From Optimal transportation to Variational Mean Field Games

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Brownian motion with drift: Eulerian version

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Generalized OT, with mean-field and noise

$$I(\mu_0, \mu_T) = \inf\{\int_0^T \mathcal{F}(\mu_t, q_t)dt, \quad \partial_t \mu_t + \nabla \cdot q_t = \nu \Delta \mu_t, \quad q_t = v_t \mu_t \},$$

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where μ_0, μ_T are given and $\mathcal{F}$ is defined by duality as a convex lsc functional

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$$\mu\geq 0,\quad q<<\mu,\quad q=v\mu,\quad v\in L^2(D,d\mu;\ \mathbf{R}^d),\quad \mathcal{F}(\mu,q)=\int_D\frac{v^2}{2}d\mu.$$

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$$\mu \geq 0, \quad q << \mu, \quad q = v\mu, \quad v \in L^2(D, d\mu;\ \mathbf{R}^d), \quad \mathcal{F}(\mu, q) = \int_D \frac{v^2}{2} d\mu.$$

Then, the optimal value $I(\mu_0, \mu_T)$ corresponds to the OT problem with quadratic cost.

Saddle-point and dual formulations

$$I(\mu_0, \mu_T) = \inf_{(\mu,q)} \sup_{(A,,B,\phi)} \int_0^T (< \mu_t, A_t - \partial_t \phi_t - \nu \Delta \phi_t > + < q_t, B_t - \nabla \phi_t >) dt$$

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$$= \sup_\phi \int_0^T \int_D -G(\partial_t \phi + \nu \Delta \phi, \nabla \phi)(t, x) dx dt \; + < \mu_T, \phi_T > - < \mu_0, \phi_0 > .$$

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We may now change the time boundary conditions, prescribing ϕ_T instead of μ_T, with optimal value $J(\mu_0, \phi_T)$.

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We may now change the time boundary conditions, prescribing ϕ_T instead of μ_T, with optimal value $J(\mu_0,\phi_T)$. This way, we have just shifted from OT to a variational MFG!

Formal optimality equations: the MFG system

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Assume we are given g, convex super-linear and non decreasing, with G of form:

$$G(A, B) = g(A + B^2/2), \quad g(a) = \sup_{w \in \mathbf{R}} \; wa - f(w), \quad f(w) = \sup_{a \in \mathbf{R}} \; wa - g(a)$$

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(for instance $g(a) = \exp(a)$, $f(w) = w \log w - w$).

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(for instance $g(a)=\exp(a)$, $f(w)=w\log w-w$). Then we obtain:

$$\partial_t\mu+\nabla\cdot(\mu\nabla\phi)=\nu\Delta\mu,\quad \partial_t\phi+\frac{1}{2}|\nabla\phi|^2+\nu\Delta\phi=f'(\mu).$$

Formal proof:

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$$\mu = g'(A+B^2/2) \ \longrightarrow \ A + B^2/2 = f'(\mu) \ \longrightarrow \ \partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + \nu\Delta\phi = f'(\mu).$$

A robust and simple numerical scheme

(Method introduced for OT by Benamou-B. in 2000.)

We use the augmented Lagrangian trick (which does not change the problem):

$$\inf_{(\mu,q)} \sup_{(A,,B,\phi)} \int_0^T (< \mu_t, A_t - \partial_t\phi_t - \nu\Delta\phi_t > + < q_t, B_t - \nabla\phi_t >)dt$$

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(Method introduced for OT by Benamou-B. in 2000.)

We use the augmented Lagrangian trick (which does not change the problem):

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Iteratively, we optimize, first in (A, B) (this is a purely local optimization problem, for each grid point (t, x), entirely parallelizable).

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