PDE Methods for Mean Field Games with Non-Separable Hamiltonian: Data in Sobolev Spaces

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Introduction

- Last time we saw some existence theorems for non-separable mean field games with data in the Wiener algebra.
- Sobolev spaces are more commonly used than the Wiener algebra, and have some good properties (e.g., composition estimates).
- Thus we want to develop a Sobolev version of this existence theory.
- We will use the energy method, to get an existence result and a uniqueness result.

Steps of the energy method

- Introduce an approximate problem.
- Prove existence of solutions for the approximate problem.
- Prove an estimate for these solutions which is uniform in the approximation parameter.
- Use this estimate to help pass to the limit as the approximation parameter(s) vanish.
- Prove that the limiting solution solves the original problem.
- Make similar estimates for uniqueness.

The mean field games system

- For reference, we write down the mean field games system we are considering:

$$u_t + \varepsilon H(t, x, m, Du) = -\Delta u, \tag{1}$$

$$m_t + \varepsilon \mathrm{div}(m H_p(t, x, m, Du)) = \Delta m, \tag{2}$$

in the context of the planning problem, i.e., with boundary conditions

$$m(0, \cdot) = m_0, \qquad u(T, \cdot) = u_T. \tag{3}$$

- The payoff problem could also be considered.

A small reformulation

- As in the previous lecture, we replace (u, m) with (w, μ), where (w, μ) have zero mean.
- This is because the mean of u doesn't affect the evolution, and the mean of m is fixed.
- The system for (w, μ) is

$$w_t + \varepsilon P\Theta(t, x, \mu, Dw) + \Delta w = 0,$$

$$\mu_t + \varepsilon \mathrm{div}\left((\mu + \bar{m})\Theta_p(t, x, \mu, Dw) \right) - \Delta \mu = 0.$$

- If you solve for (w, μ), then you can find (u, m) just by adding back the means.

The approximate problems

- We set up the approximate problems with two approximating parameters, n and δ.
- Let $\mathbb{P}_\delta$ be a mollifier in the x variable; a good choice is truncation of the Fourier series at the level $1/\delta$.
- Then we mollify the data:

$$\mu^{n,\delta}(0,\cdot) = \mathbb{P}_\delta \mu_0, \qquad w^{n,\delta}(T,\cdot) = \mathbb{P}_\delta w_T.$$

- We set up an iterative scheme:

$$w_t^{n+1,\delta} + \varepsilon P\Theta(t,x,\mu^{n,\delta}, Dw^{n,\delta}) + \Delta w^{n+1,\delta} = 0,$$

$$\mu_t^{n+1,\delta} + \varepsilon \mathrm{div}\left((\mu^{n,\delta} + \bar{m})\Theta_p(t,x,\mu^{n,\delta}, Dw^{n,\delta}) \right) - \Delta \mu^{n+1,\delta} = 0.$$

- We initialize with

$$w^{0,\delta} = \mu^{\delta,0} = 0.$$

Solution of the approximate problems

- These equations are linear, forced heat equations.
- The data (initial or terminal) is infinitely smooth.
- Thus the solutions exist and are infinitely smooth (or, are only limited by the smoothness of the Hamiltonian).
- We can write explicit formulas, even:

$$\mu^{n+1,\delta}(t,\cdot) = e^{\Delta t}\mathbb{P}_\delta \mu_0$$
$$-\varepsilon \int_0^t e^{\Delta(t-s)}\mathrm{div}\left((\mu^{n,\delta}(s,\cdot)+\bar{m})\Theta_p(s,\cdot,\mu^{n,\delta}(s,\cdot), Dw^{n,\delta}(s,\cdot) \right) ds,$$

$$w^{n+1,\delta}(t,\cdot) = e^{\Delta(T-t)}\mathbb{P}_\delta w_T$$
$$- \varepsilon P \int_t^T e^{\Delta(s-t)}\Theta(s,\cdot,\mu^{n,\delta}(s,\cdot), Dw^{n,\delta}(s,\cdot))\, ds.$$

Let's skip the uniform estimate for now

- The most important step in the energy method is finding an estimate, uniform in the approximation parameter(s).
- Let's skip that for the moment.
- We'll get back to it soon, but let's see how the argument goes once we have it.
- For now, let us assume that we can establish that

$$\sup_{t\in[0,T]} \|\mu^{n,\delta}\|_{s-1}^2 + \sup_{t\in[0,T]} \|w^{n,\delta}\|_s^2 + \int_0^T \|\mu^{n,\delta}\|_s^2 + \|w^{n,\delta}\|_{s+1}^2 \, ds$$

is bounded by some constant (independent of n and δ).

After the uniform estimate is established

- We have a sequence $(\mu^{n,\delta}, w^{n,\delta})$ uniformly bounded in the space $C([0,T]; H^{s-1} \times H^s)$.
- For sufficiently large s, then, by Sobolev embedding, we have that the first spatial derivatives are uniformly bounded.
- Inspection of the evolution equations, with this regularity, for sufficiently large s, implies that the first time derivatives are also uniformly bounded.
- Therefore, $(\mu^{n,\delta}, w^{n,\delta})$ form an equicontinuous family.
- We have a bounded equicontinuous family on a compact domain, so we can use the Arzela-Ascoli Theorem to extract a uniformly convergent subsequence.
- We get convergence to a limit $(\mu, w) \in C([0,T] \times \mathbb{T}^d)$.

Further regularity

- So far, our limit is only continuous. In order for this to solve our system, we need it to be more regular than this.
- We have uniform convergence on a compact domain. This implies convergence in $C([0,T]; H^0 \times H^0)$.
- This is better because we're in the L^2 setting, but we still need to get derivatives. We also have the uniform bound in the high norm.
- We use an interpolation lemma.

Lemma

Let m and s be real numbers such that $0 < m < s$. There exists $c > 0$ such that for all $f \in H^s$,

$$\|f\|_m \leq c\|f\|_s^{m/s}\|f\|_0^{1-m/s}.$$

Using the interpolation lemma

- Since we have a convergent sequence in $C([0,T]; H^0 \times H^0)$, we have a Cauchy sequence in this space.
- We also have a uniform bound in $H^{s-1} \times H^s$.
- Let $s' \in (0, s)$ be given. Then the $H^{s'-1} \times H^{s'}$ norm of the difference of two solutions can be bounded by a product of the $H^0 \times H^0$ norm and the $H^{s-1} \times H^s$ norm.
- This low norm is going to zero, and this high norm is bounded.
- Thus, we have a Cauchy sequence in $C([0,T]; H^{s'-1} \times H^{s'})$.
- The limit is in this space, then.

Still more regularity

- The unit ball of a Hilbert space is weakly compact.
- Since our sequence is uniformly bounded in $H^{s-1} \times H^s$, at each time, there is a weak limit in $H^{s-1} \times H^s$.
- By uniqueness of limits, then, our limit must be in $L^\infty([0,T]; H^{s-1} \times H^s)$.
- Our presumed uniform bound also is a uniform bound in $L^2([0,T]; H^s \times H^{s+1})$, which is again a Hilbert space. We also get a weak limit in this space, and by uniqueness of limits, we must have that our limit is in this space.
- Summary: We have a limit of our approximating sequence which is in
$$L^\infty([0,T]; H^{s-1} \times H^s) \cap L^2([0,T]; H^s \times H^{s+1}),$$
and in $C([0,T]; H^{s'-1} \times H^{s'})$ for all $0 \leq s' < s$.

Solution of the equation

- We want to know that this limit is not just a limit of our approximate solutions, but is itself a solution of the original problem.
- We use the Fundamental Theorem of Calculus. Integrating the system implies:

$$\mu^{n+1}(t,\cdot) = \mathbb{P}_\delta \mu_0$$
$$+ \int_0^t \left[\Delta\mu^{n+1}(\tau,\cdot) - \varepsilon \mathrm{div}\left((\bar{m} + \mu^n(\tau,\cdot))\,\Theta_p(\tau,\cdot,\mu^n, Dw^n)\right)\right]\, d\tau,$$

$$w^{n+1}(t,\cdot) = \mathbb{P}_\delta w_T + \int_t^T \left[\Delta w^{n+1}(\tau,\cdot) + \varepsilon\Theta(\tau,\cdot,\mu^n, Dw^n)\right]\, d\tau.$$

Solution of the equation

- We have sufficient regularity to pass to the limit here, both as $n \to \infty$ and as $\delta \to 0$:

$$\mu(t,\cdot) = \mu_0 + \int_0^t \left[\Delta\mu(\tau,\cdot) - \varepsilon \mathrm{div}\left((\bar{m} + \mu(\tau,\cdot))\,\Theta_p(\tau,\cdot,\mu,Dw)\right)\right]\,d\tau,$$

$$w(t,\cdot) = w_T + \int_t^T \left[\Delta w(\tau,\cdot) + \varepsilon\Theta(\tau,\cdot,\mu,Dw)\right]\,d\tau.$$

- Taking the time derivative, then, implies that (μ, w) solves the MFG system.

Getting (m, u)

- We now have a solution (μ, w). We want to get (m, u) instead.
- For μ, we add: $m = \mu + \bar{m}$.
- How do we know that m is a probability measure? Its equation,

$$m_t + \mathrm{div}(mH_p(t, x, m, Du)) = \Delta m,$$

  is positivity preserving and conserves mass.
- For u, we can determine the mean by integrating its equation,

$$u_t + H(t, x, m, Du) = -\Delta u = -\mathrm{div}(Du),$$

  in space and time, since m and Du are determined, starting from the terminal data for u.

Existence Theorem

Theorem

Let $T > 0$ and $\varepsilon > 0$ be given. Let $s \geq \left\lceil \frac{d+5}{2} \right\rceil$ and let $\mu_0 \in H^{s-1}(\mathbb{T}^d)$ be such that $\bar{m} + \mu_0$ is a probability measure. Let $u_T \in H^s(\mathbb{T}^d)$ be given. Assume that the conditions **(H1)** *and* **(H2)** *are satisfied. Then there exists $\mu \in L^\infty([0,T]; H^{s-1}) \cap L^2([0,T]; H^s)$ and there exists $u \in L^\infty([0,T]; H^s) \cap L^2([0,T]; H^{s+1})$ such that $\bar{m} + \mu$ is a probability measure for all $t \in [0,T]$, and such that $(u, \bar{m} + \mu)$ satisfies (1), (2), (3). Furthermore, for all $s' \in [0, s)$, we have $\mu \in C([0,T]; H^{s'-1})$ and $u \in C([0,T]; H^{s'})$.*

- **(H1)** is just an assumption about boundedness of the Hamiltonian and its derivatives with respect to its arguments.
- **(H2)** is our smallness constraint.

Still more regularity?

- Is the solution actually continuous in time in the highest norm? That is, is the solution in $C([0, T]; H^{s-1} \times H^s)$?
- We'll talk about this question in one of the recitations.

Changing gears: some ideas about energy estimates

- We will show how to establish the crucial energy estimate. But first, let's get warmed up and see what to expect for energy estimates.
- We will work through how to estimate for a few examples.
- Example 1: Say $u_t = u_{xxx}$.
- Estimate the growth of $E(t) = \int_{\mathbb{T}^d} (\partial_x^s u)^2 \, dx$, assuming that you have a sufficiently smooth solution.
- Taking the time derivative, we get

$$\frac{d}{dt}E = 2\int_{\mathbb{T}^d} (\partial_x^s u)(\partial_x^{s+3} u) \, dx = -\int_{\mathbb{T}^d} \partial_x (\partial_x^{s+1} u)^2 \, dx = 0.$$

- Therefore $E(t) = E(0)$.

Example 2

- Consider instead $u_t = u_{xx}$.
- Estimate the growth of $E(t) = \int_{\mathbb{T}^d} (\partial_x^s u)^2 \, dx$, assuming that you have a sufficiently smooth solution.
- Taking the time derivative, we get

$$\frac{d}{dt}E(t) = 2\int_{\mathbb{T}^d} (\partial_x^s u)(\partial_x^{s+2} u) \, dx = -2\int_{\mathbb{T}^d} (\partial_x^{s+1} u)^2 \, dx.$$

- On the one hand, $\frac{dE}{dt} \leq 0$, so $E(t) \leq E(0)$.
- On the other hand, we can be more specific. Integrating in time, we get

$$E(t) + 2\int_0^t \int_{\mathbb{T}^d} (\partial_x^{s+1} u)^2 \, dx d\tau = E(0).$$

- This is the usual parabolic gain of regularity: estimating a solution in H^s, we found that the solution is actually in $L^2([0,T]; H^{s+1})$.

Example 3

- Consider instead $u_t = uu_x$.
- Say we investigate the growth of $E(t) = E_0(t) + E_s(t)$, where

$$E_0(t) = \int_{\mathbb{T}^d} u^2 \; dx,$$

$$E_s(t) = \int_{\mathbb{T}^d} (\partial_x^s u)^2 \; dx. \tag{3}$$

- The time derivative of E_0 is straightforward:

$$\frac{dE_0}{dt} = 2\int_{\mathbb{T}^d} uu_t \; dx = 2\int_{\mathbb{T}^d} u^2 u_x \; dx = \frac{2}{3}\int_{\mathbb{T}^d} \partial_x u^3 \; dx = 0.$$

Bounding dE_s/dt

- We start estimating dE_s/dt :

$$\frac{dE_s}{dt} = 2\int_{\mathbb{T}^d} (\partial_x^s u)(\partial_x^s u_t)\ dx.$$

So, we need a useful expression for $\partial_x^s u_t$.

- We do this by extracting the most singular part (the term with the most derivatives:

$$\partial_x^s u_t = u\partial_x^{s+1}u + \left(\partial_x^s\left(uu_x\right) - u\partial_x^{s+1}u\right).$$

- Considering these two terms, we make the decomposition

$$\frac{dE_s}{dt} = A + B.$$

Bounding A

- A is the more important term to bound, and is straightforward:

$$A = 2\int_{\mathbb{T}^d} (\partial_x^s u) u (\partial_x^{s+1} u)\ dx = \int_{\mathbb{T}^d} u \partial_x (\partial_x^s u)^2\ dx$$
$$= -\int_{\mathbb{T}^d} (u_x)(\partial_x^s u)^2\ dx \leq |u_x|_{L^\infty} E \leq cE^{3/2},$$

  by Sobolev embedding (for sufficiently large s). (In 1D, $s = 2$ is enough.)

- This is good, and we should turn to B.

Bounding B

- B is somewhat easier to bound, because it doesn't have a high derivative.
- We can use the following lemma:

Lemma

Let $m \in \mathbb{N}$. There exists $c > 0$ such that for all $f \in L^\infty \cap H^m$ and for all $g \in L^\infty \cap H^m$,

$$\sum_{0 \leq |\alpha| \leq m} \|\partial^\alpha (fg) - f \partial^\alpha g\|_{L^2} \leq c \left(|Df|_\infty \|D^{m-1} g\|_{L^2} + \|D^m f\|_{L^2} |g|_\infty \right).$$

- We use this with $m = s$, $\partial^\alpha = \partial_x^s$, $f = u$, and $g = u_x$.
- Applying the lemma yields

$$B \leq c|u_x|_{L^\infty} E \leq cE^{3/2}.$$

which is the same bound we had for A.

Finishing up Example 3

- So, for $u_t = uu_x$, we get $\frac{dE}{dt} \leq cE^{3/2}$.
- A Gronwall-type argument can then be used to show that we have control of the H^s-norm up until some finite time $T > 0$.
- So, an initially smooth solution does not blow up immediately.

Example 4

- Consider $u_t = uu_x + u_{xx}$.
- This is Burgers' equation, and can be considered as a combination of Example 2 and Example 3.
- Certainly we could use the same techniques as we did in those examples, but I want to estimate it differently.
- Recall Young's inequality (with parameter $\sigma > 0$):

$$ab \leq \frac{a^2}{2\sigma} + \frac{\sigma b^2}{2}.$$

- Although it is quite elementary, this Young's inequality is very useful for parabolic problems.

Bounding the growth in Example 4

- We proceed as before, and find

$$E(t) + 2\int_0^t \int_{\mathbb{T}^d} (\partial_x^{s+1} u)^2 \; dxds \leq E(0) + \int_0^t A \; ds + \int_0^t B \; ds. \quad (4)$$

- As before, the term B is fine, and we can estimate it with $E^{3/2}$.
- It is the term A that I want to estimate differently than before (although we can do what we did before as well).
- Recall that

$$A = 2\int_{\mathbb{T}^d} u(\partial_x^s u)(\partial_x^{s+1} u) \; dx.$$

Using Young's inequality to bound A

- We proceed as follows:

$$|A| \leq 2|u|_{L^\infty} \int_{\mathbb{T}^d} (\partial_x^s u)(\partial_x^{s+1} u)\ dx$$
$$\leq cE^{1/2} \int_{\mathbb{T}^d} \frac{\sigma(\partial_x^s u)^2}{2} + \frac{(\partial_x^{s+1} u)^2}{2\sigma}\ dx.$$

- Choose $\sigma = cE^{1/2}$, say.
- Then we have the following:

$$|A| \leq cE + \frac{1}{2} \int_{\mathbb{T}^d} (\partial_x^{s+1} u)^2\ dx.$$

- We conclude the following:

$$E(t) + \frac{3}{2} \int_0^t \int_{\mathbb{T}^d} (\partial_x^{s+1} u)^2\ dxds \leq E(0) + c \int_0^t (1 + E(s))^2\ ds$$

and then our Gronwall-like argument can be made.

Next time

- Next time, we will carry out the details of the energy estimate.
- It follows the lines we just outlined, but is more complicated.

Thanks for your attention.