PDE Methods for Mean Field Games with Non-Separable Hamiltonian: Further topics (mainly uniqueness)

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Recap of my lectures so far

- I have shown existence theorems with smallness constraints, for data in the Wiener algebra.
- I have shown an existence theorem with smallness constraints, for data in Sobolev spaces.
- An advantage of Sobolev spaces is that more Hamiltonians are available because of composition estimates on Sobolev spaces.
- The key step is the energy estimate, which we went through in some detail.
- Now I will say some things about questions other than existence.

What other questions can we consider?

- Big next question: uniqueness.
- Convergence questions: connection with games with a finite number of players?
- Limit as $T \to +\infty$? Inviscid limit?
- Regularity questions: how much regularity must we require on the data? How much regularity can we infer for the solutions?
- There are of course many other questions, such as questions about applications and numerical methods, to name just a couple.

A typical uniqueness argument

- For an evolution equation $u_t = F(u)$, a typical uniqueness argument is to assume you have two solutions, u^1 and u^2. You consider a low norm, such as

$$E(t) = \|u^1(t,\cdot) - u^2(t,\cdot)\|_{L^2}^2.$$

- You have already established that the solutions satisfy a bound, so you can use that as a tool.
- You make an estimate like

$$\frac{dE}{dt} \leq cE,$$

  and then use Gronwall: $E(t) \leq E(0)e^{ct}$.
- This estimate is very similar in spirit and execution to the energy estimate you might have made for an existence argument.
- If $E(0) = 0$, then $E(t) = 0$ for all t. This is uniqueness.

What about uniqueness for MFG?

- Gronwall does not seem to make sense in the forward-backward setting. Gronwall is incremental in time, and we have to consider the whole time interval at once.
- The problem is worse than Gronwall not being available.
- Uniqueness is not true in general. There is an example by Bardi and Fischer.

The Bardi-Fischer example

- Bardi and Fischer consider a separable Hamiltonian, $H(u_x) + F(x,m)$, in one space dimension.
- The Hamiltonian satisfies

$$H(p) = \begin{cases} -bp, & p \leq 0, \\ -ap, & p > 0. \end{cases}$$

- This is taken with $a < 0 < b$.
- Payoff boundary conditions are given, so that only data for m is specified.
- If u_x is of a fixed sign, this decouples the MFG system.
- With the same data for m, two solutions are found, one with $u_x > 0$ and one with $u_x < 0$.

More on Bardi-Fischer

- This Hamiltonian is not smooth.
- Bardi and Fischer also smooth out the example; as a result, they get non-uniqueness with smooth Hamiltonian for sufficiently large values of T.
- Bardi and Cirant also have a uniqueness theorem for separable MFG with smallness conditions.

My uniqueness theorem

- I said a moment ago that your estimates for uniqueness are very similar to the estimates you make for existence.
- The estimates I showed you before for existence relied on extensive use of Young's inequality, and on a smallness assumption which unified the size of the data, the time horizon, and the strength of the coupling.
- Most of the extensive use of Young's inequality was a bit artificial: it was because of the way the iterative scheme was set up. For uniqueness, we deal with solutions of the actual equation, not the approximate problems.
- Otherwise, the uniqueness estimates are similar. Let's get started: let (u^1, μ^1), (u^2, μ^2) be two solutions.

An energy for the difference

- Define $E(t) = E_\mu(t) + E_u(t)$, with

$$E_\mu(t) = \frac{1}{2}\int_{\mathbb{T}^d} \left(\mu^1(t,x) - \mu^2(t,x)\right)^2\ dx,$$

$$E_u(t) = \frac{1}{2}\sum_{i=1}^{d}\int_{\mathbb{T}^d} \left(\partial_{x_i} u^1(t,x) - \partial_{x_i} u^2(t,x)\right)^2\ dx.$$

- So, we are estimating the difference of μ in L^2, and the difference of Du also in L^2.

Let's get estimating

- We take the time derivative of E_μ, to start:

$$\frac{dE_\mu}{dt} = \int_{\mathbb{T}^d} (\mu^1 - \mu^2)(\mu^1_t - \mu^2_t)\ dx.$$

- We do some preliminary adding and subtracting:

$$\begin{aligned}\frac{dE_\mu}{dt} &= \int_{\mathbb{T}^d} (\mu^1 - \mu^2)\Delta(\mu^1 - \mu^2)\ dx \\ &\quad - \varepsilon \int_{\mathbb{T}^d} (\mu^1 - \mu^2)\mathrm{div}\left((\mu^1 - \mu^2)\Theta_p(t,x,\mu^1,Du^1)\right)\ dx \\ &\quad - \varepsilon \int_{\mathbb{T}^d} (\mu^1 - \mu^2)\mathrm{div}\left(\mu^2\left(\Theta_p(t,x,\mu^1,Du^1) - \Theta_p(t,x,\mu^2,Du^2)\right)\right)\ dx \\ &\quad - \varepsilon\bar{m} \int_{\mathbb{T}^d} (\mu^1 - \mu^2)\mathrm{div}\left(\Theta_p(t,x,\mu^1,Du^1) - \Theta_p(t,x,\mu^2,Du^2)\right)\ dx.\end{aligned}$$

Expanding this

- We expand the divergences, and do more adding and subtracting:

$$\frac{dE_\mu}{dt} = \sum_{\ell=1}^{14} V_\ell.$$

- This is just for μ; for u, we have

$$\frac{dE_u}{dt} = \sum_{j=1}^{d} \sum_{\ell=1}^{6} W_\ell^j.$$

- I will not detail all of these terms now. Instead we will just look at a few representative terms.

Assumptions for uniqueness

- Assume that a high norm of the solutions is bounded by some constant K.
- We need a Lipschitz bound for the Hamiltonian:
  (H3) For all multi-indices β with $0 \leq |\beta| \leq 2$, for any (p^i, q^i) in a bounded subset of $\mathbb{R}^{d+1}$, there exists a constant $c > 0$ such that

$$\left|\partial^\beta \Theta(t, x, q^1, p^1) - \partial^\beta \Theta(t, x, q^2, p^2)\right| \leq c \left(|q^1 - q^2| + \sum_{i=1}^{d} |p_i^1 - p_i^2| \right).$$

First few terms:

- We have the first term:

$$V_1 = \int_{\mathbb{T}^d} (\mu^1 - \mu^2)(\Delta(\mu^1 - \mu^2))\ dx = -\int_{\mathbb{T}^d} |\nabla(\mu^1 - \mu^2)|^2\ dx.$$

- This term, V_1, is again the parabolic gain of regularity, which we will need to use.
- We look at the next term.

$$V_2 = -\varepsilon \int_{\mathbb{T}^d} (\mu^1 - \mu^2)(\nabla(\mu^1 - \mu^2)) \cdot \Theta_p(t, x, \mu^1, Du^1)\ dx.$$

- This may be integrated by parts, and since μ^1, Du^1 are bounded, we get

$$V_2 = \frac{\varepsilon}{2} \int_{\mathbb{T}^d} (\mu^1 - \mu^2)^2 \mathrm{div}(\Theta_p(t, x, \mu^1, Du^1))\ dx \leq \varepsilon G(K) E_\mu.$$

Some more terms:

- Let's skip ahead and consider V_{10} :

$$V_{10} = -\varepsilon\bar{m}\int_{\mathbb{T}^d}(\mu^1 - \mu^2)\sum_{i=1}^{d}\left[\Theta_{p_i x_i}(t,x,\mu^1,Du^1) - \Theta_{p_i x_i}(t,x,\mu^2,Du^2)\right]\,dx.$$

- This is bounded because of our Lipschitz assumption **(H3)**.

$$V_{10} \leq \varepsilon G(K)(E_\mu + E_\mu^{1/2}E_u^{1/2}).$$

- The last term we will highlight is V_{14} :

$$V_{14} = -\varepsilon\bar{m}\int_{\mathbb{T}^d}(\mu^1 - \mu^2)\sum_{i=1}^{d}\sum_{j=1}^{d}\left[\Theta_{p_i p_j}(t,x,\mu^2,Du^2)\left(\frac{\partial^2(u^1 - u^2)}{\partial x_i \partial x_j}\right)\right]\,dx.$$

- We need to use Young's inequality to bound this:

$$V_{14} \leq \varepsilon^2 G(K)E_\mu + \frac{1}{8}\sum_{j=1}^{d}\int_{\mathbb{T}^d}(\partial_{x_j}(Du^1 - Du^2))^2\,dx.$$

Our smallness constraint arises

- After estimating all the terms in our sums, integrating with respect to time (forward in time for μ and backward in time for u), we arrive at

$$E_u(t) + E_\mu(t) + \int_0^t \|D\mu^1 - D\mu^2\|_0^2 \, d\tau + \int_t^T \|D^2u^1 - D^2u^2\|_0^2 \, d\tau$$
$$\leq E_u(T) + E_\mu(0) + \varepsilon T(G(K))(E_u(t) + E_\mu(t))$$
$$+ \frac{1}{4}\int_t^T \|D\mu^1 - D\mu^2\|_0^2 \, d\tau + \frac{1}{4}\int_0^t \|D^2u^1 - D^2u^2\|_0^2 \, d\tau.$$

- We do some manipulations to this to find

$$\sup_{t\in[0,T]} (E_u(t) + E_\mu(t))$$
$$\leq \frac{3}{2}E_u(T) + \frac{3}{2}E_\mu(0) + \varepsilon T(G(K)) \left(\sup_{t\in[0,T]} (E_u(t) + E_\mu(t)) \right).$$

This gives uniqueness

- Assume $E_\mu(0) = 0$ and $E_u(T) = 0$; this means that we have the same initial and terminal data.
- Take $\varepsilon T G(K) < 1$.
- This condition again encapsulates smallness of Hamiltonian, of time horizon, and possibly of data.

Our uniqueness theorem

Theorem

Let $(u^1, \bar{m} + \mu^1)$ *and* $(u^2, \bar{m} + \mu^2)$ *be two solutions of the MFG system with planning boundary conditions, with the same data:*

$$m^1(0, \cdot) = m^2(0, \cdot), \qquad u^1(T, \cdot) = u^2(T, \cdot).$$

Assume that there exists K *such that the solutions are each bounded by* K *:*

$$\|u^i\|_{H^s} + \|\mu^i\|_{H^{s-1}} \leq K, \qquad i \in \{1, 2\},$$

for some $s > 2 + \frac{d}{2}$*. Assume* **(H3)** *holds, and let* G *be as above. Assume* $\varepsilon T(G(K)) < 1$*. Then* $(u^1, \mu^1) = (u^2, \mu^2)$*.*

Thanks for your attention.