Linear-Quadratic Mean-Field-Type Games

common noise, jump-diffusion, regime switching

Tembine Hamidou

Learning & Game Theory Laboratory, NYUAD
E-mail: tembine@nyu.edu
Wuchen Li, UCLA, E-mail: wcli@math.ucla.edu
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Acknowledgments

- Collaborators:
  - Jian Gao, Yida Xu, Michail Smyrnakis, Massa Ndong, Julian Barreiro-Gomez,
  - Eitan Altman (INRIA), Tamer Başar (UIUC), Jean-Yves LeBoudec (EPFL), Alain Bensoussan (UT), Boualem Djehiche (KTH), Tyrone E. Duncan (Kansas)
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Agenda

1. What is a Game ?
2. What is a Mean-Field-Type Game ?
3. Learning Goal

Games in Strategic/Normal Form

Strategic-form games[1]

- set of agents
- set of actions for each agent
- objective (pay-off) functions for each agent

The **payoff function** is determined by the **realized action profile**.

$$\textbf{Payoff:}\ \ L_i(u_1, u_2, \ldots),\ \ i \in \{1, 2, \ldots\}$$

$$\textbf{Nash Equilibrium:}\ \ L_i(u_1^*, u_2^*, \ldots) = \min_{u_i \in \mathcal{U}_i(u_{-i}^*)} L_i(u_1^*, \ldots, u_{i-1}^*, u_i, u_{i+1}^*, \ldots),$$

$$i \in \{1, 2, \ldots\}$$

[1] *Cournot 1838, Bertrand 1883, Borel 1921, Edgeworth'25, Roos'25, von Neumann'28, Fisher'30, Stackelberg'34, Ville'38, Nash'50, Isaacs'65, Hamilton'67, etc*

State dependence

State-dependent strategic-form games:

- set of **states**, $s_+ \sim \mathbb{P}(.|s,u)$
- set of agents,
- set of actions for each agent (may depend on the state),
- objective (pay-off) functions for each agent.

The instant **payoff function** is determined by the realized **state-action** profile.

Instant Payoff: $l_i(t, s(t), u_1(t), u_2(t), \ldots), \ i \in \{1, 2, \ldots\}$

Shapley, L. S. (1953). Stochastic games. PNAS 39 (10): 1095-1100

What is a Mean-Field-Type Game ?

Mean-Field-Type Game

Any game in which the payoffs and/or the state dynamics coefficient functions involve not only the [state and action profiles] but also the distribution of state-action pairs.

payoff(state,action, distribution)

$l_i(t, s, a_1, \ldots, a_n, D_{(s,a)})$, kernel : $\mathcal{K}(t, s, a, D_{(s,a)}; ds')$

The number of interacting agents is not necessarily large.

Games with Drift-Jump-Diffusion-Regime Switching

- Multiple agents $\mathcal{I} := \{1, 2, \ldots\}$
- state dynamics: drift, diffusion-jump-regime switching
- agent i′s cost functional is: L_i

Strategies

- Information structure
- Open-loop strategy: measurable function of time and initial data
- State-and-[conditional expectation of the state] feedback strategy,
- Closed-loop strategy
- Randomized non-anticipative strategy (with time delay)

open-loop and closed-loop optimal strategies may be DIFFERENT !

Difference between "Control Action" and "Strategy" (Policy) ?

- $\mathcal{U}_i$: set of pure controls of i : measurable map $u_i: \ [0,T] \to U_i$
- Γ_i : set of pure strategies of i : $\gamma_i : \mathbb{R} \times \prod_{j \neq i} \mathcal{U}_i \ni (s_0, u_{-i}) \to \mathcal{U}_i$
- Let $\mathcal{S} = \{(\Omega, \mathcal{F}, \mathbb{P}) \mid \mathcal{F} \subset Borel(\Omega), \mathbb{P} \ prob.on \ (\Omega, \mathcal{F}\}$.
  Random strategy of i : $\{(\Omega_i, \mathcal{F}_i, \mathbb{P}_i), \gamma_i\}$
  - $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i) \in \mathcal{S}$
  - $\gamma_i : \ \mathbb{R} \times \Omega_i \times \prod_{j \neq i} \mathcal{U}_i \to \mathcal{U}_i$ Borel-measurable
  - For any $s_0, \omega_i, u_{-i} \mapsto \gamma_i(s_0, \omega_i, u_{-i})$ is non-anticipative with delay for some delay $\tau_i > 0$

$$u_{-i} = v_{-i} \ a.e. \ on \ [0,t] \implies \gamma_i(., u_{-i}) = \gamma_i(., v_{-i}) \ a.e., on \ \ [0, t+\tau_i].$$

- Behavior mixed strategy considers $\mathcal{P}(U_i)$ instead of U_i following the standard randomization procedure

R. Aumann, Mixed and behavior strategies in infinite extensive games. 1964 Advances in Game Theory pp. 627-650 Princeton Univ. Press, Princeton, N.J.

Solution Concept

- $\mathcal{U}_i$: set of strategies of agent i
- $\mathbb{E}L_i$ Expected value of the objective functional is considered

Nash Equilibrium

No agent has incentive to deviate unilaterally
$u_i^* \in \arg\min_{u_i \in \mathcal{U}_i} \mathbb{E}[L_i(\ldots, u_i^*, u_i, u_{i+1}^*, \ldots)], \ i \in \mathcal{I}$

open-loop and closed-loop equilibrium may be DIFFERENT !

Bayes-Nash Equilibrium

Type distribution: $\xi \in \mathcal{P}(\prod_j Z_j)$
Include the available information: $Z_i \mapsto U_i$
$u_i^* \in \arg\min_{u_i \in \mathcal{U}_i} \mathbb{E}[\int L_i(\ldots, u_i^*, u_i, u_{i+1}^*, \ldots; z_i, z_{-i}) \xi(dz_{-i}|z_i)], \ i \in \mathcal{I}$

Learning Goal

Solve the following problem without using sophisticated PIDEs or SMPs.

$$
\begin{aligned}
&\textbf{Problem}: \ \inf_{u_i} \ \mathbb{E}L_i \\
&L_i = \tfrac{1}{2}q_{iT}var(s(T)) + \tfrac{1}{2}\bar{q}_{iT}\bar{s}^2(T) + \varepsilon_{i3}(T)\bar{s}(T) \\
&+\tfrac{1}{2}\int_0^T \left(q_i var(s) + \bar{q}_i\bar{s}^2 + r_i var(u_i) + \bar{r}_i\bar{u}_i^2\right)dt \\
&+\int_0^T \left(\epsilon_{i1}\ cov(s,u_i) + \bar{\epsilon}_{i1}\bar{s}\bar{u}_i + \varepsilon_{i2}\bar{u}_i + \varepsilon_{i3}\bar{s}\right)dt, \\
&ds = [b_0 + b_1(s-\bar{s}) + \bar{b}_1\bar{s} + \textstyle\sum_{j=1}^n b_{j2}(u_j - \bar{u}_j) + \sum_{j=1}^n \bar{b}_{j2}\bar{u}_j]dt \\
&+[\sigma_0 + \sigma_1(s-\bar{s}) + \bar{\sigma}_1\bar{s} + \textstyle\sum_{j=1}^n \sigma_{j2}(u_j - \bar{u}_j) + \sum_{j=1}^n \bar{\sigma}_{j2}\bar{u}_j]dB \\
&+\int_\Theta[\mu_0 + \mu_1(s-\bar{s}) + \bar{\mu}_1\bar{s} + \textstyle\sum_{j=1}^n \mu_{j2}(u_j - \bar{u}_j) + \sum_{j=1}^n \bar{\mu}_{j2}\bar{u}_j]\tilde{N}(dt,d\theta), \\
&\ s(0) \perp\!\!\!\perp \{B,N\}, \\
&\tilde{s}(t) \sim \tilde{q}_{\tilde{s}\tilde{s}'}.
\end{aligned} \tag{1}
$$

$$\bar{s} := \mathbb{E}[s\,|\mathcal{F}_t^{\tilde{s}}], \ \ \bar{u}_j := \mathbb{E}[u_j\,|\mathcal{F}_t^{\tilde{s}}],$$

$$q_k(t,\tilde{s}), b_k(t,\tilde{s}); \sigma_k(t,\tilde{s}); \bar{q}_k(t,\tilde{s}), \bar{b}_k(t,\tilde{s}); \bar{\sigma}_k(t,\tilde{s}) \in \mathbb{R},$$

$$\mu_k(t,\tilde{s},\theta) \in \mathbb{R},\ \tilde{s}(t) \in \tilde{\mathcal{S}}(\text{finite}),$$

Model specification

$$\begin{array}{cccc} \textit{Drift} & \textit{Diffusion} & \textit{Jump} & \textit{Regime Switching} \\ b(.) & B & N & \tilde{s} \end{array} \tag{2}$$

Conditional McKean-Vlasov

Coefficients: functions of

$$(t, \tilde{s}(t), s(t), \mathbb{E}[s \,|\mathcal{F}_t^{\tilde{s}}], \{u_j(.), \mathbb{E}[u_j(.) \,|\mathcal{F}_t^{\tilde{s}}]\}_{\{j\in\{1,2,\ldots\}\}}),$$

McKean-Vlasov $\neq$ Liouville dynamical systems.

Discrete time game problem

$$
\begin{array}{l}
\textbf{Problem} : \ \inf_{u_i} \ \mathbb{E}L_i \\
L_i = q_{iK}s_K^2 + \bar{q}_{iK}(E[s_K])^2 + \epsilon_{i3K}E[s_K] \\
+ \sum_{k=0}^{K-1} q_{ik}s_k^2 + \bar{q}_{ik}(E[s_k])^2 + r_{ik}u_{ik}^2 + \bar{r}_{ik}(E[u_{ik}])^2 \\
+2\sum_{k=0}^{K-1} \epsilon_{i1k}s_k u_{ik} + \bar{\epsilon}_{i1k}(E[s_k])E[u_{ik}] + \epsilon_{i2k}E[u_{ik}] + \epsilon_{i3k}E[s_k], \\
\text{subject to} \\
s_{k+1} = \left(b_{0k} + b_{1k}s_k + \bar{b}_{1k}E[s_k] + \sum_{j=1}^{n} b_{j2k}u_{jk} + \sum_{j=1}^{n} \bar{b}_{j2k}E[u_{jk}]\right) \\
+\left(\sigma_{0k} + \sigma_{1k}s_k + \bar{\sigma}_{1k}E[s_k] + \sum_{j=1}^{n} \sigma_{j2k}u_{jk} + \sum_{j=1}^{n} \bar{\sigma}_{j2k}E[u_{jk}]\right) w_{k+1}, \\
s(0) = s_0,
\end{array}
\tag{3}
$$

w_k is a (possibly) correlated noise process $E[w_k^2] < +\infty$. It does not need to be Brownian, Gaussian ...

How do we solve Stochastic Games?

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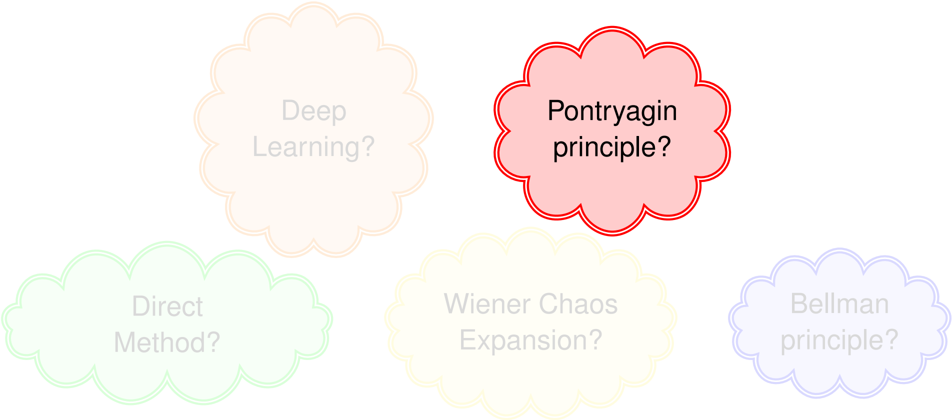

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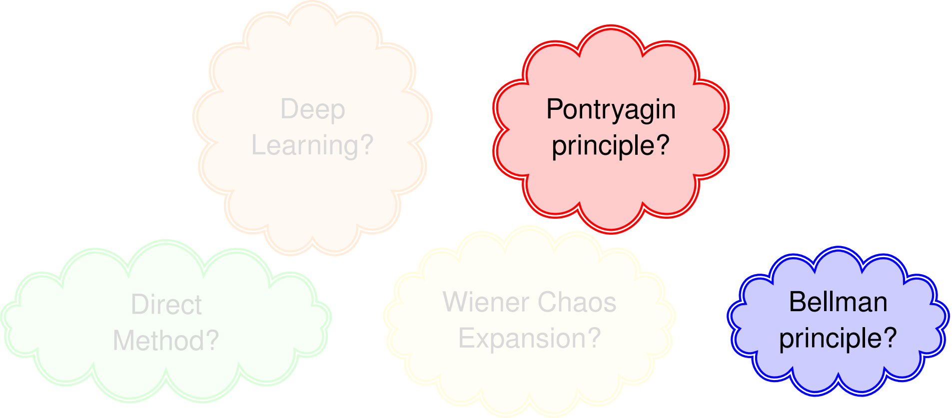

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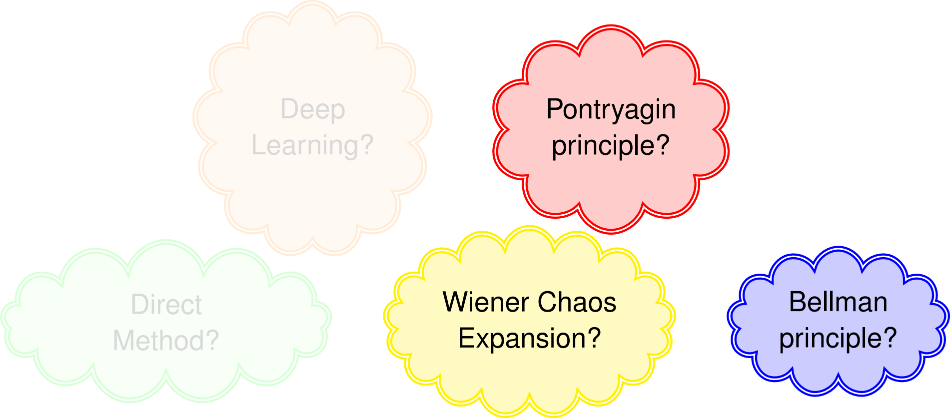

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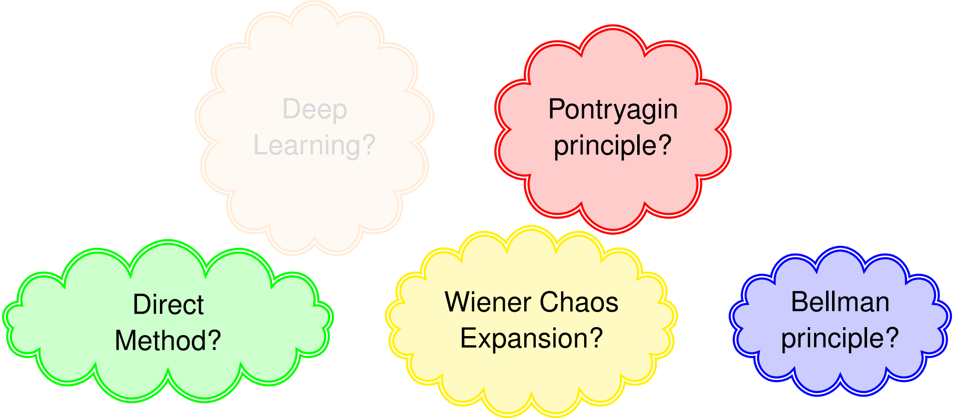

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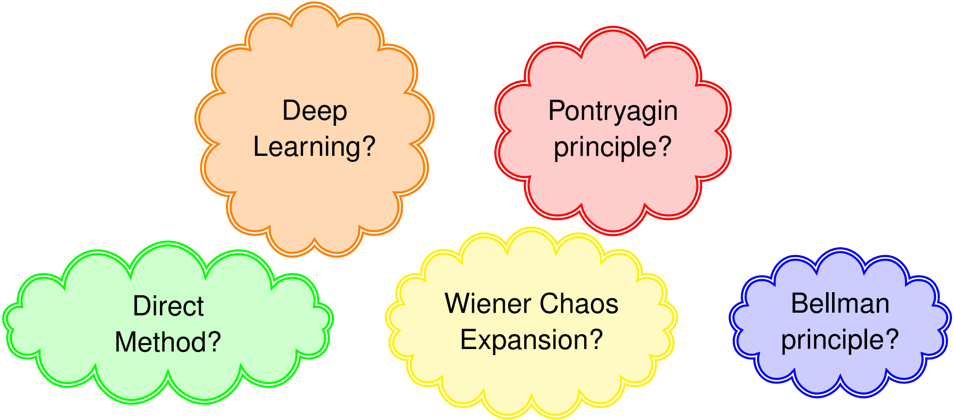
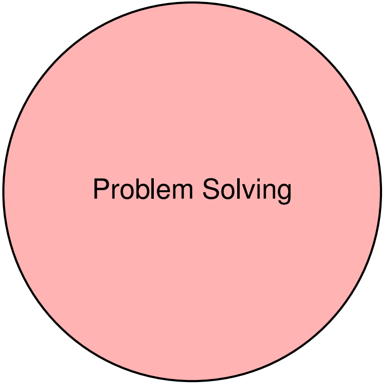


Input from Engineers, Students and Beginners

What tools do you use?



Other tools

- Direct method
- Polynomial chaos
- Learning
- Efficient approximation

Input from Engineers, Students and Beginners

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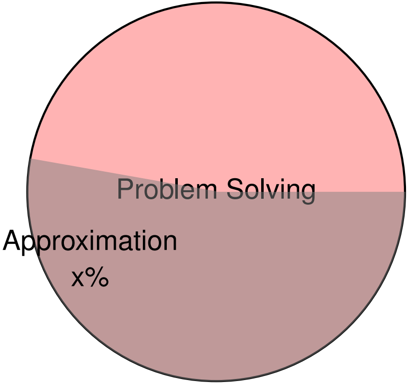

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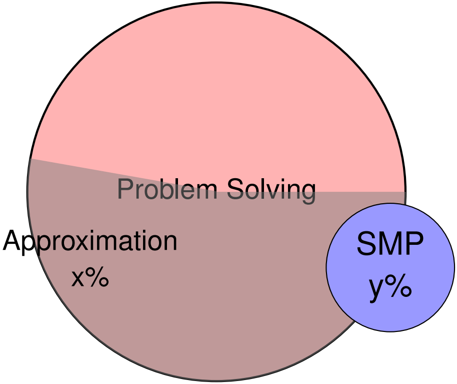

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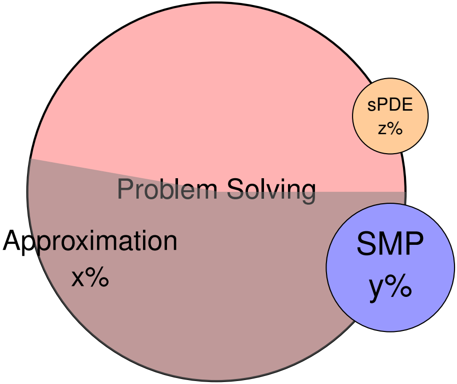

Other tools

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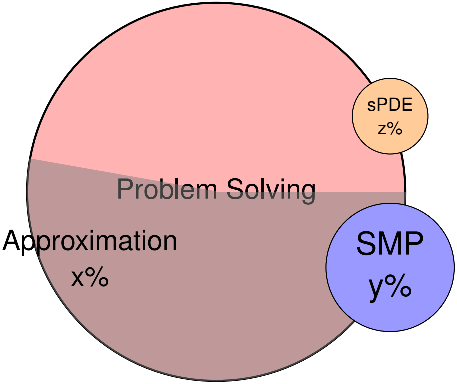


Other tools

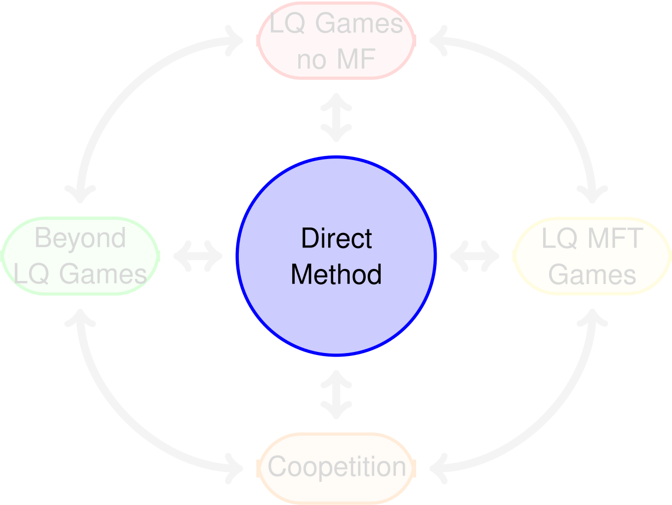
- Direct method
- Polynomial chaos
- Learning
- Efficient approximation

How to solve the discrete or continuous time game problem?

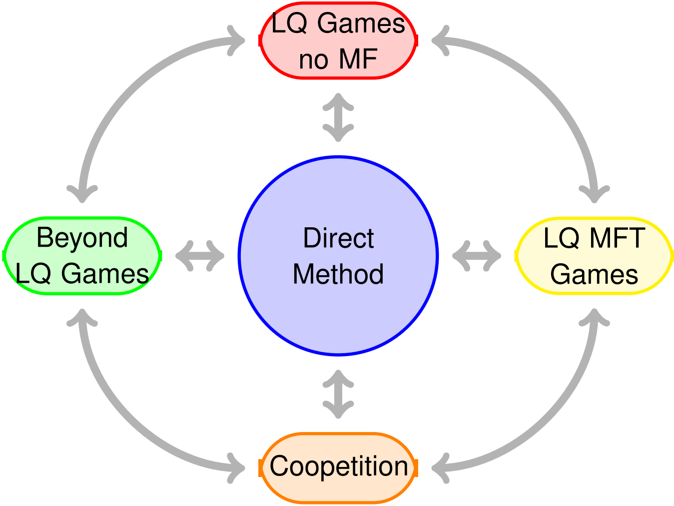
No PDE no SMP

- No Bellman-Shapley operator,
- No Lagrange multipliers
- No PDE - No SPIDEs - no Bellman system
- No MP - no SMP - no Pontryagin system

Agenda


LQ Games
no MF
Beyond
LQ Games
Direct
Method
LQ MFT
Games
Coopetition

Agenda


LQ Games no MF
Beyond LQ Games
Direct Method
LQ MFT Games
Coopetition

Agenda of Lecture - Recitation

Lecture 1: linear-quadratic control and games

Recitation 1: Roos 1925

Lecture 2: linear-quadratic control and games of mean-field type

Recitation 2: Price formation in the smart grid

Lecture 3: "Social" Cost, Robustness, Bargaining, Empathy, Coopetition

Recitation 3: Blockchain-based technologies

Lecture 4: How far can we go without PDEs without SMPs ?

Recitation 4: Discrete-time and beyond LQ

Agenda: next step

- introduction to linear-quadratic control and games

THANK YOU

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Lecture 1 : Introduction to
Linear-Quadratic Control and Games

Part I: Direct Method

Agenda

1. Single Agent Decision-Making: Deterministic
2. Single Agent Decision-Making: Stochastic
3. Multiple Agent Decision-Making: Deterministic
4. Multi-Agent Decision-Making: Jump-Diffusion

One Agent: LQ Deterministic case

$$L_0 := q_0(T)s^2(T) + \int_0^T u_0^2(t)dt$$

Decision-maker 0 : $\inf_{u_0} L_0$ subject to (1)
$\dot{s} = u_0, \quad s(0) = s_0 \in \mathbb{R},$
$u_0(.) \in U_0 = \mathbb{R}$ unconstrained.

Direct method

$$\alpha(T)s^2(T) = \alpha(0)s^2(0) + \int_0^T \dot{\alpha}(t)s^2(t) + 2\alpha(t)s(t)u_0(t)dt,$$

$$\begin{aligned}
&L_0 - \alpha(0)s^2(0) \\
&= [q_0(T) - \alpha(T)]s^2(T) + \int_0^T [u_0^2(t) + 2\alpha(t)s(t)u_0(t) + \dot{\alpha}(t)s^2(t)]dt \\
&= \{q_0(T) - \alpha(T)\}s^2(T) + \int_0^T \{u_0(t) + \alpha(t)s(t)\}^2 + \{\dot{\alpha}(t) - \alpha^2(t)\}s^2(t)dt
\end{aligned}$$

One Agent: explicit solution

One Agent

$L_0 := q_0(T)s^2(T) + \int_0^T u_0^2(t)dt$
Decision-maker 0 : $\inf_{u_0} L_0$ subject to
$\dot{s} = u_0, \quad s(0) \in \mathbb{R}.$

One Agent: explicit solution by means of Direct method

$\inf_{u_0 \in \mathcal{U}_0} \mathbb{E}[L_0] = \alpha(0)s^2(0),$
state-feedback: $u_0^* = -\alpha s,$
Riccati: $\dot{\alpha} - \alpha^2 = 0, \quad \alpha(T) = q_0(T), \quad \Longrightarrow \quad \alpha(t) = \frac{1}{\frac{1}{q_0} + T - t},$ (2)
$s(t) = s(0)e^{-\int_0^t \alpha(t')dt'} = s(0)(1 - \frac{t}{\frac{1}{q_0} + T}).$

Constrained case: non-convex action constraint

One Agent with non-convex action set

$L_{00} := \int_0^T s^2(t)dt$
Decision-maker 0 : $\inf_{u_0} L_{00}$ subject to
$\dot{s} = u_0, \quad s(0) = 0,$
$u_0(.) \in \{-1, +1\} := U_{00}.$

Randomized strategy (see also Borel 1921 on colonel Blotto games)

$\hat{L}_{00} := \int_0^T \hat{s}^2(t)dt$
Decision-maker 0 : $\inf_{\hat{u}} \hat{L}_{00}$ subject to
$\dot{\hat{s}} = \hat{u}_+ - \hat{u}_-,$
$u_0 \sim \hat{u}_+\delta_{+1} + \hat{u}_-\delta_{-1}, \quad \hat{u}_\pm \in [0, 1], \ \hat{u}_+ + \hat{u}_- = 1,$
$\hat{s}(0) = 0$

Non-convex action constraint

One Agent with non-convex action set

$\hat{L}_{00} := \int_0^T \hat{s}^2(t)dt$

Decision-maker 0 : $\inf_{\hat{u}_0} \hat{L}_{00}$ subject to

$\dot{\hat{s}} = \hat{u}_+ - \hat{u}_-,$

$u_0 \sim \hat{u}_-\delta_{-1} + \hat{u}_+\delta_{+1}, \ \ \hat{u}_\pm \in [0,1], \ \hat{u}_+ + \hat{u}_- = 1,$

$\hat{s}(0) = 0$

A trivial solution

$$\hat{u}^*_+ = \hat{u}^*_- = \frac{1}{2},$$
$$u^*_0 \sim \frac{1}{2}[\delta_{-1} + \delta_{+1}],$$
$$\hat{L}^*_{00} = 0$$

Non-convexity of U_{00} may create non-existence of optimal pure strategy

One Agent: LQ Stochastic (mean-field free) case

$$
\begin{array}{l}
L_0 := q_0 s^2(T) + \int_0^T u_0^2(t)dt \\
\text{Decision-maker } 0: \quad \inf_{u_0} \mathbb{E}[L_0] \text{ subject to} \\
ds = u_0 dt + \sigma dB, \quad s(0) \in \mathbb{R}.
\end{array} \tag{3}
$$

Direct method

$$\alpha(T)s^2(T) = \alpha(0)s^2(0) + \int_0^T \dot{\alpha}s^2 + 2\alpha s u_0 dt + \int_0^T \sigma^2 \alpha dt + \int_0^T 2\sigma\alpha s dB,$$

$$
\begin{array}{l}
L_0 - \alpha(0)s^2(0) = \{q_0(T) - \alpha(T)\}s^2(T) \\
+ \int_0^T \{u_0 + \alpha s\}^2 + \{\dot{\alpha} - \alpha^2\}s^2 dt + \int_0^T \sigma^2 \alpha dt + \int_0^T 2\sigma\alpha s dB
\end{array}
$$

One Agent: LQ Stochastic (mean-field free) case

Stochastic optimal control problem

$$L_0 := q_0 s^2(T) + \int_0^T u_0^2(t)dt$$
Decision-maker 0 : $\inf_{u_0} \mathbb{E}[L_0]$ subject to
$$ds = u_0 dt + \sigma dB, \quad s(0) \in \mathbb{R}. \qquad (4)$$

Explicit solution by means of Direct method

$$\inf_{u_0 \in \mathcal{U}_0} \mathbb{E}[L_0] = \alpha(0)[s(0)]^2 + \int_0^T \sigma^2 \alpha dt,$$
state-feedback: $u_0^* = -\alpha s,$
Riccati: $\dot{\alpha} - \alpha^2 = 0, \quad \alpha(T) = q_0(T),$
$$s(t) = s(0)e^{-\int_0^t \alpha(t')dt'} + e^{-\int_0^t \alpha(t')dt'} * \int_0^t \sigma dB(t') e^{+\int_0^{t'} \alpha(t'')dt''} \qquad (5)$$

Direct Method: mean-field free case

Risk-Neutral

Duncan T. E. and Pasik-Duncan B., A direct method for solving stochastic control problems, Commun. Info. Systems, (special issue for H. F. Chen), 12 (2012), 1-14.

T. E. Duncan, Linear-quadratic stochastic differential games with general noise processes, Models and Methods in Economics and Management Science: Essays in Honor of Charles S. Tapiero, (eds. F. El Ouardighi and K. Kogan) , Operations Research and Management Series, Springer Intern. Publishing, Switzerland, Vol.198, 2014, 17-26.

Risk-Sensitive: Linear Exponential-Quadratic

Tyrone E. Duncan: Linear Exponential Quadratic Stochastic Differential Games, IEEE Transactions on Automatic Control, vol. 61, no 9, Sept. 2016

Duncan T. E. and Pasik-Duncan B., Linear-quadratic fractional Gaussian control, SIAM J. Control Optim. 51 (2013), 4604-4619.

Duncan T. E. and Pasik-Duncan B. , Linear-exponential-quadratic control for stochastic equations in a Hilbert space, Dynamic Systems and Applications 21 (2012), 407-416. 111.

Duncan T. E. , Linear-exponential-quadratic Gaussian control, IEEE Trans. Autom. Control, 58 (2013), 2910-2911

Direct Method: mean-field free case

Risk-Neutral Non-Quadratic

- T. E. Duncan, Some solvable stochastic control problems in noncompact rank one symmetric spaces, Stochastics Stochastic. Repts 35 (1991), 129-142.
- T. E. Duncan and B. Pasik-Duncan, Explicit strategies for some linear and nonlinear stochastic differential games, J. Math. Engrg. Sci. Aerospace, 2016.
- T. E. Duncan and B. Pasik-Duncan, A solvable stochastic differential game in the two-sphere, 52nd Proc. IEEE Conf. Decision and Control, 7833-7837, Firenze, 2013.

Witsenhausen: two decision-makers and two stages

$$\begin{cases} \begin{array}{c|c|c|c} \textit{State} & \textit{Observation} & \textit{Decision} & \textit{Time stepcost} \\ \hline x_0 = X, & y_0 = x_0, & u_0 & \text{first stage cost } k_0 u_0^2 \\ x_1 = x_0 + u_0, & y_1 = x_1 + w_1, & u_1 & \text{second stage } 0 \\ x_2 = x_1 - u_1, & & & \text{terminal cost : } x_2^2 \end{array} \end{cases} \quad (6)$$

Information structure based optimal strategies

Find $u_0(y_0), u_1(y_1)$ such that $E[L]$ is minimized.

H.S. Witsenhausen, A counterexample in stochastic optimal control,SIAM J. Contr., 6(1):131-147, 1968.

R. Bansal and T. Başar, Stochastic team problems with nonclassical information revisited: When is an affine law optimal? IEEE Trans. Automat. Contr., AC-32(6):554-559, June 1987.

LQ Game Problem: deterministic case

$$L_i := \frac{1}{2} q_i(T) s^2(T) + \frac{1}{2} \int_0^T [q_i(t) s^2(t) + r_i(t) u_i^2(t)] dt$$

$$\begin{array}{l} \text{Decision-maker } i: \quad \inf_{u_i} L_i \text{ subject to} \\ \dot{s} = as + \sum_{i=1}^n b_i u_i, \\ s(0) \in \mathbb{R}. \end{array} \tag{7}$$

Semi-explicit solution

$$\begin{array}{l} \inf_{u_i \in \mathcal{U}_i} L_i = \frac{1}{2} \alpha_i(0) [s(0)]^2 \\ \text{state-feedback: } u_i^* = -\frac{b_i}{r_i} \alpha_i s, \\ \text{coupled Riccati: } \dot{\alpha}_i + 2a\alpha_i - \alpha_i^2 \frac{b_i^2}{r_i} - 2\alpha_i \sum_{j \neq i} \frac{b_j^2}{r_j} \alpha_j + q_i = 0, \\ \qquad \alpha_i(T) = q_i(T). \end{array} \tag{8}$$

T. Başar and G. J. Olsder. Dynamic Noncooperative Game Theory. Academic Press, London/New York, 1982;

T. Başar , On the uniqueness of the Nash solution in Linear-Quadratic differential Games, International Journal of Game Theory June 1976, Volume 5, Issue 2-3, pp 65-90

Poisson

- t_k i.i.d exponentially distributed with intensity ν
- $N(t) = \max\{n : \sum_{k=1}^{n} t_k \leq t\}$

Poisson

- $N(0) = 0$
- $N(t + \Delta) - N(t) \sim Poisson(\nu\Delta)$
- $N(t)$ has independent increments

Poisson random measure

- (Θ, ν) Radon measure
- If $S_1, \ldots, S_k$ are mutually disjoint set , the random variables $N(S_1), \ldots, N(S_k)$ are independent (defined over $(\Omega, \mathcal{F}, \mathbb{P})$)
- For each S, $N(S)$ is Poisson distributed with intensity $\nu(S)$
- N is $\mathbb{P}$-a.s. measure

Compensated

- $\tilde{N}(dt, d\theta) = N(dt, d\theta) - dt\nu(d\theta)$
- $\int_{[0,t]\times\Theta} \theta N(dt, d\theta) - \int_{[0,t]\times\Theta} dt\theta\nu(d\theta) = \int_{[0,t]\times\Theta} \theta\tilde{N}(dt, d\theta)$ is a $\mathbb{P}$-martingale w.r.t the filtration $\sigma(N(S),\ S \in \mathcal{B}[0,t] \times \mathcal{B}(\mathbb{R}))$

LQ Game Problem: mean-field free case

$$L_i := \frac{1}{2}q_i(T)s^2(T) + \frac{1}{2}\int_0^T [q_i(t)s^2(t) + r_i(t)u_i^2(t)]dt$$

Decision-maker i : $\inf_{u_i} \mathbb{E}[L_i]$ subject to (9)
$ds = \left\{as + \sum_{i=1}^n b_i u_i\right\} dt + \sigma dB + \int_\Theta \mu(t,\theta)\tilde{N}(dt, d\theta),$
$s(0) \in L^2(\Omega, \mathbb{R}).$

Semi-explicit solution

$\inf_{u_i \in \mathcal{U}_i} \mathbb{E}[L_i] = \mathbb{E}\left\{\frac{1}{2}\alpha_i(0)[s(0)]^2 + \int_0^T [\sigma^2 + \int_\Theta \mu^2(t,\theta)\nu(d\theta)]\frac{1}{2}\alpha_i dt\right\}$
state-feedback: $u_i^* = -\frac{b_i}{r_i}\alpha_i s,$ (10)
coupled Riccati: $\dot{\alpha}_i + 2a\alpha_i - \alpha_i^2\frac{b_i^2}{r_i} - 2\alpha_i \sum_{j\neq i}\frac{b_j^2}{r_j}\alpha_j + q_i = 0,$
$\alpha_i(T) = q_i(T).$

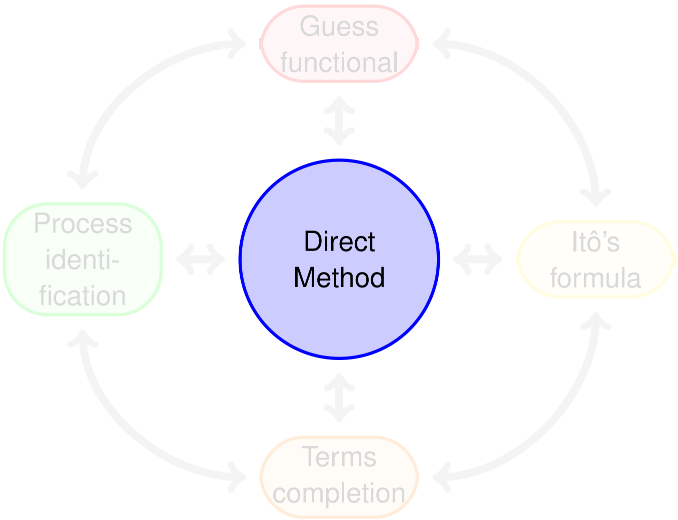
Key ingredient of the Proof

Nothing but Itô's formula applied to: $\frac{1}{2}\alpha_i(t)s^2(t)$

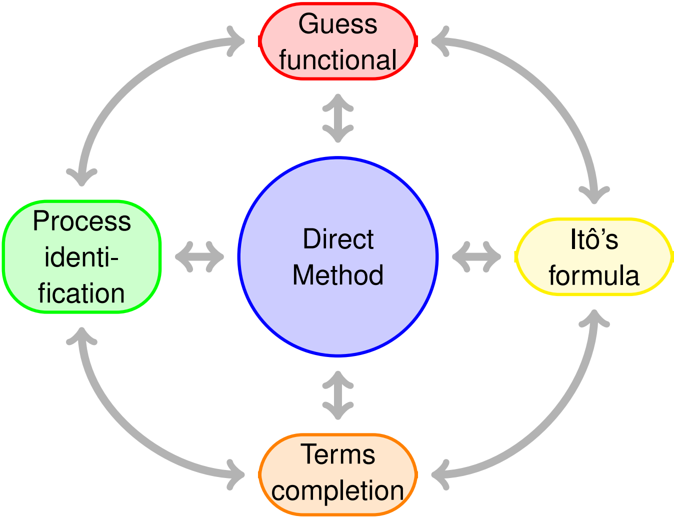
$$
\begin{aligned}
& f_i(T, s(T)) \\
& = f_i(0, s(0)) + \int_0^T [f_{i,t} + f_{i,s}D + f_{i,ss}\frac{\sigma^2}{2}]dt + \int_0^T \sigma f_{i,s} dB \\
& + \int_0^T \int_\Theta [f_i(t, s + \mu(t, \theta)) - f_i(t, s) - f_{i,s}\mu(t, \theta)]\nu(d\theta)dt \\
& + \int_0^T \int_\Theta [f_i(t_-, s + \mu(t_-, \theta)) - f_i(t_-, s)]\tilde{N}(dt, d\theta).
\end{aligned}
\tag{11}
$$

D is the drift term $D := as + \sum_{i=1}^n b_i u_i$.
see Cox & Miller'75, Kurtz, Oksendal, Protter.

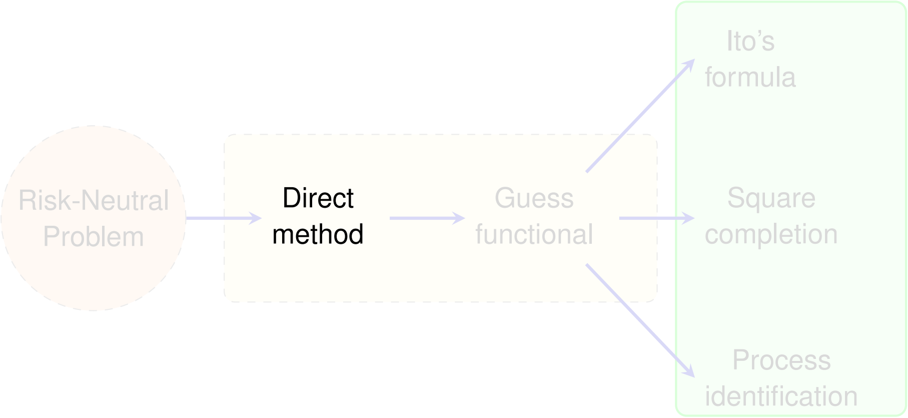
Direct Method: How does it work?



Direct Method: How does it work?

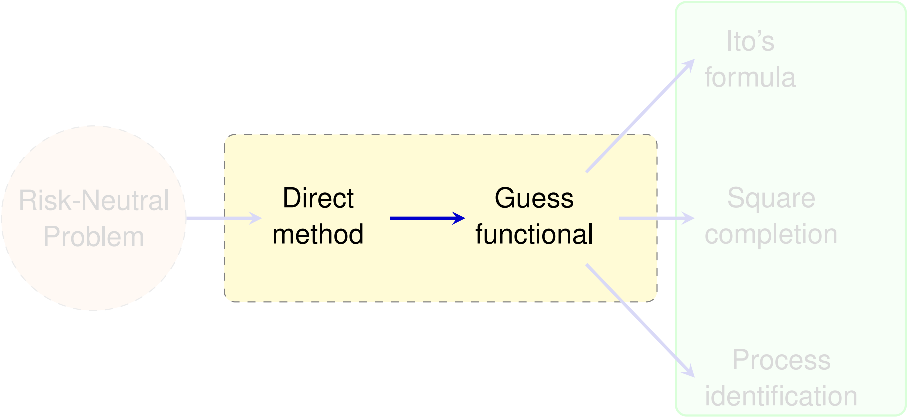


Procedure



Risk-Neutral Problem
Direct method
Guess functional
Ito's formula
Square completion
Process identification

Procedure


Risk-Neutral Problem
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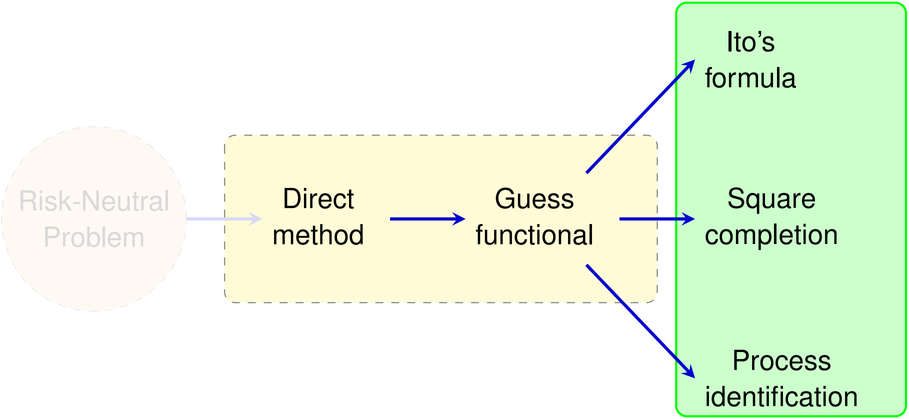
Risk-Neutral
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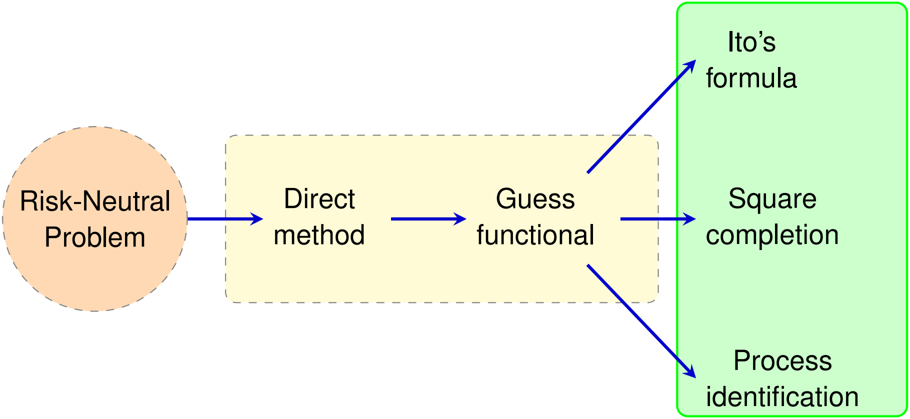
Risk-Neutral Problem
Direct method
Guess functional
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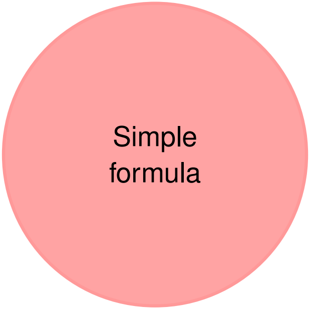
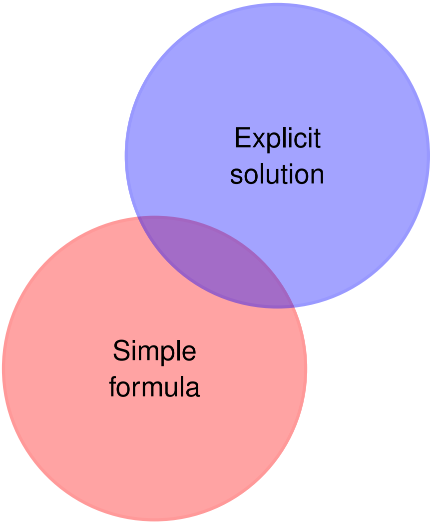
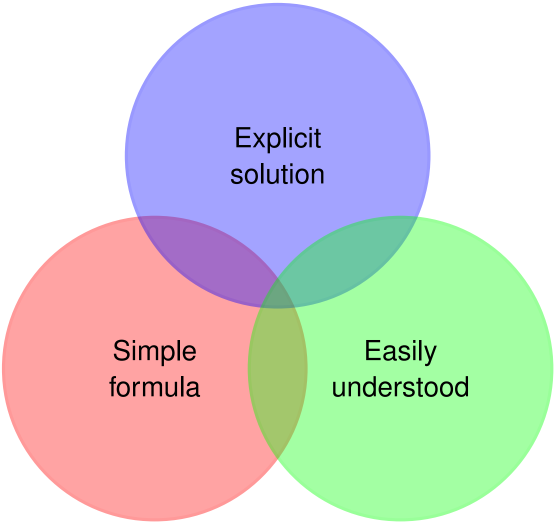
Direct method is key



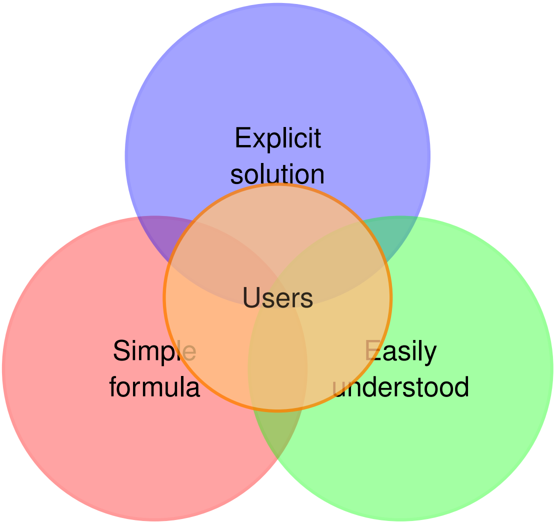
Direct method is key



Direct method is key



Direct method is key



Agenda: next step

- Recitation 1: Roos'1925-27 model of interaction.

THANK YOU

Linear-Quadratic Mean-Field-Type Games

common noise, jump-diffusion, regime switching

Tembine Hamidou

Learning & Game Theory Laboratory, NYUAD
Wuchen Li, UCLA
IPAM Graduate Summer School

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Recitation 1: Roos'1925-27 model of interaction.

Agenda

1. Roos'1925
2. Price Formation: Bayesian case
3. Network Security as a Public Good

Roos'1925-27 model of interaction.

Augustin Cournot, Recherches sur les principes mathematiques de la theorie des richesses, Paris, 1838; translated by N. T. Bacon, Researches into the Mathematical Principles of the Theory of Wealth, London, 1897; see chapter VII.

Griffith C. Evans, A Simple Theory of Competition, The American Mathematical Monthly, Vol. 29, No. 10 (Nov. - Dec., 1922), pp. 371- 380

Roos C.F. (1925): A Mathematical Theory of Competition, American Journal of Mathematics, 47, 163-175

Roos C.F. (1927): A Dynamic Theory of Economics, Journal of Political Economy 35, 632-656

Simaan M., and T. Takayama (1978): Game Theory Applied to Dynamic Duopoly with Production Constraints,Automatica, 14, 161-166

Chaim Fershtman and Morton I. Kamien: Dynamic Duopolistic Competition with Sticky Prices, Econometrica, Vol. 55, No. 5 (Sep., 1987), pp. 1151-1164

Price Formation

Dynamic price model taken from Roos'1925:

$$
\begin{aligned}
&\sup_{u_i} \int_{t_0}^{t_1} e^{-\lambda_i t} \left[s u_i - C_i(u_i)\right] dt, \\
&\dot{s} = \eta[a - \textstyle\sum_{i=1}^{n} u_i - s], \\
&s(t_0) = s_0, \\
&a > 0, \eta > 0, \lambda_i > 0, t_0 < t_1,
\end{aligned} \tag{1}
$$

Tentative Interpretation

- $C_i(u_i) := c_i u_i + \frac{r_i + \bar{r}_i}{2} u_i^2$, cost of agent i
- s : price
- u_i : quantity produced by i, $s u_i$: gain of i

Questions

- Find state-feedback [Cournot]-Nash equilibria using Direct Method
- Find fully cooperative solution using Direct Method

Price Formation Game with incomplete information

Bayesian game with incomplete information (modified from Roos'1925):

$$\begin{array}{l} \sup_{u_i} \int_{t_0}^{t_1} e^{-\lambda_i t}\,[s u_i - C_i(u_i)]\,dt, \\ \dot{s} = \eta[a - \sum_{i=1}^{n} u_i - s], \\ s(t_0) = s_0, \end{array} \quad (2)$$

Incomplete information

$C_i(u_i) := c_i u_i + \frac{r_i + \bar{r}_i}{2} u_i^2,$

- Agent i knows her type $(c_i, \frac{r_i+\bar{r}_i}{2})$
- i does not know the type of the others $(\mathbf{c_j}, \frac{\mathbf{r_j}+\bar{\mathbf{r}}_\mathbf{j}}{\mathbf{2}})_{\mathbf{j}\neq\mathbf{i}}$

Questions

- Find state-feedback [Cournot-Bayes]-Nash equilibria using Direct Method
- Find fully cooperative solution using Direct Method
- Conclude

Network Security as a Public Good

Level of Contribution for Security

$$\begin{aligned} &\sup_{u_i} \int_{t_0}^{t_1} e^{-\lambda_i t} \left[s(1-\epsilon s) - C_i(u_i) \right] dt, \\ &\dot{s} = -as + \textstyle\sum_{i=1}^{n} u_i, \\ &s(t_0) = s_0, \\ &\lambda_i > 0, \ t_0 < t_1 \end{aligned} \tag{3}$$

- $C_i(u_i)$: effort cost
- $s(1-\epsilon s)$: public good when $0 < s < \frac{1}{\epsilon}$.
- λ_i : discount factor

Questions

- Find state-feedback Nash equilibria using Direct Method
- Find fully cooperative solution using Direct Method
- Conclude

Agenda: next step

- Lecture 2: LQ games (with mean-field dependence)

THANK YOU