incentive to deviate unilaterally + consistency. **(see Jovanovic'82)**

Mean-Field Equilibrium System (with Jump-Diffusion)

- Pontryagin's approach:

$$\begin{cases} m(0),\ m_t = -(Dm)_s + (\sigma^2 m)_{ss} + J^*[m], \\ dp = -H_{s_i}dt + qdB_i + \int_\Theta \tilde{q}(t,\theta)\tilde{N}_i(dt,d\theta), \\ p(T) = h_{s_i}(s_i(T), m(T)) \\ s_i(0),\quad ds_i = Ddt + \sigma dB_i + \int_\Theta \mu(t,\theta)\tilde{N}_i(dt,d\theta), \end{cases}$$

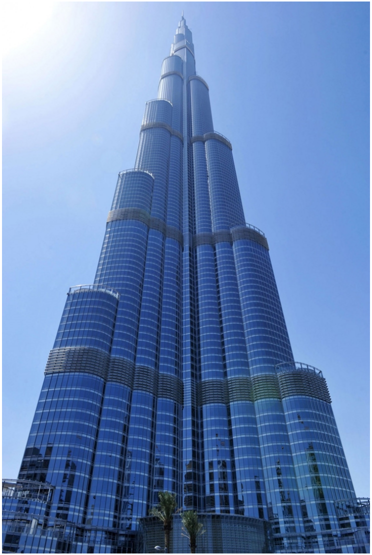
- Bellman's approach:

$$\begin{cases} m(0),\quad m_t = -(Dm)_s + (\sigma^2 m)_{ss} + J^*[m], \\ v_t + H(t, s_i, m, v_{s_i}) + \frac{\sigma^2}{2} v_{s_i s_i} + J[v] = 0, \\ v(T, s_i(T)) = h(s_i(T), m(T)), \\ J[v] := \int_\Theta [v(t, s_i+\mu) - v(t,s_i) - \mu v_{s_i}(t,s_i)]\nu(d\theta) \end{cases}$$

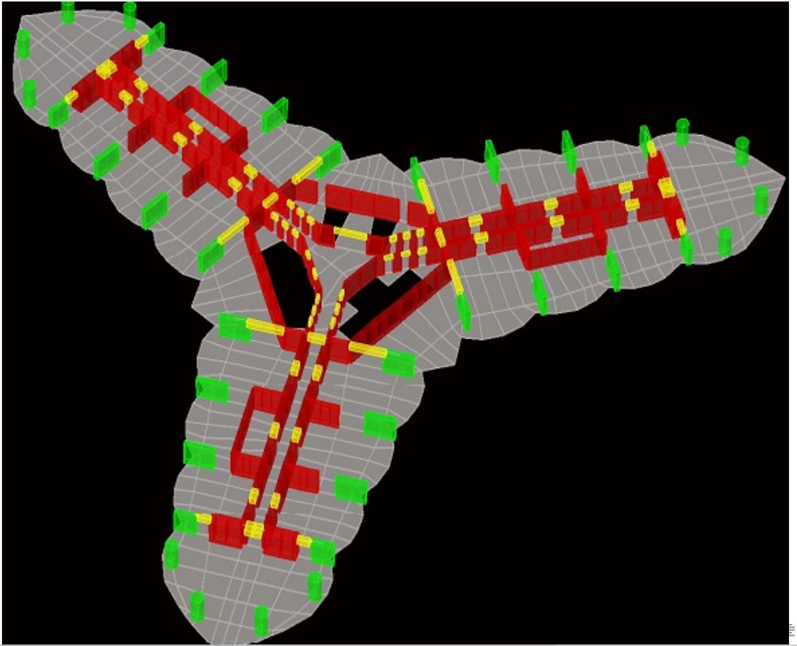
Q: Existence? Uniqueness? Implementation?

See Benamou-Brenier'00, Lasry & Lions'06-, Gomes et al'10-,Bensoussan et al'12-, Kolokoltsov'12-, Carmona & Delarue'12-

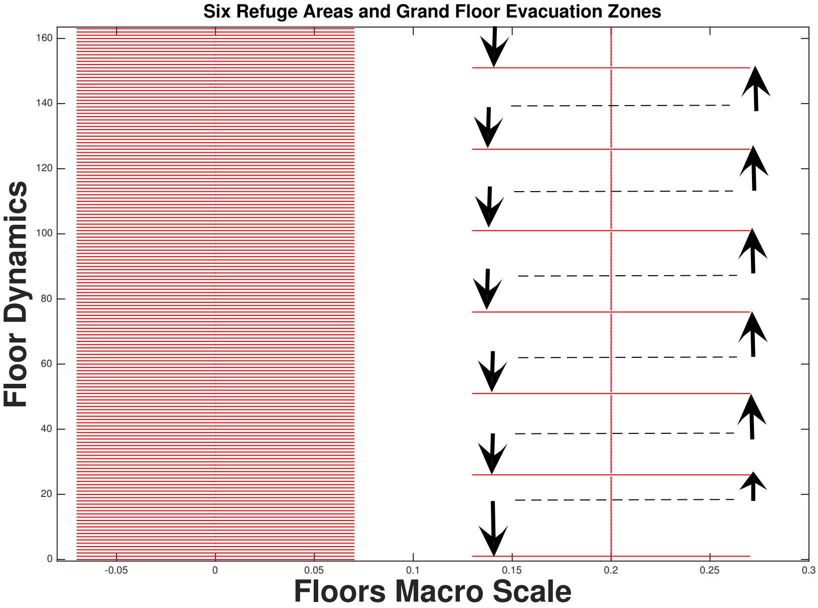
Evacuation of Multi-Level Building



Ground Floor



163 floors: dynamics and sky-bridges

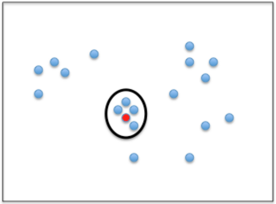


Constrained mean-field game: an example

h_1, h_2 are non-decreasing functions of $G_{n,\epsilon}$

- State of agent i : $\dot{s}_i = u_i, \;\; s_i(t) \in D \subset \mathbb{R}^3, \;\; u_{i,\,|\partial D} = 0$
- Local congestion measure: $G_{i,n,\epsilon}(s_i)$ number of people inside the $m-$ball $B(s_i, \epsilon_i)$
- An agent is represented by a box (which occupies a certain "space" in the building !)
- ϵ_i : between 0.5 and 2 meters, $||u_i||$: between 0 and 0.8 m/s
- Instantaneous cost of i: $h_1(G_{i,n,\epsilon}(s_i, m_{i,n}))||u_i||^2 + h_2(G_{i,n,\epsilon}(s_i, m_{i,n}))$

$$m_{i,n} = \sum_{j \neq i} \delta_{s_j}.$$



Constrained Best Response Problem

Given $m_{i,n}$, $\begin{cases} \inf_{u_i} \{ g(s_i(T), G_{i,n,\epsilon}(s_i(T), m_{i,n}(T))) \\ + \int_0^T h_1(G_{n,\epsilon}(s_i, m_{i,n})) \|u_i\|^2 + h_2(G_{n,\epsilon}(s_i, m_{i,n}))\, dt \} \\ \dot{s}_i = u_i; \quad u_i(.) \in U, \; s_i(0) \in D \subset \mathbb{R}^3; \\ u_{i, \, |\partial D} = 0. \end{cases}$

Difficulty: $G_{i,n,\epsilon}(s_i, .) := \sum_{j \neq i} 1\!\!1_{\, d(s_j, s_i) \leq \epsilon_i} = \int 1\!\!1_{B(s_i, \epsilon_i)}(y) m_{i,n}(dy)$ is discontinuous.

$$
(mfs)\begin{cases}
v_t + H(s, v_s, G(.)) = 0, \quad \text{on } (0,T)\times D \\
H(s,p,G) = \frac{\|p\|^2}{4h_1(G(s))} - h_2(G(t,s)), \\
v(T,s) = g(s), \ \text{on } D \\
G(t, s(t), m) = \int_{y\in B(s(t),\epsilon)} m(t,dy) \\
m_t + div_s(mH_p) = 0, \quad m_0(.) \text{ on } D \\
u = 0, \quad \text{on } \partial D \\
u = k(.), \quad \text{at exits}
\end{cases}
$$

Second order system

$$
(mfss)\begin{cases}
v_t + H(s, v_s, G(s)) + \frac{1}{2}\sigma^2 v_{ss} = 0, \quad \text{on } (0,T)\times D\\
H(s,p,G) = \frac{\|p\|^2}{4h_1(G(s))} - h_2(G(s)),\\
v(T,s) = g(s), \ \text{on } D\\
G(t, s(t)) = \int_{y\in B(s(t),\epsilon)} m(t, dy)\\
m_t + div_s(mH_p) - \frac{1}{2}(\sigma^2 m)_{ss} = 0, \quad m_0(.) \text{ on } D\\
u = 0, \quad \text{on } \partial D\\
u = k, \quad \text{at exits}
\end{cases}
$$

Constrained mean-field game: a basic example

Evacuation Queues

Djehiche B., Tcheukam A., and Tembine H.: A Mean-Field Game of Evacuation in a Multi-Level Building, IEEE Transactions on Automatic Control, October 2017

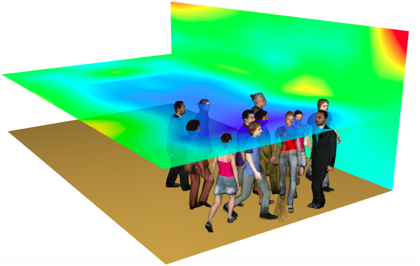
Evacuation and Smoke

A simple mean-field game model:

- local congestion measure : counting process
- each agent makes strategic decisions under constraints
- strategic behavior and queues at the stairs/exits

Ongoing:

- combine the mobility patterns with smoke and flame dynamics



LQ mean-field games: Direct method

No PDEs No SMPs:

Does the Direct method extend to LQ mean-field games ?

Partial answer in: B. Djehiche, H. Tembine: On the Solvability of Risk-Sensitive Linear-Quadratic Mean-Field Games, Preprint 2015.

A simple example of mean-field game

$$
\begin{aligned}
&L_i := q(T)s_i^2(T) + \bar{q}(T)[\bar{m}(T)]^2 + \int_0^T u_i^2(t)dt,\\
&q(T) > 0,\ q(T) + \bar{q}(T) \geq 0.
\end{aligned} \tag{1}
$$

LQ MFG: a trivial example

Decision-maker i :

$$
\begin{aligned}
&\inf_{u_i} \mathbb{E}[L_i] \text{ subject to}\\
&ds_i(t) = u_i(t)dt + s_i(t)[\sigma dB_i(t) + \int_\Theta \mu(\theta)\tilde{N}_i(dt, d\theta)],\\
&s_i(0) \in L^2_{m_0}(\mathbb{R}),\ s_i(0) \perp\!\!\!\perp \{B_i, N_i\}.
\end{aligned} \tag{2}
$$

$\bar{m}$: the average state created by others (infinite number of other decision-makers).

Mean-field equilibrium cost

$$\text{MFE cost: } L^*_{\text{mfg}} = \left[\bar{q}e^{-2\int_0^t \alpha(t')\, dt'} + \alpha(0)\right] s^2(0),$$
$$\alpha(t) = c\left[1 - \frac{1}{1+\frac{q}{c-q}e^{c(T-t)}}\right], \quad c = \sigma^2 + \int_{\theta\in\Theta}\mu^2(\theta)\nu(d\theta), \tag{3}$$

Mean-field equilibrium strategy

$$\text{Best response strategy: } u_i(t) = -\alpha(t)s_i(t),$$
$$s_i(t) = \left(s(0)e^{\int_0^t -\alpha(t)\mathrm{d}t}\right)e^{\sigma B_i(t)+\int_0^t\int_{\theta\in\Theta}\mu(\theta)\tilde{N}_i(\mathrm{d}t,\mathrm{d}\theta)}, \tag{4}$$
$$\bar{m}(t) = s(0)e^{-\int_0^t \alpha(t')\, dt'}.$$

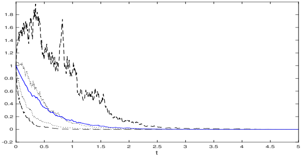


Figure: State dynamics.

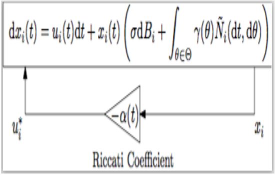


Figure: Equilibrium control action in feedback form

From mean-field games to mean-field-type games

Current mean-field games literature need to be extended to include

- non-symmetry, non-negligible effect of individual decision on the mean-field term

- motivation: variance reduction (Markowitz'1952)

Markowitz, H.M. Portfolio Selection. *J. Financ.* **1952**, *7*, 77–91.

Markowitz, H.M. The Utility of Wealth. *J. Politi. Econ.* **1952**, *60*, 151–158.

Markowitz, H.M. *Portfolio Selection: Efficient Diversification of Investments*; John Wiley & Sons: New York, NY, USA, 1959.

Mean-Field-Type Control

Some prior works on Mean-Field-Type Control:

A. Bensoussan, B. Djehiche, H. Tembine, P. Yam: Risk-Sensitive Mean-Field-Type Control, CDC 2017.

M. Lauriere, O. Pironneau, Dynamic programming for mean-field type control, C. R. Acad. Sci. Paris, 2014

A. Bensoussan and J. Frehse and S.C.P. Yam, Mean-Field Games and Mean-field Type Control, Springer Briefs in Mathematics, 2014.

R. Buckdahn, B. Djehiche, J. Li, (2011) A general stochastic maximum principle for SDEs of mean-field type, Appl. Math. Optim., 64: 197-216.

D. Andersson and B. Djehiche, A Maximum Principle for SDEs of Mean-Field Type, Appl. Math. Optim. 63 (2011) 341-356.

R. Buckdahn, B. Djehiche, J. Li, S. Peng, (2009) Mean-field backward stochastic differential equations: a limit approach, The Annals of Probability, 37(4): 1524-1565.

What is a Mean-Field-Type Game ?

Mean-Field-Type Game

Any game in which the payoffs and/or the state dynamics coefficient functions involve not only the [state and action profiles] but also the distribution of state-action pairs.

payoff(state,action, distribution)

$l_i(s, u_1, \ldots, u_n, D_{(s,u)})$, kernel : $\mathcal{K}(s, u, D_{(s,u)}; ds')$

The number of agents is not necessarily large.

Quantity-of-Interest

variance, skewness, kurtosis, value-at-risk, success probability, mean-variance payoff etc

Mean-field-type game theory

Mean-field-type game theory is the multiple agent generalization of single agent mean-field-type control.

Key change:

- Agents' payoffs and state transitions can now depend on other agents as well.

Example: comfort temperature control in a multi-level building

What is a Mean-Field-Type Game?

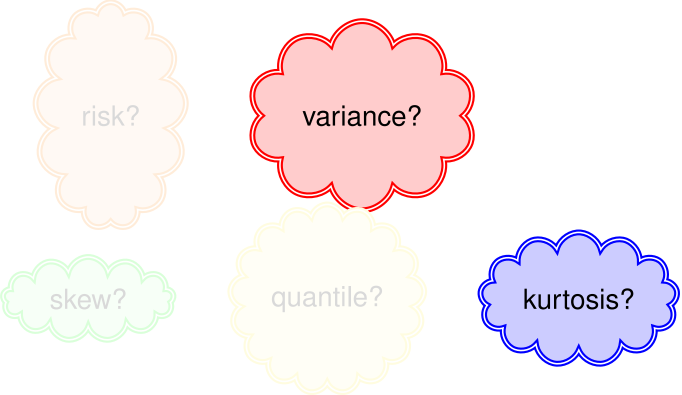
Game with **distribution-dependent** quantity-of-interest

What is a Mean-Field-Type Game?

Game with **distribution-dependent** quantity-of-interest

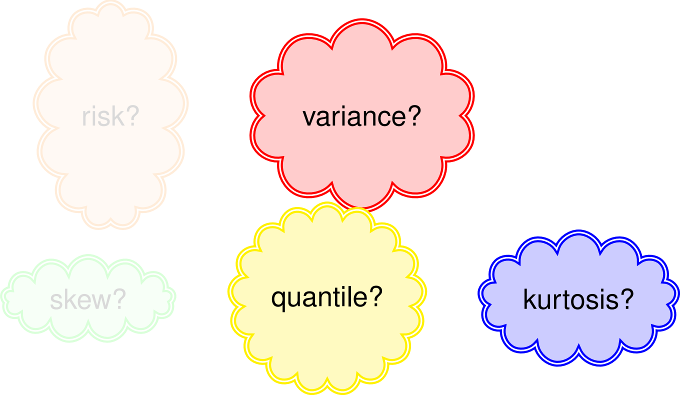
What is a Mean-Field-Type Game?

Game with **distribution-dependent** quantity-of-interest



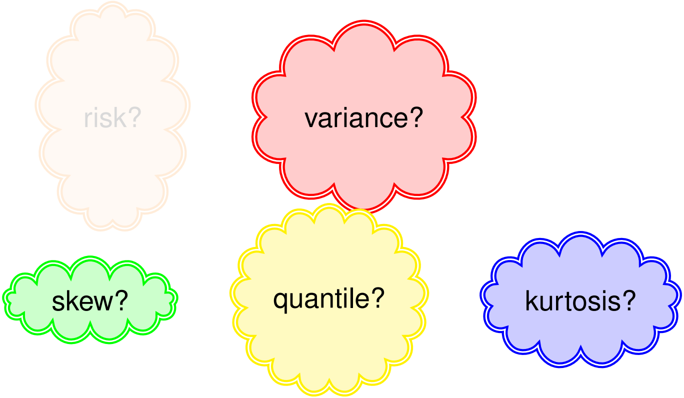
What is a Mean-Field-Type Game?

Game with **distribution-dependent** quantity-of-interest



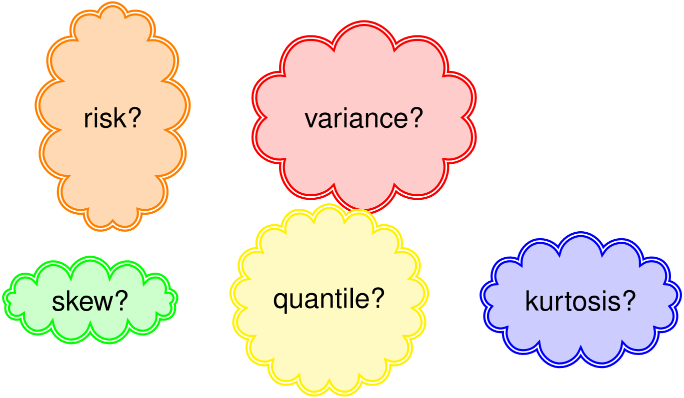
What is a Mean-Field-Type Game?

Game with **distribution-dependent** quantity-of-interest



What is a Mean-Field-Type Game?

Game with **distribution-dependent** quantity-of-interest



Agenda

1. Jovanovic & Rosenthal 1982-1988's Model
2. Mean-Field Games (with Jump-Diffusion)
3. LQ mean-field games: Direct method
4. LQ mean-field-type games: Direct method
5. Games with common noise

LQ mean-field-type games: Direct method

No PDEs No SMPs:
Does the Direct method extend to mean-field-type games ?
Partial answer in this talk

A simple example of mean-field-type game

$$
\begin{aligned}
&L_i := q(T)s^2(T) + \bar{q}(T)[\mathbb{E}s(T)]^2 + \int_0^T u_i^2(t)dt,\\
&q(T) > 0, q(T) + \bar{q}(T) \geq 0.
\end{aligned} \tag{5}
$$

LQ MFTG

Decision-maker i :

$$
\begin{aligned}
&\inf_{u_i} \ \mathbb{E}[L_i] \ \text{ subject to}\\
&ds(t) = u_i(t)dt + s(t)[\sigma dB(t) + \int_\Theta \mu(\theta)\tilde{N}(dt, d\theta)],\\
&s(0) \in L^2(\Omega, \mathbb{R}), \ s(0) \perp\!\!\!\perp \{B, N\}.
\end{aligned} \tag{6}
$$

MFTG equilibrium cost

$$L^*_{\text{mftg}} = \beta(0)[\mathbb{E}s(0)]^2,$$
$$\dot{\beta}(t) + c\alpha(t) - \beta^2(t) = 0, \;\; \beta(T) = q + \bar{q}. \tag{7}$$

MFTG: equilibrium strategy

$$u_i(t) = -\alpha(t)(s(t) - \mathbb{E}s(t)) - \beta(t)\mathbb{E}s(t),$$
$$\alpha(t) = c\left[1 - \frac{1}{1+\frac{q}{c-q}e^{c(T-t)}}\right],$$
$$c = \sigma^2 + \int_{\theta\in\Theta} \mu^2(\theta)\nu(d\theta),$$
$$\mathbb{E}s(t) = s(0)e^{-\int_0^t \beta(t')\, dt'}, \tag{8}$$

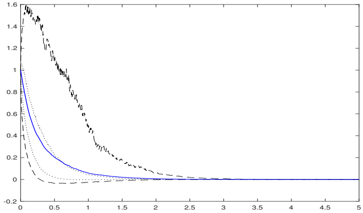


Figure: MFTG: More cautions ?

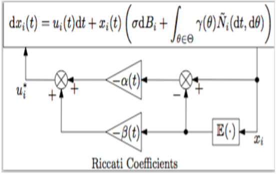
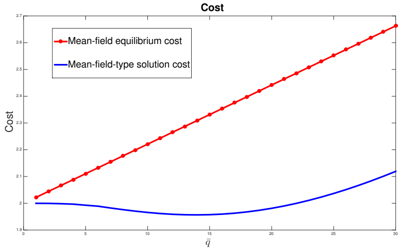
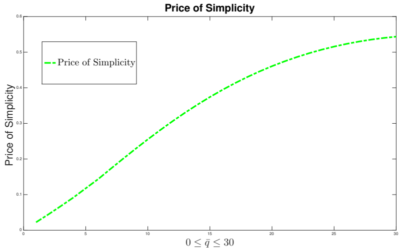


Figure: Equilibrium control action in feedback form: MFTG

The price of simplicity:

$$L^*_{\text{mfg}} - L^*_{\text{mftg}} = \bar{q}s^2(0)e^{-\int_0^T \alpha(t)dt}\left[-e^{-\int_0^T \beta(t)dt} + e^{-\int_0^T \alpha(t)dt}\right] \geq 0,$$





Game Problem I

$$L_i := \frac{1}{2}q_i(T)s^2(T) + \frac{1}{2}\bar{q}_i(T)[\mathbb{E}(s(T))]^2 \\ +\frac{1}{2}\int_0^T \left\{q_i(t)s^2(t) + \bar{q}_i(t)(\mathbb{E}[s(t)])^2 + r_i(t)u_i^2(t) + \bar{r}_i(t)[\mathbb{E}(u_i(t))]^2\right\} dt. \quad (9)$$

Decision-maker i :

$\inf_{u_i} \mathbb{E}[L_i]$ subject to

$$ds(t) = \left\{a(t)s(t) + \bar{a}(t)\mathbb{E}[s(t)] + \sum_{j=1}^n b_j(t)u_j(t) + \sum_{j=1}^n \bar{b}_j(t)\mathbb{E}[u_j(t)]\right\} dt \\ +\sigma(t)dB(t) + \int_\Theta \mu(t,\theta)\tilde{N}(dt,d\theta), \\ s(0) \in L^2(\Omega,\mathbb{R}),\ s(0) \perp\!\!\!\perp \{B,N\}. \quad (10)$$

T. Başar,B. Djehiche, H. Tembine: Mean-field-type game theory, under preparation

T. E. Duncan, H. Tembine: Linear-Quadratic Mean-Field-Type Games: A Direct Method, Games Journal, Jan. 2018

Solution to Game Problem I

$A: \ \mathbb{E}s^2(0) < \infty, \ r_i(t) > 0, r_i(t) + \bar{r}_i(t) > 0, q_i(t) > 0, q_i(t) + \bar{q}_i(t) > 0.$

Theorem

Mean-Field-Type Nash Equilibrium

$$\begin{array}{l} \inf_{u_i \in \mathcal{U}_i} \mathbb{E}[L_i] = \mathbb{E}\left\{\frac{1}{2}\alpha_i(0)(s(0) - \bar{s}(0))^2 + \frac{1}{2}\beta_i(0)[\bar{s}(0)]^2 \right. \\ \qquad\qquad \left. + \gamma_i(0)\bar{s}(0) + \delta_i(0)\right\} \\ u_i^* = -\frac{b_i}{r_i}\alpha_i(s - \bar{s}) - \frac{(b_i + \bar{b}_i)}{r_i + \bar{r}_i}(\beta_i \bar{s} + \gamma_i), \\ \frac{d\bar{s}}{dt} = [a + \bar{a} - \sum_{i=1}^n \frac{(b_i + \bar{b}_i)^2}{r_i + \bar{r}_i}\beta_i]\bar{s} + \sum_{i=1}^n \frac{(b_i + \bar{b}_i)^2}{r_i + \bar{r}_i}\gamma_i, \\ \bar{s}(0) = \bar{s}_0. \end{array} \quad (11)$$

Riccati System for Nash Equilibria

$$
\begin{aligned}
&\dot{\alpha}_i + 2a\alpha_i - \tfrac{b_i^2}{r_i}\alpha_i^2 - 2\alpha_i \textstyle\sum_{j\neq i} \tfrac{b_j^2}{r_j}\alpha_j + q_i = 0,\\
&\alpha_i(T) = q_i(T),\\
&\dot{\beta}_i + 2(a+\bar{a})\beta_i - \beta_i^2 \tfrac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i} - 2\beta_i \textstyle\sum_{j\neq i} \tfrac{(b_j+\bar{b}_j)^2}{r_j+\bar{r}_j}\beta_j + q_i + \bar{q}_i = 0,\\
&\beta_i(T) = q_i(T) + \bar{q}_i(T),\\
&\dot{\gamma}_i + (a+\bar{a})\gamma_i - \tfrac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i}\beta_i\gamma_i - \beta_i \textstyle\sum_{j\neq i} \tfrac{(b_j+\bar{b}_j)^2}{r_j+\bar{r}_j}\gamma_j - \gamma_i \textstyle\sum_{j\neq i} \tfrac{(b_j+\bar{b}_j)^2}{r_j+\bar{r}_j}\beta_j = 0,\\
&\gamma_i(T) = 0,\\
&\dot{\delta}_i - \gamma_i \textstyle\sum_{j\neq i} \tfrac{(b_j+\bar{b}_j)^2}{r_j+\bar{r}_j}\beta_j - \tfrac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i}\tfrac{1}{2}\gamma_i^2 + [\sigma^2 + \int_\Theta \mu^2(t,\theta)\nu(d\theta)]\tfrac{1}{2}\alpha_i = 0\\
&\delta_i(T) = 0
\end{aligned}
\tag{12}
$$

Game Problem (with Common Noise)

$$L_i := \frac{1}{2}q_i(T)s^2(T) + \frac{1}{2}\bar{q}_i(T)[\mathbb{E}(s(T)|\mathcal{F}_T^{B_o})]^2$$
$$+\frac{1}{2}\int_0^T \Big\{q_i(t)s^2(t) + \bar{q}_i(t)(\mathbb{E}[s(t)\ |\mathcal{F}_t^{B_o}\ ])^2 \qquad (13)$$
$$+r_i(t)u_i^2(t) + \bar{r}_i(t)[\mathbb{E}(u_i(t)|\mathcal{F}_t^{B_o})]^2\Big\}\,dt.$$

Decision-maker i :

$\inf_{u_i}\ \mathbb{E}[L_i]$ subject to

$$ds(t) = \Big\{a(t)s(t) + \bar{a}(t)\mathbb{E}[s(t)\mid \mathcal{F}_t^{B_o}]$$
$$+\sum_{i=1}^n b_i(t)u_i(t) + \sum_{i=1}^n \bar{b}_i\mathbb{E}[u_i(t)\mid \mathcal{F}_t^{B_o}]\Big\}\,dt \qquad (14)$$
$$+\sigma(t)dB(t) + \int_\Theta \mu(t,\theta)\tilde{N}(dt,d\theta) + \sigma_o(t)dB_o(t),$$
$$s(0)\in L^2_{m_0}(\mathbb{R}),\quad s(0) \perp\!\!\!\perp \{B,N\}.$$

T. E. Duncan, H. Tembine: Linear-quadratic mean-field-type games with jump-diffusion and an observable common noise: A direct method, Preprint, 2018

Solution to Game Problem (with Common Noise)

$A: \ \mathbb{E}s^2(0) < +\infty, \ r_i(t) > 0, r_i(t) + \bar{r}_i(t) > 0, q_i(t) > 0, q_i(t) + \bar{q}_i(t) \geq 0.$

Theorem

Mean-Field-Type Nash Equilibrium

$$
\begin{aligned}
&\inf_{u_i\in\mathcal{U}_i} \mathbb{E}[L_i] = \mathbb{E}\Big\{\tfrac{1}{2}\alpha_i(0)(s(0)-\hat{s}(0))^2 + \tfrac{1}{2}\beta_i(0)[\hat{s}(0)]^2 \\
&\qquad\qquad +\gamma_i(0)\hat{s}(0) + \delta_i(0)\} \\
&u_i^* = -\tfrac{b_i}{r_i}\alpha_i(s-\hat{s}) - \tfrac{(b_i+\bar{b}_i)}{r_i+\bar{r}_i}(\beta_i\hat{s}+\gamma_i), \\
&d\hat{s} = \{[a+\bar{a} - \textstyle\sum_{i=1}^n \tfrac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i}\beta_i]\hat{s} + \textstyle\sum_{i=1}^n \tfrac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i}\gamma_i\}dt + \sigma_o dB_o, \\
&\hat{s}(0) = \hat{s}_0, \\
&\hat{X}(t) = \mathbb{E}[X(t) \mid \mathcal{F}_t^{B_o}].
\end{aligned}
\tag{15}
$$

Stochastic Riccati System for Nash Equilibria

$$
\begin{aligned}
&d\alpha_i = -\left\{2a\alpha_i - \frac{b_i^2}{r_i}\alpha_i^2 - 2\alpha_i \sum_{j\neq i} \frac{b_j^2}{r_j}\alpha_j + q_i\right\} dt + [\alpha_{i,o}]dB_o,\\
&\alpha_i(T) = q_i(T),\\
&d\beta_i = -\left\{2\beta_i(a+\bar{a}) - \beta_i^2 \frac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i} - 2\beta_i \sum_{j\neq i} \frac{(b_j+\bar{b}_j)^2}{r_j+\bar{r}_j}\beta_j + q_i + \bar{q}_i\right\} dt\\
&\qquad +[\beta_{i,o}]dB_o,\\
&\beta_i(T) = q_i(T) + \bar{q}_i(T),\\
&d\gamma_i = -\left\{(a+\bar{a})\gamma_i + \beta_{i,o}\sigma_o - \frac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i}\beta_i\gamma_i\right.\\
&\qquad \left. -\beta_i \sum_{j\neq i} \frac{(b_j+\bar{b}_j)^2}{r_j+\bar{r}_j}\gamma_j - \gamma_i \sum_{j\neq i} \frac{(b_j+\bar{b}_j)^2}{r_j+\bar{r}_j}\beta_j\right\} dt - [\sigma_o\beta_i]dB_o,\\
&\gamma_i(T) = 0,\\
&d\delta_i = -\left\{\frac{1}{2}\sigma_o^2\beta_i + \gamma_{i,o}\sigma_o - \gamma_i \sum_{j\neq i} \frac{(b_j+\bar{b}_j)^2}{r_j+\bar{r}_j}\beta_j - \frac{(b_i+\bar{b}_i)^2}{r_i+\bar{r}_i}\frac{1}{2}\gamma_i^2\right.\\
&\qquad \left. + [\sigma^2 + \int_\Theta \mu^2(t,\theta)\nu(d\theta)]\frac{1}{2}\alpha_i\right\} dt - [\sigma_o\gamma_i]dB_o\\
&\delta_i(T) = 0
\end{aligned}
\tag{16}
$$

Summary: Multi-agent purely-selfish approach

- the common noise brings stochastic Riccati system
- $\hat{s}$ is an $\mathcal{F}^{B_o}$-adapted process
- Regularize the processes $(\gamma_i)_i$ and $(\delta_i)_i$
- the purely-selfish approach leads to a coupled Riccati system
- simple use of Itô's formula for jump-diffusion process
    - No Bellman principle, No Pontryagin principle were employed
    - no SPDEs and no SMPs

Agenda: next step

- Recitation 2: Price Formation in the Smart Grid

THANK YOU

Linear-Quadratic Mean-Field-Type Games

common noise, jump-diffusion, regime switching

Tembine Hamidou

Learning & Game Theory Laboratory, NYUAD
Wuchen Li, UCLA
IPAM Graduate Summer School

June 18-29, 2018

Acknowledgments

- Collaborators:
  - Jian Gao, Yida Xu, Michail Smyrnakis, Massa Ndong, Julian Barreiro-Gomez,
  - Eitan Altman (INRIA), Tamer Başar (UIUC), Jean-Yves LeBoudec (EPFL), Alain Bensoussan (UT), Boualem Djehiche (KTH), Tyrone E. Duncan (Kansas)
- We appreciate support from U.S. Air Force Office of Scientific Research under grant number FA9550-17-1-0259.

Recitation 2: Price Formation

Agenda

1. Price Formation in the Smart Grid: complete information

2. Price Formation in the Smart Grid: INcomplete information

Price Formation in the Smart Grid

Dynamic price model taken from Roos'1925:

Dynamic log-price adjustment

Roos C.F. (1925): A Mathematical Theory of Competition, American Journal of Mathematics, 47, 163-175

Roos C.F. (1927): A Dynamic Theory of Economics, Journal of Political Economy 35, 632-656

Log-price adjustment with Jump-Diffusion-Common Noise

B. Djehiche, J. Barreiro-Gomez, H. Tembine: Electricity Price Dynamics in the Smart Grid: A Mean-Field-Type Game Perspective, accepted and to appear in International Symposium on Mathematical Theory of Networks and Systems MTNS 2018

Price Formation in the Smart Grid

$$
\begin{aligned}
&\sup_{u_i} \ \mathbb{E}\{-\tfrac{q}{2}e^{-\lambda_i t_1}\,(s(t_1)-\hat{s}(t_1))^2 + \int_{t_0}^{t_1} e^{-\lambda_i t}\,[\hat{s}u_i - C_i(u_i)]\,dt\},\\
&ds = \eta[a-\textstyle\sum_{i=1}^n u_i - s]dt + [\sigma dB + \int_\Theta \mu(\theta)\tilde{N}(dt,d\theta)] + \sigma_o dB_o,\\
&s(t_0) = s_0,\\
&q>0, \eta>0, \lambda_i>0, t_0<t_1,
\end{aligned}
\tag{1}
$$

Tentative Interpretation

- B_o : common noise
- $\hat{s}(t) = \mathbb{E}[s(t) \mid \mathcal{F}_t^{B_o}]$, conditional expectation
- $\hat{s}u_i$: gain
- $C_i(u_i) = c_i u_i + \frac{1}{2} r_i u_i^2 + \frac{1}{2}\bar{r}_i \hat{u}_i^2$, cost
- c_i : type of decision maker i : setup marginal cost

Equilibrium strategy of i :
$u_i^* = -\frac{\eta\tilde{\alpha}_i}{r_i}(s-\hat{s}) + \frac{\hat{s}(1-\eta\tilde{\beta}_i)-(c_i+\eta\tilde{\gamma}_i)}{r_i+\bar{r}_i}$,
Conditional equilibrium price of i :

$$d\hat{s} = \eta\left\{a + \sum_{j=1}^{n}\frac{c_j+\eta\tilde{\gamma}_j}{r_j+\bar{r}_j} - \hat{s}\left(1+\sum_{j=1}^{n}\frac{1-\eta\tilde{\beta}_j}{r_j+\bar{r}_j}\right)\right\}dt + \sigma_o dB_o, \quad (2)$$

$\hat{s}(t_0) = \hat{s}_0$,
Equilibrium revenue of i :
$\mathbb{E}\frac{1}{2}\alpha_i(t_0)(s(t_0)-\hat{s}_0)^2 + \frac{1}{2}\beta_i(t_0)\hat{s}_0^2 + \gamma_i(t_0)\hat{s}_0 + \delta_i(t_0)$.

$$
\begin{aligned}
& \mathrm{d}\tilde{\alpha}_i = \left\{(\lambda_i + 2\eta)\tilde{\alpha}_i - \frac{\eta^2}{r_i}\tilde{\alpha}_i^2 - 2\eta^2\tilde{\alpha}_i \sum_{j\neq i} \frac{\tilde{\alpha}_j}{r_j}\right\} \mathrm{d}t + \tilde{\alpha}_{i,o}\mathrm{d}B_o, \\
& \tilde{\alpha}_i(t_1) = -q_i, \\
& \mathrm{d}\tilde{\beta}_i = \left\{(\lambda_i + 2\eta)\tilde{\beta}_i - \frac{(1-\eta\tilde{\beta}_i)^2}{r_i + \bar{r}_i} + 2\eta\tilde{\beta}_i \sum_{j\neq i} \frac{1-\eta\tilde{\beta}_j}{r_j + \bar{r}_j}\right\} \mathrm{d}t + \tilde{\beta}_{i,o}\mathrm{d}B_o, \\
& \tilde{\beta}_i(t_1) = 0,
\end{aligned}
\tag{3}
$$

$$
\begin{aligned}
\mathrm{d}\tilde{\gamma}_i &= \Big\{(\lambda_i+\eta)\tilde{\gamma}_i - \eta\tilde{\beta}_i a - \tilde{\beta}_{i,o}\sigma_o + \frac{(1-\eta\tilde{\beta}_i)(c_i+\eta\tilde{\gamma}_i)}{r_i+\bar{r}_i} + \eta\tilde{\gamma}_i\sum_{j\neq i}\frac{1-\eta\tilde{\beta}_j}{r_j+\bar{r}_j} \\
&\quad -\eta\tilde{\beta}_i\sum_{j\neq i}\frac{c_j+\eta\tilde{\gamma}_j}{r_j+\bar{r}_j}\Big\}\mathrm{d}t - \tilde{\beta}_i\sigma_o\mathrm{d}B_o, \\
\tilde{\gamma}_i(0) &= 0, \\
\mathrm{d}\tilde{\delta}_i &= -\Big\{-\lambda_i\tilde{\delta}_i + \frac{1}{2}\sigma_o^2\tilde{\beta}_i + \frac{1}{2}\tilde{\alpha}_i\left(\sigma^2+\int_\Theta \mu^2(\theta)\nu(d\theta)\right) + \eta\tilde{\gamma}_i a \\
&\quad +\tilde{\gamma}_{i,o}\sigma_o + \frac{1}{2}\frac{(c_i+\eta\tilde{\gamma}_i)^2}{r_i+\bar{r}_i} + \eta\tilde{\gamma}_i\sum_{j\neq i}\frac{c_j+\eta\tilde{\gamma}_j}{r_j+\bar{r}_j}\Big\}\mathrm{d}t - \sigma_o\tilde{\gamma}_i\mathrm{d}B_o, \\
\tilde{\delta}_i(t_1) &= 0,
\end{aligned} \tag{4}
$$

Numerical example

Table: Parameters Scenarios 2 and 3

Parameter	**value**
t_1	1.5
p_0	50
$\bar{p}_0$	50
c_1	1
c_2	5
c_3	10
$\lambda_1, \ldots, \lambda_3$	0.1
s	0.5
a	1
$r_1, \bar{r}_1$	1
$r_2, \bar{r}_2$	2
$r_3, \bar{r}_3$	3
q	1
D	$u_1 + u_2 + u_3$

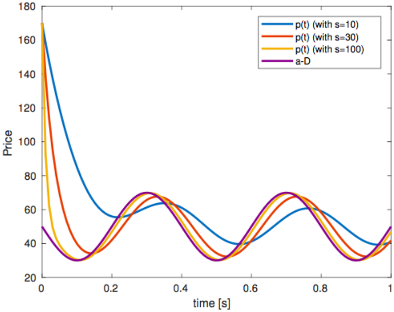


Figure: Sample Price dynamics.

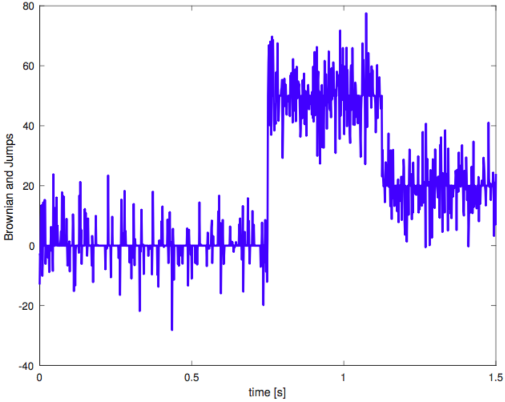


Figure: Sample jump-diffusion

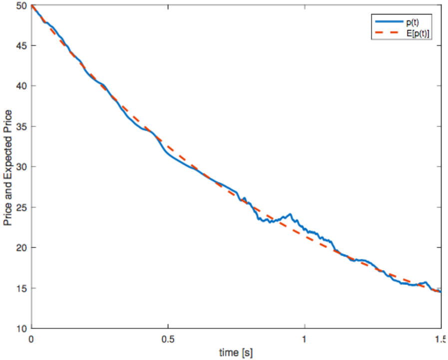


Figure: Equilibrium price.

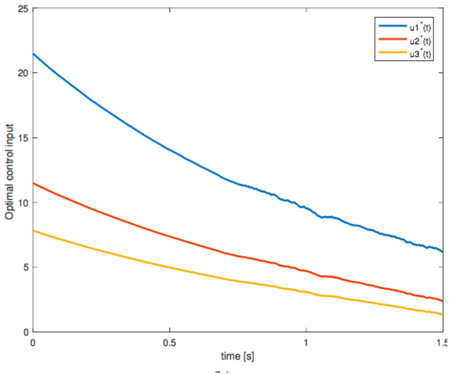


Figure: Equilibrium strategies.

Bayesian Mean-Field-Type Game

Dynamic price model under incomplete information about the type of other agents

$$
\begin{array}{l}
\sup_{u_i}\ \mathbb{E}\{-\frac{q}{2}e^{-\lambda_i t_1}\left(s(t_1)-\hat{s}(t_1)\right)^2+\int_{t_0}^{t_1} e^{-\lambda_i t}\left[\hat{s}u_i - C_i(u_i)\right]dt\},\\
ds=\eta[a-\sum_{i=1}^n u_i - s]dt+[\sigma dB+\int_\Theta \mu(\theta)\tilde{N}(dt,d\theta)]+\sigma_o dB_o,\\
s_0,\\
q>0, \eta>0, \lambda_i>0, t_0<t_1,\\
\hat{s}(t)=\mathbb{E}[s(t)\mid \mathcal{F}_t^{B_o}],\\
C_i(u_i)=c_i u_i+\frac{1}{2}r_i u_i^2+\frac{1}{2}\bar{r}_i\hat{u}_i^2,\\
(c_i, r_i, \bar{r}_i):\text{ type of decision maker } i
\end{array}
\tag{5}
$$

Agent i does not know $(c_{-i}, r_{-i}, \bar{r}_{-i})$

but has $\xi_{-i}(.|c_i, r_i, \bar{r}_i)$ the conditional probability law over $(c_{-i}, r_{-i}, \bar{r}_{-i})$

Equilibrium strategy of i :

$$\tilde{u}_i^* = -\frac{\eta\hat{\alpha}_i}{r_i}(s-\hat{s}) + \frac{\hat{s}(1-\eta\hat{\beta}_i)-(c_i+\eta\hat{\gamma}_i)}{r_i+\bar{r}_i},$$

Conditional equilibrium price of i :

$$\begin{aligned} d\hat{s} &= \eta\left\{a + \int \sum_{j=1}^n \frac{c_j+\eta\hat{\gamma}_j}{r_j+\bar{r}_j} d\xi_{-i}(.|c_i, r_i, \bar{r}_i) \right. \\ &\left. -\hat{s}\left(1 + \int \sum_{j=1}^n \frac{1-\eta\hat{\beta}_j}{r_j+\bar{r}_j} d\xi_{-i}(.|c_i, r_i, \bar{r}_i)\right)\right\} dt + \sigma_o dB_o, \\ \hat{s}(t_0) &= \hat{s}_0, \end{aligned} \tag{6}$$

$$
\begin{aligned}
&\mathrm{d}\hat{\alpha}_i = \left\{(\lambda_i + 2\eta)\hat{\alpha}_i - \frac{\eta^2}{r_i}\hat{\alpha}_i^2 - 2\eta^2\hat{\alpha}_i \int \sum_{j\neq i} \frac{\hat{\alpha}_j}{r_j} d\xi_{-i}(.|c_i, r_i, \bar{r}_i)\right\} \mathrm{d}t + \hat{\alpha}_{i,o}\mathrm{d}B_o,\\
&\hat{\alpha}_i(t_1) = -q_i,\\
&\mathrm{d}\hat{\beta}_i = \left\{(\lambda_i + 2\eta)\hat{\beta}_i - \frac{(1-\eta\hat{\beta}_i)^2}{r_i+\bar{r}_i} + 2\eta\hat{\beta}_i \int \sum_{j\neq i} \frac{1-\eta\hat{\beta}_j}{r_j+\bar{r}_j} d\xi_{-i}(.|c_i, r_i, \bar{r}_i)\right\} \mathrm{d}t\\
&+\hat{\beta}_{i,o}\mathrm{d}B_o,\\
&\hat{\beta}_i(t_1) = 0,
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
d\hat{\gamma}_i &= \Big\{(\lambda_i+\eta)\hat{\gamma}_i - \eta\hat{\beta}_i a - \hat{\beta}_{i,o}\sigma_o + \tfrac{(1-\eta\hat{\beta}_i)(c_i+\eta\hat{\gamma}_i)}{r_i+\bar{r}_i} + \eta\hat{\gamma}_i \int \sum_{j\neq i} \tfrac{1-\eta\hat{\beta}_j}{r_j+\bar{r}_j} d\xi_{-i}(.|c_i, r_i, \bar{r}_i) \\
&-\eta\hat{\beta}_i \int \sum_{j\neq i} \tfrac{c_j+\eta\hat{\gamma}_j}{r_j+\bar{r}_j} d\xi_{-i}(.|c_i, r_i, \bar{r}_i)\Big\} dt - \hat{\beta}_i\sigma_o dB_o, \\
\hat{\gamma}_i(0) &= 0, \\
d\hat{\delta}_i &= -\Big\{ -\lambda_i\hat{\delta}_i + \tfrac{1}{2}\sigma_o^2\hat{\beta}_i + \tfrac{1}{2}\hat{\alpha}_i\left(\sigma^2 + \int_\Theta \mu^2(\theta)\nu(d\theta)\right) + \eta\hat{\gamma}_i a \\
&+\hat{\gamma}_{i,o}\sigma_o + \tfrac{1}{2}\tfrac{(c_i+\eta\hat{\gamma}_i)^2}{r_i+\bar{r}_i} + \eta\hat{\gamma}_i \int \sum_{j\neq i} \tfrac{c_j+\eta\hat{\gamma}_j}{r_j+\bar{r}_j} d\xi_{-i}(.|c_i, r_i, \bar{r}_i)\Big\} dt - \sigma_o\hat{\gamma}_i dB_o, \\
\hat{\delta}_i(t_1) &= 0,
\end{aligned}
\tag{8}
$$

Agenda: next step

- Lecture 3: Social cost, Robustness, Bargaining, Empathy, Coopetition

THANK YOU