Link between MFG and initial value problems

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Last week, we discussed the variational MFG

$$\partial_t \mu + \nabla \cdot (\mu \nabla \phi) = \nu \Delta \mu, \quad \partial_t \phi + \frac{1}{2}|\nabla \phi|^2 + \nu \Delta \phi = f'(\mu),$$

$t \in [0, T]$, $x \in D = \mathbb{T}^d$, $\mu(t, x) \geq 0$, $\phi(t, x)$, respectively prescribed at $t = 0$ and $t = T$.

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With $\nu = 0$ and written in terms of $v = \nabla \phi$, these equations read

$$\partial_t \mu + \nabla \cdot (\mu v) = 0, \quad \partial_t v + (v \cdot \nabla) v = \nabla(f'(\mu)),$$

and looks like the equations written by Euler in 1755-57 for compressible fluids.

The Euler equations written in conservation form

$$\partial_t \mu + \nabla \cdot q = 0, \quad q = \mu v,$$

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If f is CONCAVE, we get a WELL-POSED INITIAL VALUE PROBLEM, with boundary conditions only at $t = 0$ and none at $t = T$, in sharp contrast with MFG.

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We want to solve the initial value problem for a large class of equations including Euler's ones by a variational approach based on convexity.

This turns out to be doable through a GENERALIZED MFG, involving vector-potentials (and measures taking values in the cone of semi-definite positive matrices).

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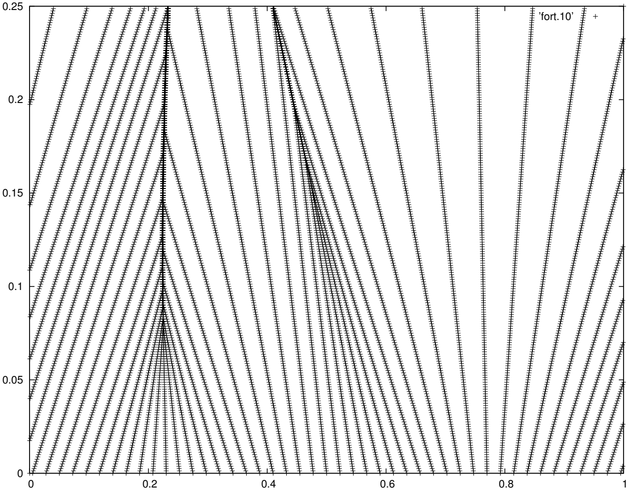
$$\partial_t U + \nabla \cdot (F(U)) = 0, \quad U = U(t,x) \in \mathcal{W} \subset \mathbb{R}^m, \quad x \in D$$

(where F is given so that $\sum_{\beta=1}^{m} \partial_\beta \mathcal{E}(W) \partial_\alpha F^{i\beta}(W) = \partial_\alpha Q^i(W), \ \forall W \in \mathcal{W}$,

for some $(\mathcal{E}, Q) : \mathcal{W} \to \mathbb{R}^{1+d}$, with $\mathcal{W}$ open convex and "entropy" $\mathcal{E}$ strictly convex,

which implies: $\partial_t(\mathcal{E}(U)) + \nabla \cdot (Q(U)) = 0$, for all smooth solutions U)

contains the Euler equations,
for which: $\mathcal{E}(\mu, q) = \frac{|q|^2}{2\mu} - f(\mu), \quad \mu > 0$, with f concave.



Inviscid Burgers equation : $\partial_t u + \partial_x(u^2/2) = 0$, $u = u(t,x)$, $x \in \mathbb{R}/\mathbb{Z}$, $t \geq 0$.
Formation of two shock waves. (Vertical axis: $t \in [0, 1/4]$, horizontal axis: $x \in \mathbb{T}$.)

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The problem is not trivial since there may be many weak solutions starting from U_0

which are not entropy-preserving (by "convex integration" à la De Lellis-Székelyhidi).

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N.B. The supremum in A exactly encodes that U is a weak solution with initial condition U_0, all test functions A acting like Lagrange multipliers.

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$$= \sup_{A(T,\cdot)=0} \int_0^T \int_D -G(\partial_t A, \nabla A) - \int_D A(0,\cdot)\cdot U_0$$

$$G(E,B) = \sup_{V\in\mathcal{W}\subset\mathbb{R}^m} E\cdot V + B\cdot F(V) - \mathcal{E}(V), \; (E,B)\in\mathbb{R}^m\times\mathbb{R}^{m\times d}.$$

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Notice that G is automatically convex.

Comparison with the MFG considered last week

$$\sup_{\phi} \int_0^T \int_D -G(\partial_t \phi + \nu \Delta \phi, \nabla \phi) - < \mu_0, \phi_0 >$$

was our variational MFG with μ_0 and ϕ_T prescribed.

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Thus our dual maximization problem to solve the initial value problem can be interpreted as a generalized variational 1st order-MFG with vector-valued potential.

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Theorem 2: For the Burgers equation, all entropy solutions can be recovered, for arbitrarily large T.

(*) more precisely if, $\forall\ t, x, V \in \mathcal{W}$, $\mathcal{E}''(V) - (T-t)F''(V) \cdot \nabla(\mathcal{E}'(U(t,x))) > 0$.

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$$\partial_t \mu + \nabla \cdot q = 0, \quad \partial_t q + \nabla \cdot (\frac{q \otimes q}{\mu}) = -\nabla(p(\mu))$$

where

$$f(w) = w - \log w, \quad p'(w) = -wf''(w) \longrightarrow p(w) = w.$$

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Then, the generalized MFG amounts to minimizing

$$\int_{[0,T]\times D} \exp(u)\exp(\frac{1}{2}Q\cdot M^{-1}\cdot Q) + \int_D \sigma_0\rho_0 + w_0\cdot q_0,$$

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among all fields $u = u(t,x) \in \mathbb{R}$, $Q = Q(t,x) \in \mathbb{R}^d$, $M = M(t,x) = M^t(t,x) \in \mathbb{R}^{d\times d}$, $M \geq 0$,

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$M = M(t,x) = M^t(t,x) \in \mathbb{R}^{d\times d}$, $M \geq 0$, of form:

$$u = \partial_t \sigma + \partial^i w_i,\ \ Q_i = \partial_t w_i + \partial_i \sigma,\ \ M_{ij} = \delta_{ij} - \partial_i w_j - \partial_j w_i,$$

where σ and w must vanish at $t = T$.

The Euler equations of incompressible fluids

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The same method also applies to the Euler equations of incompressible fluids ("saturated congestion")

$$\partial_t q + \nabla \cdot (q \otimes q) = -\nabla p, \quad \nabla \cdot q = 0,$$

where q is prescribed at $t = 0$ and p is now a Lagrange multiplier ("price") for constraint $\nabla \cdot q = 0$.

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We get again a generalized MFG for measures valued in the cone of semi-definite symmetric matrices:

Generalized MFG for incompressible fluids

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This generalized (variational) MFG reads

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$$M_{ij}(T,\cdot) = \delta_{ij}, \quad \partial_t M_{ij} = \partial_j Q_i + \partial_i Q_j + 2\partial_i \partial_j (-\triangle)^{-1} \partial_k Q^k.$$

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Then, the maximization problem in A simply reads

$$\sup_A \int_{[0,T]\times\mathbb{T}} -\frac{(\partial_t A)^2}{2(1-\partial_x A)} - \int_{\mathbb{T}} A(0,\cdot)u_0.$$

with $A = A(t,x) \in \mathbb{R}$ subject to $A(T,\cdot) = 0, \ \ \partial_x A \leq 1.$

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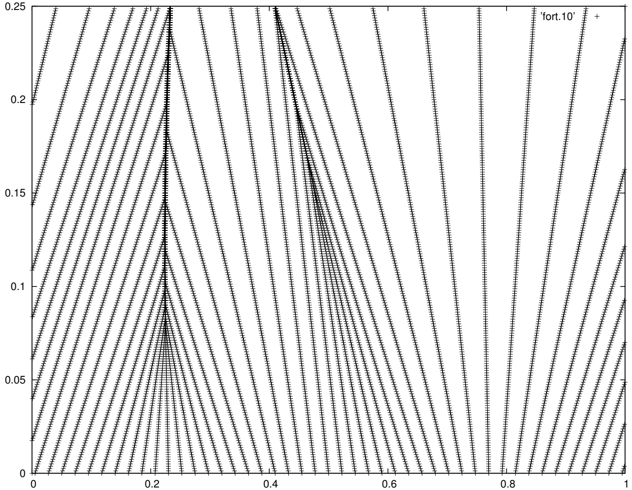
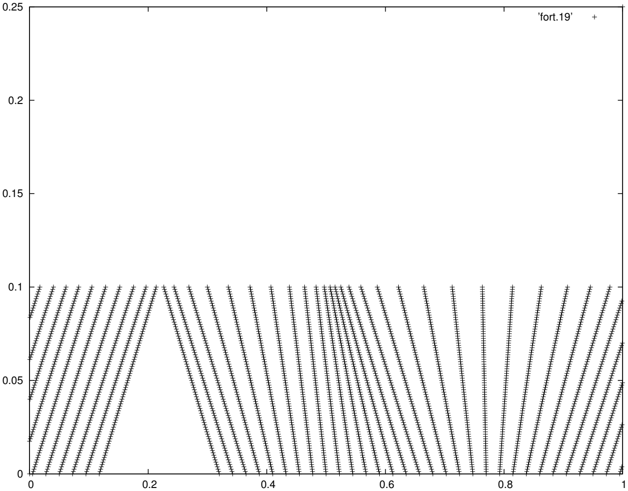
Introducing $\mu = 1 - \partial_x A \geq 0$, $q = \partial_t A$, we get the MFG

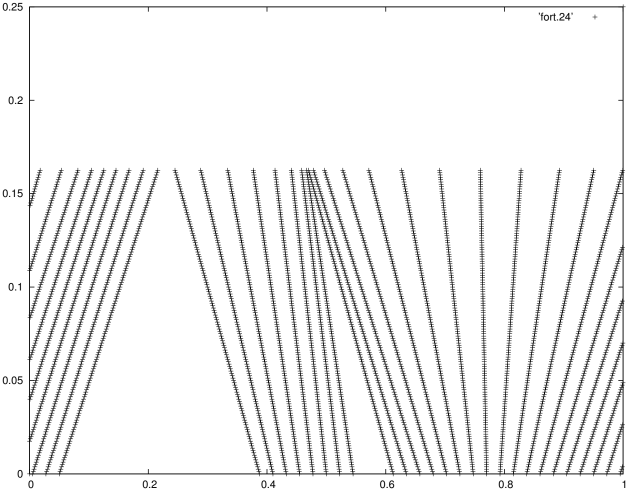
$$\sup_{(\mu,q)}\{ \int_{[0,T]\times\mathbb{T}} -\frac{q^2}{2\mu} - qu_0 \mid \partial_t \mu + \partial_x q = 0, \;\; \mu(T,\cdot) = 1\}.$$



Inviscid Burgers equation : $\partial_t u + \partial_x(u^2/2) = 0$, $u = u(t, x)$, $x \in \mathbb{R}/\mathbb{Z}$, $t \geq 0$.
Formation of two shock waves. (Vertical axis: $t \in [0, 1/4]$, horizontal axis: $x \in \mathbb{T}$.)



Inviscid Burgers equation : $\partial_t u + \partial_x(u^2/2) = 0$, $u = u(t,x)$, $x \in \mathbb{R}/\mathbb{Z}$, $t \geq 0$.
Recovery of the solution at time T=0.1 by convex optimization.
Observe the formation of a first vacuum zone as the first shock has formed.



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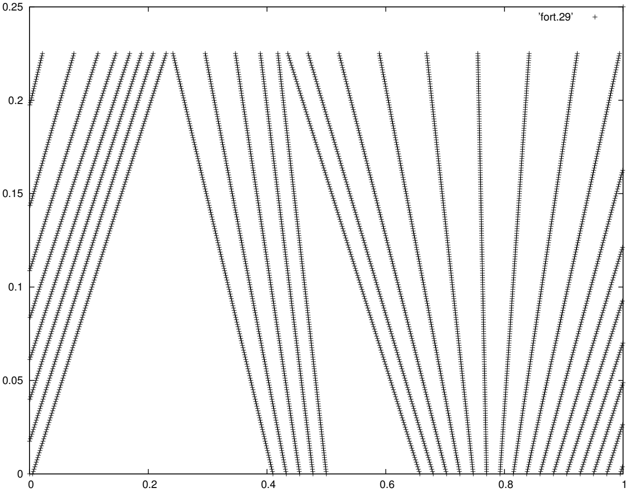
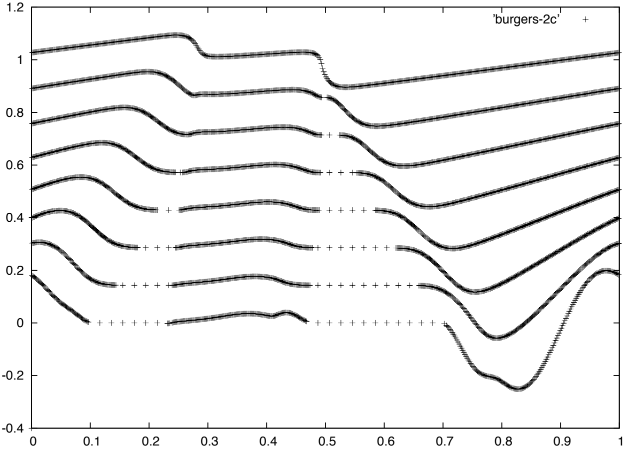
Recovery of the solution at time T=0.16 by convex optimisation.
Observe the formation of a second vacuum zone as the second shock has formed.



Inviscid Burgers equation : $\partial_t u + \partial_x(u^2/2) = 0$, $u = u(t, x)$, $x \in \mathbb{R}/\mathbb{Z}$, $t \geq 0$.
Recovery of the solution at time T=0.225 by convex optimisation.
Observe the extension of the two vacuum zones.



Numerics: 2 lines of code differ from a standard (Benamou-B.) OT solver!

Analogy with mountain climbing: going from Everest to Lhotse without following the crest! (Partial credit to Thomas Gallouët for this analogy.)

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For more details, voir Y.B. ArXiv Oct. 2017.