جامعة الملك عبدالله
للعلوم والتقنية
King Abdullah University of
Science and Technology

# Explicit solutions of mean-field games

Diogo A. Gomes

- Mean-field games (MFGs) are models for systems with a large number of rational agents who seek to minimize a cost functional and have access to statistical information on the distribution of the population.
- These models were introduced in the engineering community by Caines, Huang and Malhamé and in the mathematical community by Lasry and Lions.

The workhorse of MFG theory is the system:

$$\begin{cases} -u_t + \frac{|Du|^2}{2} + V(x) = \epsilon \Delta u + F(x,m), \\ m_t - \operatorname{div}(mDu) = \epsilon \Delta m, \end{cases}$$

with initial and terminal conditions

$$\begin{cases} u(x,T) = u_T(x) \\ m(x,0) = m_0(x). \end{cases}$$

Here, $m_0$ and $u_T$ are given, $m_0 \geq 0$ with $\int_{\mathbb{R}^d} m_0 dx = 1$. To avoid technical difficulties, we work with periodic boundary conditions; the domain of $u$ and $m$ is $\mathbb{T}^d \times [0,T]$.

The corresponding stationary MFG is

$$\begin{cases} \frac{|Du|^2}{2} + V(x) = \epsilon \Delta u + F(x,m) + \overline{H}, \\ -\operatorname{div}(mDu) = \epsilon \Delta m, \end{cases}$$

and the solution is a triplet $(u, m, \overline{H})$. We require $m \geq 0$ and $\int m \mathrm{d}x = 1$.
In the periodic case, $u, m : \mathbb{T}^d \to \mathbb{R}$ and $\overline{H} \in \mathbb{R}$.

# Optimal control and Hamilton-Jacobi equations

- We fix $T > 0$ and consider an agent whose state is $\mathbf{x}(t) \in \mathbb{R}^d$ for $0 \leq t \leq T$.
- Agents can change their state by choosing a control in $\mathcal{W} = L^\infty([t, T], \mathbb{R}^d)$.
- the state of an agent evolves according to

$$\dot{\mathbf{x}}(t) = \mathbf{v}(t). \tag{1}$$

- We fix a Lagrangian $\tilde{L} : \mathbb{R}^d \times \mathbb{R}^d \times [0, T] \to \mathbb{R}$, with $v \mapsto L(x, v, t)$ uniformly convex. For example,

$$\tilde{L}(x, v, t) = \frac{|v|^2}{2} - V(x) + \tilde{F}(x, t),$$

  with $\tilde{F} : \mathbb{R}^d \times [0, T] \to \mathbb{R}$ a continuous function bounded from below.
- We select a bounded continuous function, $u_T : \mathbb{R}^d \to \mathbb{R}$, called the terminal cost.

- Agents have preferences that are encoded in the action functional,

$$J(\mathbf{v}; x, t) = \int_t^T \tilde{L}(\mathbf{x}(s), \mathbf{v}(s), s)ds + u_T(\mathbf{x}(T)),$$

where $\mathbf{x}$ solves (1) with the initial condition $\mathbf{x}(t) = x$.

- Each agent seeks to minimize $J$ among all possible controls in $\mathcal{W}$. The value function is infimum of $J$ over all controls,

$$u(x, t) = \inf_{\mathbf{v} \in \mathcal{W}} J(\mathbf{v}; x, t).$$

The Legendre transform, $\tilde{H}$, of $\tilde{L}$ is the Hamiltonian

$$\tilde{H}(x,p,t) = \sup_{v \in \mathbb{R}^d} \left[ -p \cdot v - \tilde{L}(x,v,t) \right].$$

By the uniform convexity of $\tilde{L}$ in the second coordinate, the maximum is achieved at a unique point, $v^*$ given by

$$v^* = -D_p \tilde{H}(x,p,t).$$

For $\tilde{L}$ as before,

$$\tilde{H}(x,p,t) = \frac{|p|^2}{2} + V(x) - \tilde{F}(x,t).$$

A classical result in control theory states that if $u \in C^1(\mathbb{R}^d \times [t_0, T])$, then $u$ solves the Hamilton-Jacobi equation,

$$-u_t(x,t) + \tilde{H}(x, D_x u(x,t), t) = 0.$$

Further, as we prove next, the optimal control, $\mathbf{v}^*(t)$, is determined in feedback form by

$$\mathbf{v}^*(t) = -D_p \tilde{H}(\mathbf{x}^*(t), D_x u(\mathbf{x}^*(t), t), t).$$

### Theorem (Verification Theorem)

*Let $\tilde{u} \in C^1(\mathbb{R}^d \times [t_0, T])$ solve the Hamilton–Jacobi equation with the terminal condition $u_T(x)$. Let*

$$\mathbf{v}^*(t) = -D_p \tilde{H}(\mathbf{x}^*(t), D_x \tilde{u}(\mathbf{x}^*(t), t), t)$$

*and $\mathbf{x}^*(t)$ be the corresponding trajectory. Then, $\mathbf{v}^*(t)$ is an optimal control and $\tilde{u}(x, t) = u(x, t)$, where $u$ is the value function.*

# Proof

For any $\mathbf{v}(s)$ and any trajectory $\mathbf{x}$ we have

$$\tilde{u}(\mathbf{x}(T), T) = \int_t^T (D_x\tilde{u}(\mathbf{x}(s), s) \cdot \mathbf{v}(s) + \tilde{u}_s(\mathbf{x}(s), s))\, ds + \tilde{u}(\mathbf{x}(t), t).$$

In addition,

$$(D_x\tilde{u}(\mathbf{x}(s), s) \cdot \mathbf{v}(s) \geq -\tilde{H}(x, D_x\tilde{u}(\mathbf{x}(s), s) - \tilde{L}(\mathbf{x}(s), \mathbf{v}(s),$$

and the previous inequality is an identity for $\mathbf{v} = \mathbf{v}^*$. Hence,

$$\tilde{u}(x, t) \leq \int_t^T \tilde{L}(\mathbf{x}(s), \mathbf{v}(s), s) ds + u_T(\mathbf{x}(T)),$$

and the previous inequality is an identity if $\mathbf{v} = \mathbf{v}^*$.

# Transport equation

Let $b : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ be a Lipschitz vector field. The ODE

$$\begin{cases} \dot{\mathbf{x}}(t) = b(\mathbf{x}(t), t) & t > 0, \\ \mathbf{x}(0) = x \end{cases}$$

induces a flow, $\Phi^t$, in $\mathbb{R}^d$ that maps the initial condition, $x \in \mathbb{R}^d$, at $t = 0$ to the solution at time $t > 0$.

Fix a probability measure, $m_0 \in \mathcal{P}(\mathbb{R}^d)$. For $0 \leq t \leq T$, let $m(\cdot, t)$ be the push-forward,$\Phi^t \sharp m_0$,, by $\Phi^t$ of $m_0$ given by

$$\int_{\mathbb{R}^d} \phi(x) m(x,t) dx = \int_{\mathbb{R}^d} \phi\left(\Phi^t(x)\right) m_0 dx. \tag{2}$$

For $0 \leq t \leq T$, $m(\cdot, t)$ is a probability measure.

## Proposition

*Assume that $b(x,t)$ is Lipschitz continuous in $x$. Let $\Phi^t$ be the corresponding flow and $m = \Phi_t \sharp m_0$. Then, $m \in C(\mathbb{R}_0^+, \mathcal{P}(\mathbb{R}^d))$ and*

$$\begin{cases} m_t(x,t) + \operatorname{div}(b(x,t)m(x,t)) = 0, & (x,t) \in \mathbb{R}^d \times [0,T], \\ m(x,0) = m_0(x), & x \in \mathbb{R}^d, \end{cases}$$

*in the distributional sense.*

# Proof

We recall that $m$ is a solution in the sense of distributions if

$$-\int_0^T \int_{\mathbb{R}^d} m(x,t)\left(\phi_t(x,t)+b(x,t)\phi_x(x,t)\right)dxdt = \int_{\mathbb{R}^d} m_0(x)\phi(x,0)dx,$$

for every $\phi \in C_c^\infty(\mathbb{R}^d \times [0,T))$. Differentiating both sides of (2) gives

$$\int_{\mathbb{R}^d} \phi(x)m_t(x,t)dx = \int_{\mathbb{R}^d} \left(b(\Phi^t(x),t)D_x\phi\left(\Phi^t(x)\right)\right)m_0(x)dx.$$

Thus,

$$\int_{\mathbb{R}^d} \phi(x)m_t(x,t)dx = \int_{\mathbb{R}^d} \left(b(x,t)D_x\phi(x)\right)m(x,t)dx,$$

using the definition of $\Phi^t$. To conclude the proof, we integrate by parts the right-hand side.

# Mean-field models I

- The mean-field game framework was developed to study systems with an infinite number of rational agents in competition.
- Each agent seeks to optimize an individual control problem that depends on statistical information about the whole population.
- The only information available to the agents is the probability distribution of the agents' states.

- For each time $t$, $m(x,t)$ is a probability density in $\mathbb{R}^d$ that gives the distribution of the agents
- We set
$$\tilde{L}(x,v,t) = L(x,v,m(\cdot,t)).$$
and denote the Legendre transform of $L$ by $H$.
- Each agent seeks to minimize a control problem whose value function solves
$$-u_t + H(x, D_x u, m) = 0.$$

According to the Verification Theorem, if the previous equation has a solution, $u$, the vector field, $b = -D_p H(x, D_x u(x,t), m)$, gives an optimal strategy. Because all agents are rational, they use this strategy.

Hence, $u$ and $m$ are determined by

$$\begin{cases} -u_t + H(x, D_x u, m) = 0 \\ m_t - \operatorname{div}(D_p H m) = 0. \end{cases}$$

We supplement this system with initial-terminal determine by the terminal value function for the agents $u_T : \mathbb{R}^d \to \mathbb{R}$ and their initial distribution is $m_0 : \mathbb{R}^d \to \mathbb{R}_0^+$.

We begin our study of explicit solutions by considering a first-order quadratic MFG with a logarithmic nonlinearity. This game is given by

$$\begin{cases} \frac{|u_x|^2}{2} + V(x) + b(x)u_x = \ln m + \overline{H}, \\ -(m(Du + b(x)))_x = 0, \end{cases} \tag{3}$$

with, $u, m : \mathbb{T} \to \mathbb{R}$, $m \geq 0$,

$$\int_{\mathbb{T}} m dx = 1,$$

and, for definiteness,

$$\int_{\mathbb{T}} u \, \mathrm{d}x = 0.$$

Moreover, we suppose that

$$\int_{\mathbb{T}} b(y) \mathrm{d}y = 0.$$

If $Du + b = 0$, the second equation in (3) holds immediately. This suggests that we set

$$u(x) = -\int_0^x b(y)\,\mathrm{d}y + \int_{\mathbb{T}} \int_0^z b(y)\,\mathrm{d}y\,\mathrm{d}z.$$

Using the previous formula in the first equation, we get

$$m(x) = \frac{e^{V(x) - \frac{b^2(x)}{2}}}{\int_{\mathbb{T}} e^{V(y) - \frac{b^2(y)}{2}}\,\mathrm{d}y}.$$

Let $\psi : \mathbb{T} \to \mathbb{R}$ and $b(x) = \psi_x(x)$. Then,

$$u(x) = -\psi(x), \qquad m(x) = \frac{e^{V(x) - \frac{\psi_x^2(x)}{2}}}{\int_{\mathbb{T}} e^{V(y) - \frac{\psi_x^2(y)}{2}} \, \mathrm{d}y},$$

and

$$\overline{H} = \ln\left[\int_{\mathbb{T}} e^{V(y) - \frac{\psi_x^2(y)}{2}} \, \mathrm{d}y\right]$$

is a solution of the corresponding MFG.

The preceding ideas can be extended in various ways:

- In the one-dimensional case, the approach can be made systematic through the current method, examined later.
- For $d > 1$, a similar idea can be used for the MFG

$$\begin{cases} \frac{|Du|^2}{2} + V(x) + D\psi \cdot Du = \ln m + \overline{H}, \\ -\operatorname{div}(m(Du + D\psi)) = 0, \end{cases}$$

For $P \in \mathbb{R}^d$, consider the system

$$\begin{cases} -\Delta u + \frac{1}{2}|P + Du|^2 + V(x) = \ln m \\ -\Delta m - \operatorname{div}((P + Du)m) = 0. \end{cases}$$

Define $m$ by the Hopf-Cole transform

$$m = e^{\frac{v-u}{2}},$$

where $u$ and $v$ solve

$$\begin{cases} -\Delta u + \frac{1}{2}|P + Du|^2 + V(x) = \frac{v-u}{2} \\ \quad \Delta v + \frac{1}{2}|P + Dv|^2 + V(x) = \frac{v-u}{2}. \end{cases}$$

By a direct computation, $m$ solves

$$-\Delta m - \operatorname{div}((P + Du)m) = 0.$$

To check this, it is enough to observe that

$$\begin{aligned}
-\Delta m &= m\left[\frac{1}{2}\Delta u - \frac{1}{2}\Delta v - \frac{|Du - Dv|^2}{4}\right] \\
&= m\Delta u - \frac{m}{4}\left[|P + Du|^2 - |P + Dv|^2 + |Du - Dv|^2\right] \\
&= (P + Du)\cdot Dm + m\Delta u = \operatorname{div}((P + Du)m).
\end{aligned}$$

For $P = 0$, we obtain

$$\begin{cases} -\Delta u + \frac{1}{2}|Du|^2 + V(x) = \frac{v-u}{2} \\ \quad \Delta v + \frac{1}{2}|Dv|^2 + V(x) = \frac{v-u}{2}. \end{cases}$$

Hence, $v = -u$ and, thus, we get the scalar PDE

$$u - \Delta u + \frac{1}{2}|Du|^2 + V(x) = 0.$$

# Time-independent

We consider the MFG

$$\begin{cases} -\Delta u + \frac{1}{2}|Du|^2 + \beta|x|^2 = \ln m + \overline{H} \\ -\Delta m - \operatorname{div}(mDu) = 0. \end{cases}$$

For $m = \mu e^{-u}$ the second equation holds trivially. Next, for

$$u = \alpha|x|^2,$$

the first equation gives

$$2\alpha^2 + \alpha + \beta = 0.$$

If $\beta < 0$, the preceding equation has a solution, $\alpha > 0$. Finally, $\mu$ is given by the condition, $\int_{\mathbb{R}} m dx = 1$. To find $\overline{H}$, we use the expressions for $u$ and $m$ in the first equation.

# Time-dependent I

For the time-dependent case,

$$\begin{cases} -u_t + \frac{u_x^2}{2} + \beta x^2 = \ln m \\ m_t - (mu_x)_x = 0, \end{cases}$$

we select

$$m(x,t) = \sqrt{c(t)}e^{-c(t)x^2} \qquad u(x,t) = a(t) + b(t)x^2.$$

Then,

$$\begin{cases} -\dot{a} - \dot{b}x^2 + 2b^2x^2 + \beta x^2 = \frac{1}{2}\ln c - cx^2 \\ e^{-cx^2}\left(\frac{\dot{c}}{2c^{1/2}} - \dot{c}x^2c^{1/2}\right) - \left(2bxc^{1/2}e^{-cx^2}\right)_x = 0. \end{cases}$$

# Time-dependent II

Hence, matching powers, we get

$$\begin{cases} -\dot{a} = \frac{1}{2}\ln c \\ -\dot{b} + 2b^2 + \beta = -c \\ \dot{c} - 4bc = 0. \end{cases}$$

Now, we consider the mean-field game

$$\begin{cases} -u_t + H(x, Du) = F(\eta * m) \\ m_t - \operatorname{div}(mD_pH) = 0, \end{cases}$$

where $\eta$ is a standard mollifier. We take singular initial data for $m$

$$m(x,0) = \bar{m}(x) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}$$

and terminal data for $u$

$$u(x,T) = \bar{u}(x)$$

The transport equation

$$m_t - \operatorname{div}(m D_p H) = 0$$

can be solved by tracking the flow of each of the points $x_i$ through the ODE

$$\dot{\mathbf{x}}_i(t) = D_p H(\mathbf{x}_i(t), Du(\mathbf{x}_i(t), t));$$

that is,

$$m(x,t) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\mathbf{x}_i(t)}(x).$$

Next, we define

$$\mathbf{p}_i(t) = Du(\mathbf{x}_i(t), t).$$

By the method of characteristics, we have

$$\dot{\mathbf{p}}_i = -D_{x_i}\left(H(\mathbf{x}_i, \mathbf{p}_i) - F\left(\frac{1}{N}\sum_{j=1}^{N}\eta(\mathbf{x}_i(t) - \mathbf{x}_j(t))\right)\right).$$

Therefore, we reduce the solution of the original MFG to the system of ODEs

$$\begin{cases} \dot{\mathbf{x}}_i = D_p H(\mathbf{x}_i, \mathbf{p}_i) \\ \dot{\mathbf{p}}_i = -D_{x_i} \left( H(\mathbf{x}_i, \mathbf{p}_i) - F\left( \frac{1}{N} \sum_{j=1}^N \eta(\mathbf{x}_i(t) - \mathbf{x}_j(t)) \right) \right) \end{cases}$$

with the following initial-terminal conditions

$$\begin{cases} \dot{\mathbf{x}}_i(0) = x_i \\ \dot{\mathbf{p}}_i(T) = D_x \bar{u}(\mathbf{x}_i(T)). \end{cases}$$

Here, we consider the mean-field game

$$\begin{cases} \frac{(u_x+p)^2}{2} + V(x) = g(m) + \overline{H}, \\ -(m(u_x+p))_x = 0, \end{cases} \tag{4}$$

According to the second equation, the *current*,

$$j = m(u_x + p),$$

is constant.

If $j \neq 0$, $m(x) \neq 0$, and $u_x + p = j/m$. Thus

$$\begin{cases} F_j(m) = \overline{H} - V(x), \\ m > 0, \ \int\limits_{\mathbb{T}} m dx = 1, \\ \int\limits_{\mathbb{T}} \frac{1}{m} dx = \frac{p}{j}, \end{cases}$$

where $F_j(m) = \frac{j^2}{2m^2} - g(m)$.
For each $x$, the first equation in is algebraic.

- If $g$ is increasing and $g(+\infty) = +\infty$, for each $x \in \mathbb{T}$ and $\overline{H} \in \mathbb{R}$, there exists a unique solution.
- If $g$ is not increasing, there may exist multiple solutions.

## Proposition

*Let $g$ be increasing. Then, for every $j \neq 0$, (4) has a unique smooth solution, $(u_j, m_j, \overline{H}_j)$, with current $j$. This solution is given by*

$$m_j(x) = F_j^{-1}(\overline{H}_j - V(x)), \quad u_j(x) = \int_0^x \frac{j}{m_j(y)} dy - p_j x,$$

*where $p_j = \int_{\mathbb{T}} \frac{j}{m_j(y)} dy$ and $\overline{H}_j$ is such that $\int_{\mathbb{T}} m_j(x) dx = 1$.*

For $j = 0$, we have

$$\begin{cases} \frac{(u_x+p)^2}{2} - g(m) = \overline{H} - V(x), \\ m \geq 0, \int\limits_{\mathbb{T}} m dx = 1, \\ m(u_x + p) = 0. \end{cases}$$

Here, we consider the case where $g(m) = m$; the analysis is similar for $g$ increasing.

Accordingly, we have

$$\begin{cases} \frac{(u_x+p)^2}{2} - m = \overline{H} - V(x); \\ m \geq 0, \ \int\limits_{\mathbb{T}} m dx = 1; \\ m(u_x + p) = 0. \end{cases}$$

Thus

- $m(x) = (V(x) - \overline{H})^+$
- $\frac{(u_x+p)^2}{2} = (V(x) - \overline{H})^-$

The map $\overline{H} \mapsto \int\limits_{\mathbb{T}} (V(x) - \overline{H})^+ dx$ is decreasing thus there exists a unique number, $\overline{H}_V$, such that $\int\limits_{\mathbb{T}} m(x) dx = 1$.

i. If $\min_{\mathbb{T}} V < \overline{H}_V < \max_{\mathbb{T}} V$, $m$ is non-smooth and there are regions where it vanishes. Moreover, there are $C^1$ solutions:

$$u^{\pm}(x) = \pm \int_0^x \sqrt{2(V(y) - \overline{H}_V)^-}\, dy - px,$$

with $p = \pm \int_{\mathbb{T}} \sqrt{2(V(y) - \overline{H}_V)^-}\, dy$. Additionally, there exist Lipschitz solutions given by:

$$(u^{x_0})_x(x) = \sqrt{2(V(x) - \overline{H}_V)^-}\, \chi_{[0,x_0)} - \sqrt{2(V(x) - \overline{H}_V)^-}\, \chi_{(x_0,1)} - p^{x}$$

where

$$p^{x_0} = \int_{y<x_0} \sqrt{2(V(y) - \overline{H}_V)^-}\, dy - \int_{y>x_0} \sqrt{2(V(y) - \overline{H}_V)^-}\, dy,$$

and $x_0 \in \mathbb{T}$ is such that $V(x_0) < \overline{H}_V$.

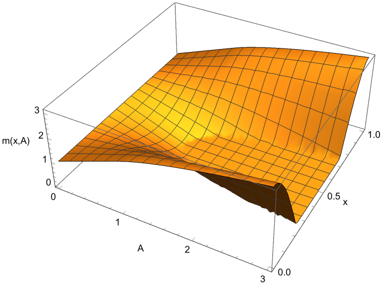
ii. If $\overline{H}_V < \min_{\mathbb{T}} V$, $m$ is smooth and positive. Moreover,

$$m(x) = V(x) - \overline{H}_V, \; u(x) = 0, \; p = 0, \; \overline{H}_V = \int_{\mathbb{T}} V - 1.$$

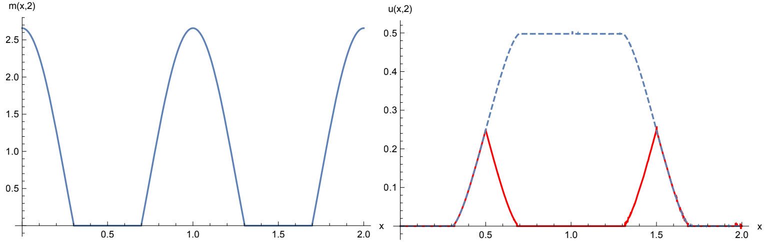
Thus if $j = 0$, we have a unique, smooth solution if and only if $u_x + p \equiv 0$ or, equivalently, $m(x) = V(x) - \overline{H}_V$. Equivalently,

$$\int_{\mathbb{T}} V(x)dx \leq 1 + \min_{\mathbb{T}} V.$$

This is the case for $V$ with small oscillation; that is, $\operatorname{osc} V \leq 1$.




$$m(x, A) \text{ for } V_A(x) = A\sin(2\pi(x + \tfrac{1}{4})), \; .$$




$m(x, 2)$ (left) and two distinct solutions $u(x, 2)$ (right).

Now, we consider MFGs with decreasing $g$. To simplify, we assume that $g(m) = -m$.

- In contrast with the monotone case, $m$ may not be unique.
- $m$ can be discontinuous

Here, we look at semiconcave solutions:

a. $u$ solves the equation at the points where it is $C^1$ and $m$ is continuous;

b. $\lim_{x \to x_0^-} u_x(x) \geq \lim_{x \to x_0^+} u_x(x)$ at points of discontinuity of $u_x$.

# $j \neq 0$, $g$ decreasing

With $g(m) = -m$, we have

$$\begin{cases} \frac{j^2}{2m^2} + m = \overline{H} - V(x); \\ m > 0, \ \int\limits_{\mathbb{T}} m dx = 1; \\ \int\limits_{\mathbb{T}} \frac{1}{m} dx = \frac{p}{j}. \end{cases} \tag{5}$$

The minimum of $t \mapsto j^2/2t^2 + t$ is attained at $t_{min} = j^{2/3}$. Thus, $j^2/2t^2 + t \geq 3j^{2/3}/2$ for $t > 0$ and hence

$$\overline{H} \geq \overline{H}_j^{cr} = \max_{\mathbb{T}} V + \frac{3j^{2/3}}{2}.$$

The function $t \mapsto j^2/2t^2 + t$ is decreasing on the interval $(0, t_{min})$ and increasing on the interval $(t_{min}, +\infty)$. For $\overline{H} \geq \overline{H}_j^{cr}$, let $m_{\overline{H}}^-$ and $m_{\overline{H}}^+$ solve

$$\frac{j^2}{2(m_{\overline{H}}^{\pm}(x))^2} + m_{\overline{H}}^{\pm}(x) = \overline{H} - V(x),$$

with $0 \leq m_{\overline{H}}^-(x) \leq t_{min} \leq m_{\overline{H}}^+(x)$.
Furthermore, if $(u, m, \overline{H})$ solves (4), then $m(x)$ agrees with either $m_{\overline{H}}^+(x)$ or $m_{\overline{H}}^-(x)$, almost everywhere.

Let $m_j^- := m^-_{\overline{H}_j^{cr}}$ and $m_j^+ := m^+_{\overline{H}_j^{cr}}$. Two fundamental quantities for our analysis are

$$\begin{cases} \alpha^+(j) = \int_0^1 m_j^+(x)dx, \\ \alpha^-(j) = \int_0^1 m_j^-(x)dx. \end{cases}$$

If $V$ is not constant, we have

$$\alpha^-(j) < \alpha^+(j)$$

for $j > 0$.

## Proposition

*Suppose that* $x = 0$ *is the single maximum of* $V$. *Then, for every* $j > 0$, *there exists a unique number,* $p_j$, *such that* $(4)$ *has a semiconcave solution with a current level,* $j$. *Moreover, the solution of* $(5)$, $(u_j, m_j, \overline{H}_j)$, *is unique.*

The solution has the following form:

i. If $\alpha^+(j) \leq 1$,

$$m_j(x) = m^+_{\overline{H}_j}(x), \quad u_j(x) = \int_0^x \frac{j dy}{m_j(y)} - p_j x,$$

where $p_j = \int_{\mathbb{T}} \frac{j dy}{m_j(y)}$ and $\overline{H}_j$ is such that $\int_{\mathbb{T}} m_j(x) dx = 1$.

ii. If $\alpha^-(j) \geq 1$,

$$m_j(x) = m^-_{\overline{H}_j}(x), \quad u_j(x) = \int_0^x \frac{j dy}{m_j(y)} - p_j x,$$

where $p_j = \int_{\mathbb{T}} \frac{j dy}{m_j(y)}$ and $\overline{H}_j$ is such that $\int_{\mathbb{T}} m_j(x) dx = 1$.

iii. If $\alpha^-(j) < 1 < \alpha^+(j)$, we have that $\overline{H}_j = \overline{H}_j^{cr}$, and

$$m_j(x) = m_j^-(x)\chi_{[0,d_j)} + m_j^+(x)\chi_{[d_j,1)}, \quad u_j(x) = \int_0^x \frac{j dy}{m_j(y)} - p_j x,$$

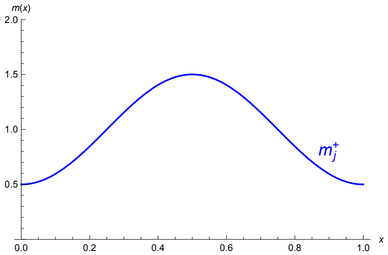
where $p_j = \int_{\mathbb{T}} \frac{j dy}{m_j(y)}$ and $d_j$ is such that

$$\int_{\mathbb{T}} m_j(x) dx = \int_0^{d_j} m_j^-(x) dx + \int_{d_j}^1 m_j^+(x) dx = 1.$$

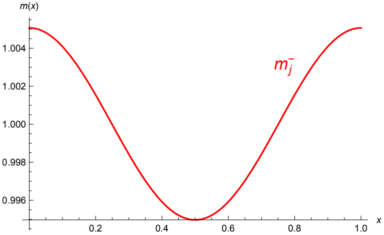
By the previous proposition, if $V$ has a single maximum point then, for every current, $j > 0$, there exists a unique $p_j$ and a unique triplet, $(u_j, m_j, \overline{H}_j)$, that solves (5) for $p = p_j$. In contrast, as we show next, if $V$ has multiple maxima and $j > 0$ is such that Case *iii* in Proposition 4 holds, there exist infinitely many solutions.

## Proposition

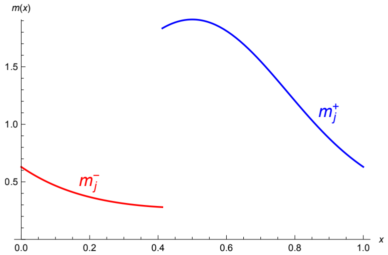
*Suppose that* $V$ *attains a maximum at* $x = 0$ *and at* $x = x_0 \in (0, 1)$. *Let* $j$ *be such that* $\alpha^-(j) < 1 < \alpha^+(j)$. *Then, there exist infinitely many numbers,* $p$, *and pairs,* $(u, m)$, *such that* $(u, m, \overline{H}_j^{cr})$ *is a semiconcave solution of* (4).



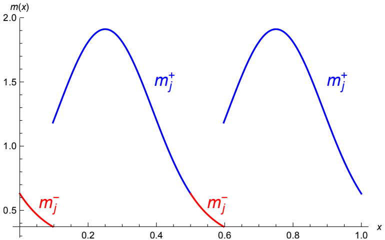

Solution $m$ for $j = 0.001$ and $V(x) = \frac{1}{2}\sin(2\pi(x + 1/4))$.

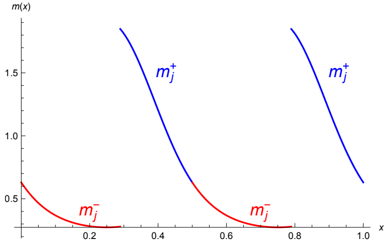



Solution $m$ for $j = 10$ and $V(x) = \frac{1}{2}\sin(2\pi(x + 1/4))$.




Solution $m_j$ for $j = 0.5$ and $V(x) = \frac{1}{2}\sin(2\pi(x + 1/4))$.




Two distinct solutions for $j = 0.5$ and $V(x) = \frac{1}{2}\sin(4\pi(x + 1/8))$.

# $j = 0$, $g$ decreasing I

Now, we examine the case $j = 0$:

$$\begin{cases} \frac{(u_x+p)^2}{2} + m = \overline{H} - V(x); \\ m \geq 0, \ \int\limits_{\mathbb{T}} mdx = 1; \\ m(u_x + p) = 0. \end{cases} \tag{6}$$

Suppose that (6) has a solution. Because $m \geq 0$, we have $\overline{H} - V(x) \geq 0$ for $x \in \mathbb{T}$. Thus, $\overline{H} \geq \max\limits_{\mathbb{T}} V$. On the other hand,

$$\int\limits_{\mathbb{T}} \left(\overline{H} - V(x)\right) dx \geq \int\limits_{\mathbb{T}} mdx = 1.$$

# $j = 0$, $g$ decreasing II

Consequently, $\overline{H} \geq 1 + \int_{\mathbb{T}} V$. Therefore,

$$\overline{H} \geq \max\left(\max_{\mathbb{T}} V, 1 + \int_{\mathbb{T}} V\right) =: \overline{H}_0.$$

## Proposition

*The MFG (6) does not have semiconcave solutions for* $\overline{H} > \overline{H}_0$.

Now, we construct solutions to (6) with $\overline{H} = \overline{H}_0$. It turns out that if $V$ has a large oscillation, then (6) has infinitely many semiconcave solutions.

i. if $1 + \int_{\mathbb{T}} V \geq \max_{\mathbb{T}} V$, then the triplet $(u_0, m_0, \overline{H}_0)$ with

$$m_0(x) = \overline{H}_0 - V(x), \;\; u_0(x) = 0,$$

solves (6) in the classical sense for $p = 0$;

ii. if $\max_{\mathbb{T}} V > 1 + \int_{\mathbb{T}} V$, define

$$m_0^{d_1,d_2}(x) = \begin{cases} \overline{H}_0 - V(x), \;\; x \in [d_1, d_2], \\ 0, \;\; x \in \mathbb{T} \setminus [d_1, d_2], \end{cases}$$

and

$$u_0^{d_1,d_2}(x) = \int_0^x (u_0^{d_1,d_2})_x(y) dy, \quad x \in \mathbb{T},$$

where

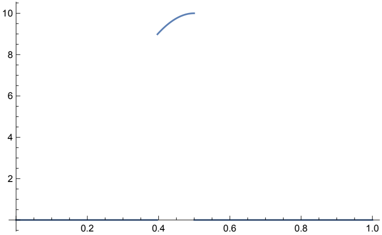
$$(u_0^{d_1,d_2})_x(x) = \begin{cases} \sqrt{2(\overline{H}_0 - V(x))} - p_0^{d_1,d_2}, & x \in [0, d_1), \\ -p_0^{d_1,d_2}, & x \in [d_1, d_2], \\ -\sqrt{2(\overline{H}_0 - V(x))} - p_0^{d_1,d_2}, & x \in (d_2, 1], \end{cases}$$

and $p_0^{d_1,d_2} = \int\limits_0^{d_1} \sqrt{2(\overline{H}_0 - V(x))}dx - \int\limits_{d_2}^{1} \sqrt{2(\overline{H}_0 - V(x))}dx.$

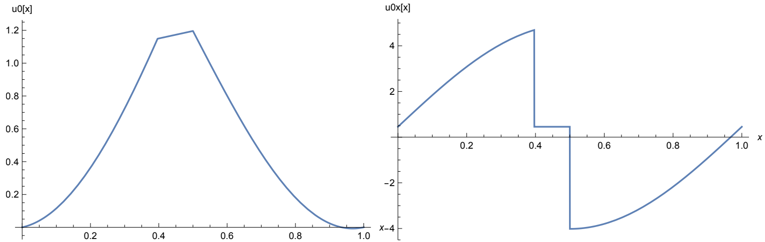
Then, for any pair, $(d_1, d_2)$, such that

$$\int\limits_{d_1}^{d_2} (\overline{H}_0 - V(x))dx = 1, \tag{7}$$

the triplet $(u_0^{d_1,d_2}, m_0^{d_1,d_2}, \overline{H}_0)$ is a semiconcave solution for (6) for $p = p_0^{d_1,d_2}$. Furthermore, there exist infinitely many pairs, $(d_1, d_2)$, such that (7) holds.




$m_0$ for $V(x) = 5\sin(2\pi(x + \frac{1}{4}))$ with $d_2 = 0.5$ and $d_1$ such that (7) holds.




$u_0$ (left) and $(u_0)_x$ (right) for $V(x) = 5\sin(2\pi(x + \frac{1}{4}))$ with $d_2 = 0.5$ and $d_1$ such that (7) holds.

Our last example is the following first-order stationary MFGs with congestion:

$$\begin{cases} \frac{|P+Du|^\gamma}{\gamma m^\alpha} + V(x) = g(m) + \overline{H} \\ -\operatorname{div}(m^{1-\alpha}|P+Du|^{\gamma-2}(P+Du)) = 0. \end{cases} \tag{8}$$

# Lack of classical solutions

In general, (8) may not have classical solutions.

- We take $P = 0$, $\gamma = 2$, and $g(m) = m$.
- By adding a constant to $V$, we can assume without loss of generality that

$$\int_{\mathbb{T}^d} V dx = 0. \tag{9}$$

Thus, we obtain

$$\begin{cases} \frac{|Du|^2}{2m^\alpha} + V(x) = m + \overline{H} \\ -\operatorname{div}(m^{1-\alpha} Du) = 0. \end{cases} \tag{10}$$

Now, we assume that $(u, m, \overline{H})$ is a classical solution to (10) with $m > 0$ and $\int_{\mathbb{T}^d} m\,dx = 1$. Then, multiplying the second equation by $u$ and integrating over $\mathbb{T}^d$, we have

$$\int_{\mathbb{T}^d} m^{1-\alpha}|Du|^2\,dx = 0.$$

Hence, because $m$ does not vanish, $u$ is constant. Accordingly, the first equation in (10) becomes

$$m = -\overline{H} + V(x).$$

Using $\int_{\mathbb{T}^d} m\,dx = 1$ and (9), we obtain

$$m = 1 + V(x).$$

However, without further assumptions, $1 + V$ may take negative values and, thus, (4) may not have a classical solution with $m > 0$.

# Critical congestion $\alpha = 1$

If $\alpha = 1$ and $\gamma = 2$, the second equation in (8) becomes $\Delta u = 0$. Hence, $u$ is constant. Therefore, the first equation in (8) is the following algebraic equation for $m$:

$$\frac{|P|^2}{2m} - g(m) = \overline{H} - V(x).$$

If $g$ is increasing and $P \neq 0$, for each $x$ and for each fixed $\overline{H}$, the preceding equation has at most one solution, $m(x) > 0$. Furthermore, the constant $\overline{H}$ is determined by the normalization condition on $m$.

# Two dimensional case I

Now, we consider the case $d = 2$. Moreover, given $Q = (q_1, q_2) \in \mathbb{R}^2$, we set $Q^\perp = (-q_2, q_1)$.
From the second equation in (8), there exists $Q \in \mathbb{R}^2$ and a scalar function, $\psi$, such that

$$m^{1-\alpha}|P + Du|^{\gamma-2}(P + Du) = Q^\perp + (D\psi)^\perp. \tag{11}$$

Consequently,

$$m^{1-\alpha}|P + Du|^{\gamma-1} = |Q + D\psi|.$$

Raising the prior expression to the power $\gamma'$, where $\gamma' = \frac{\gamma}{\gamma-1}$, and rearranging, we obtain

$$\frac{|P + Du|^\gamma}{m^\alpha} = \frac{|Q + D\psi|^{\gamma'}}{m^{\alpha-(\alpha-1)\gamma'}}.$$

Therefore,

$$\frac{|P+Du|^{\gamma}}{\gamma m^{\alpha}}+V(x)-g(m)-\overline{H}=\frac{|Q+D\psi|^{\gamma'}}{\gamma m^{\tilde{\alpha}}}+V(x)-g(m)-\overline{H},$$

with

$$\tilde{\alpha}=\alpha-(\alpha-1)\gamma'.$$

Moreover, from (11), we have

$$P^{\perp}+(Du)^{\perp}=m^{1-\tilde{\alpha}}|Q+D\psi|^{\gamma'-2}(Q+D\psi).$$

Hence $Q$ and $\psi$, we recover $P$ and $u$ from the previous equation. Moreover,

$$\operatorname{div}(m^{1-\tilde{\alpha}}|Q+D\psi|^{\gamma'-2}(Q+D\psi))=0.$$

Thus, our MFG becomes

$$\begin{cases} \frac{|Q+D\psi|^{\gamma'}}{\gamma' m^{\tilde\alpha}} + \frac{\gamma}{\gamma'}V(x) - \frac{\gamma}{\gamma'}g(m) = \frac{\gamma}{\gamma'}\overline{H} \\ \operatorname{div}(m^{1-\tilde\alpha}|Q+D\psi|^{\gamma'-2}(Q+D\psi)) = 0. \end{cases}$$

Thus, we obtain an equation of the form of (8) with exponents $\tilde\alpha$ and $\gamma'$ in the place of $\alpha$ and $\gamma$.