Strategic resource allocation

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North American Aerospace Defense Command (NORAD)

NORAD Area of Operations

AADS=Alaska Air Defense Sector
WADS=Western Air Defense Sector

CADS=Canadian Air Defence Sector
EADS=Eastern Air Defense Sector

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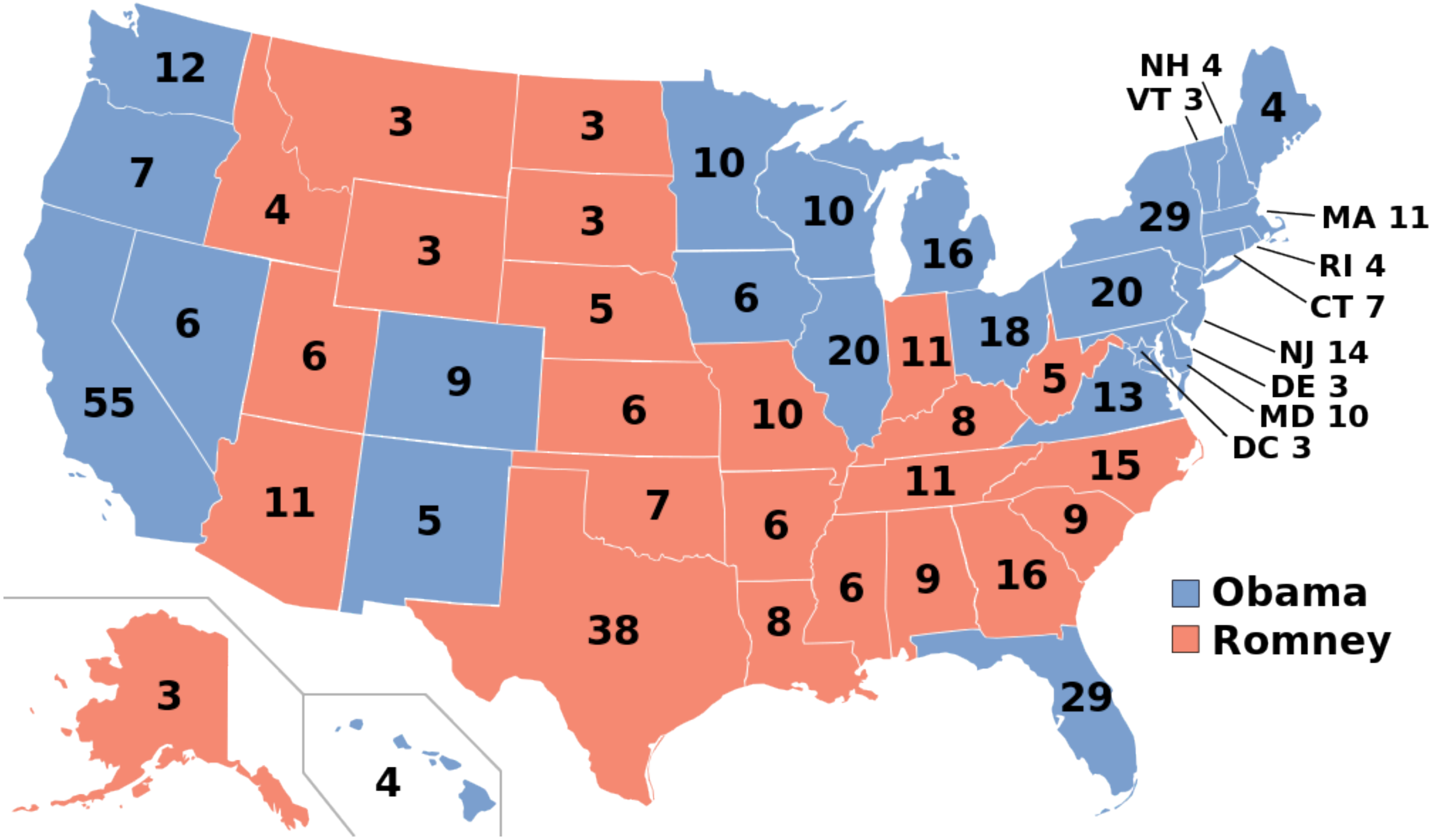
US 2012 presidential election



Advertisement by competing brands

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RENAULT

PEUGEOT

Outline

- The Colonel Blotto game
  - Model
  - Solution
- Variants and related games

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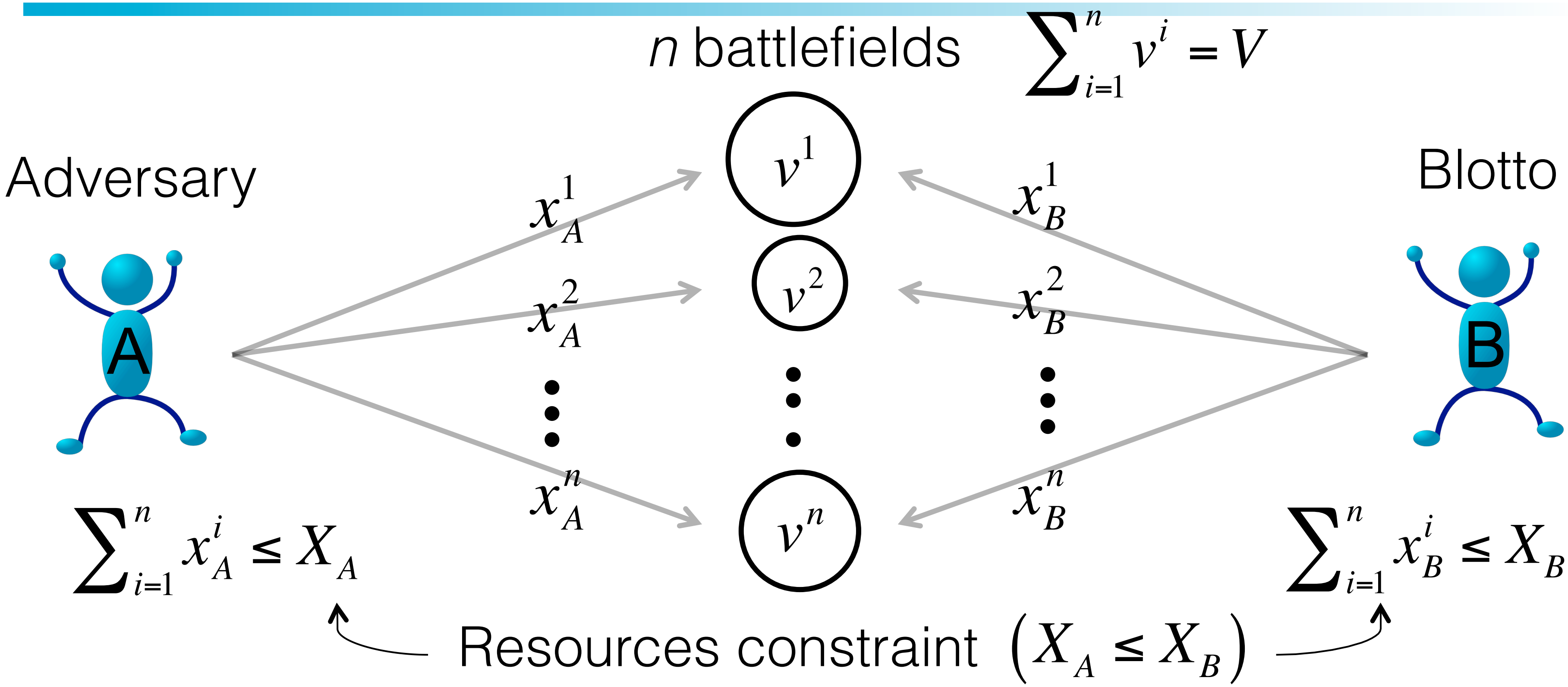
The Colonel Blotto game (Borel 1921)



- If $x_A^i \leq x_B^i$ then B wins battlefield *i*
  - A tie is resolved in favor of the stronger player
- Payoff: sum of values of battlefields won

Features of the Colonel Blotto game

- A general resource allocation game

- A simultaneous-move game
- A constant-sum game (sum of payoffs = V)
- Not a finite game
- Payoffs are not continuous

  → Nash equilibrium?

- If $X_B \geq nX_A$ there is a pure strategy equilibrium
  - B puts X_B / n on each battlefield and wins all

Applications

- The Colonel Blotto game is useful in environment where
  - Strategic attacks are present
  - Fixed resources have to be allocated
  - Players move simultaneously
- Applications
  - Information technology (IT) security: resource (human, processor) allocation across tasks.
  - Emergency relief allocation of state / federal resources: equipment, water, food, medical supplies, air fleet
  - Anti-terror defenses with fixed resources: the Colonel Blotto game allows to consider simultaneous games
  - Air space patrolling / monitoring
  - Politics, allocation of lobbying resources, advertisement, etc.

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Example: no pure-strategy NE

- 3 identical battlefields $v^1 = v^2 = v^3 = 1 \Rightarrow V = 3$
- 2 identical players $X_A = X_B = 1$
  - Ties are resolved at random (50-50)

→ There exists a symmetric equilibrium with equal payoffs

$$\Pi_A = \Pi_B = 3/2$$

- There exists no pure strategy equilibrium
  - Suppose that $x_A^{1,2,3} = 1/3$
  - Player B's best response is $x_B^{1,2} = 1/2 \ \ and \ \ x_B^3 = 0$
  - Payoffs: $\Pi_B = 2 \ \ and \ \ \Pi_A = 1$
  - Player A could do better

Example (2): mixed-strategy NE

- 3 identical battlefields $v^1 = v^2 = v^3 = 1 \Rightarrow V = 3$
- 2 identical players $X_A = X_B = 1$
  - Ties are resolved at random (50-50)

- Suppose A and B use mixed strategies such that the marginals are

$$x_A^i \sim Uniform\left(\left[0, \frac{2v^i}{V} X_B\right]\right) = Uniform\left(\left[0, \frac{2}{3}\right]\right)$$

$$x_B^i \sim Uniform\left(\left[0, \frac{2v^i}{V} X_B\right]\right) = Uniform\left(\left[0, \frac{2}{3}\right]\right)$$

  - Each field has resource 1/3 in expectation
  - Payoffs: $\Pi_A = \Pi_B = 3/2$

→ This is an equilibrium!

Example (3): discussion

- 3 identical battlefields $v^1 = v^2 = v^3 = 1 \Rightarrow V = 3$
- 2 identical players $X_A = X_B = 1$
  - Ties are resolved at random (50-50)

- On this simple example:
  - Pure strategy of 1/3 for player A: $\Pi_B = 2 \;\; and \;\; \Pi_A = 1$
  - Mixed strategy unif. on [0, 2/3]: $\Pi_A = \Pi_B = 3/2$
  - Mixing improves player A's payoff $from \;\; \Pi_A = 1 \;\; to \;\; \Pi_A = 3/2$

➢ If attacker is strategic, mixing is essential

➢ Can we find a strategy for player A/B with the correct marginals that satisfies the budget constraint?

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First solution [Gross & Wagner, 1950]: the case of two battlefields

- Complete solution for n=2 with arbitrary
  $X_B, X_A \quad and \quad v_1, v_2$

- If $X_B \geq 2X_A$, then pure strategy equilibrium

- If $X_B < 2X_A$, then mixed-strategy equilibrium
  - Finite number of mass points
  - The closer X_B, X_A are, the more mass points (i.e., closer to continuum)
  - Example: if $v^1 = v^2 = 1$, and $X_B = 2X_A - \varepsilon$
    - B mixes between X_A and $X_B - X_A$
    - A mixes between X_A and 0

First solution [Gross & Wagner, 1950]: the case of three battlefields

- Solution for n=3 with arbitrary values v_1, v_2, v_3 but only for symmetric players $X_B = X_A$

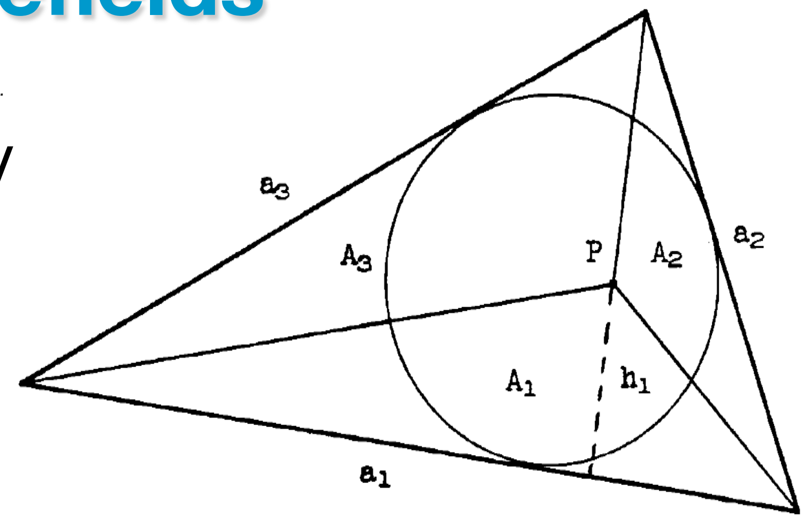


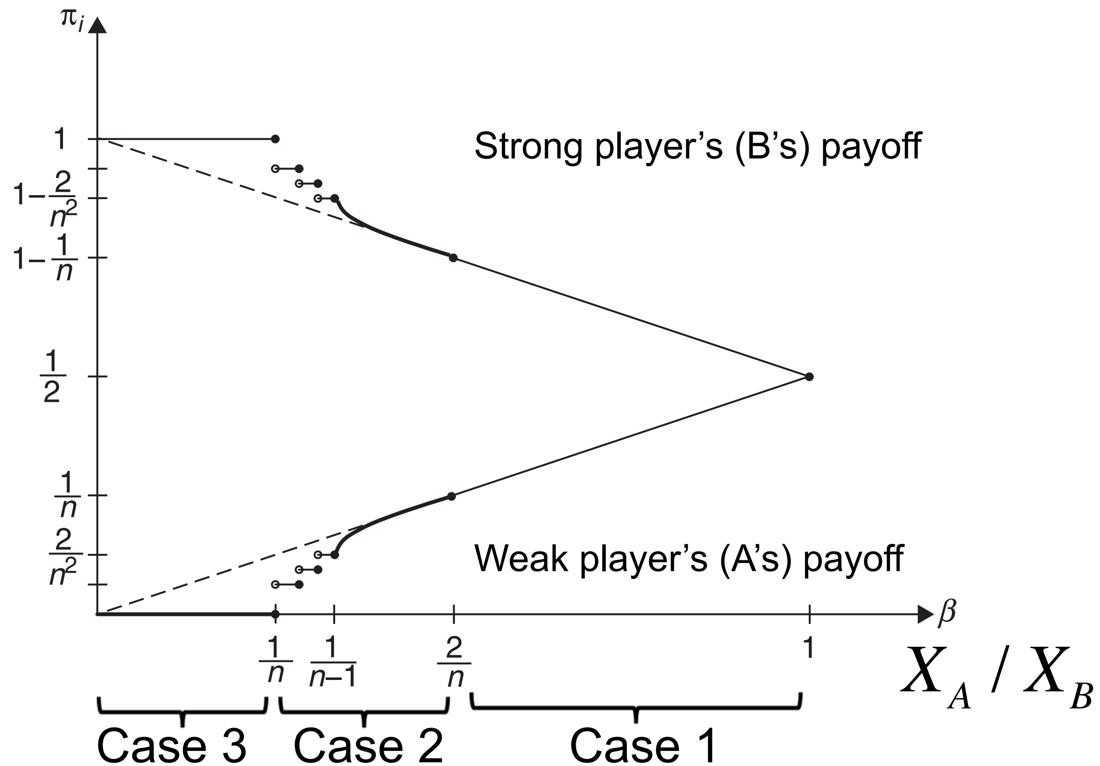
Illustration from Gross & Wagner '50

- Marginals: $x_A^i \sim Uniform[0, \frac{2v^i}{V} X_A]$, same for x_B^i
  - Easy: show that a distribution with these marginals is an equilibrium
  - Difficult: find a joint distribution with these marginals that respect the budget constraint
- Extensions of Gross & Wagner (1950)
  - [Laslier & Picard, 2002], [Thomas, 2013]

Second solution [Roberson 2006]: *n* homogeneous battlefields

- Solution arbitrary X_B, X_A and arbitrary number *n* of identical battlefields: $v^1 = v^2 = \cdots = v^n$
- Distinguishes several regimes of X_A/X_B

Illustration from [Roberson 2011]



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Different regimes

- Case 1: $\frac{2}{n} < \frac{X_A}{X_B} \leq 1$ [players with similar resources]

$$\Pi_A = \frac{X_A}{2X_B}V; \quad \Pi_B = \left(1 - \frac{X_A}{2X_B}\right)V$$

  - Mixed equilibrium
- Case 2: [resources in intermediate range]

$$\Pi_A > 0; \quad \Pi_B < V$$

  - Case 2a: $\frac{1}{n-1} \leq \frac{X_A}{X_B} \leq \frac{2}{n}$ mixed equilibrium with continuum
  - Case 2b: $\frac{1}{n} < \frac{X_A}{X_B} < \frac{1}{n-1}$ mixed equilibrium with mass points
- Case 3: $X_B \geq nX_A$ [extreme resource disparity]

$$\Pi_A = 0; \quad \Pi_B = V$$

  - Pure equilibrium (multiple)

Case 1 [players with similar resources]

- $n \geq 3$ homogeneous battlefields: $v^1 = \cdots = v^n, \quad \sum_{i=1}^{n} v^i = nv = V$

- Theorem: If $\frac{2}{n} < \frac{X_A}{X_B} \leq 1$, then in equilibrium:
  - Unique equilibrium marginals:

$$x_A^i \sim Uniform\left(\left[0, \frac{2}{n} X_B\right]\right) \text{ with proba } \frac{X_A}{X_B}; \ 0 \text{ with proba } \left(1 - \frac{X_A}{X_B}\right)$$

$$x_B^i \sim Uniform\left(\left[0, \frac{2}{n} X_B\right]\right)$$

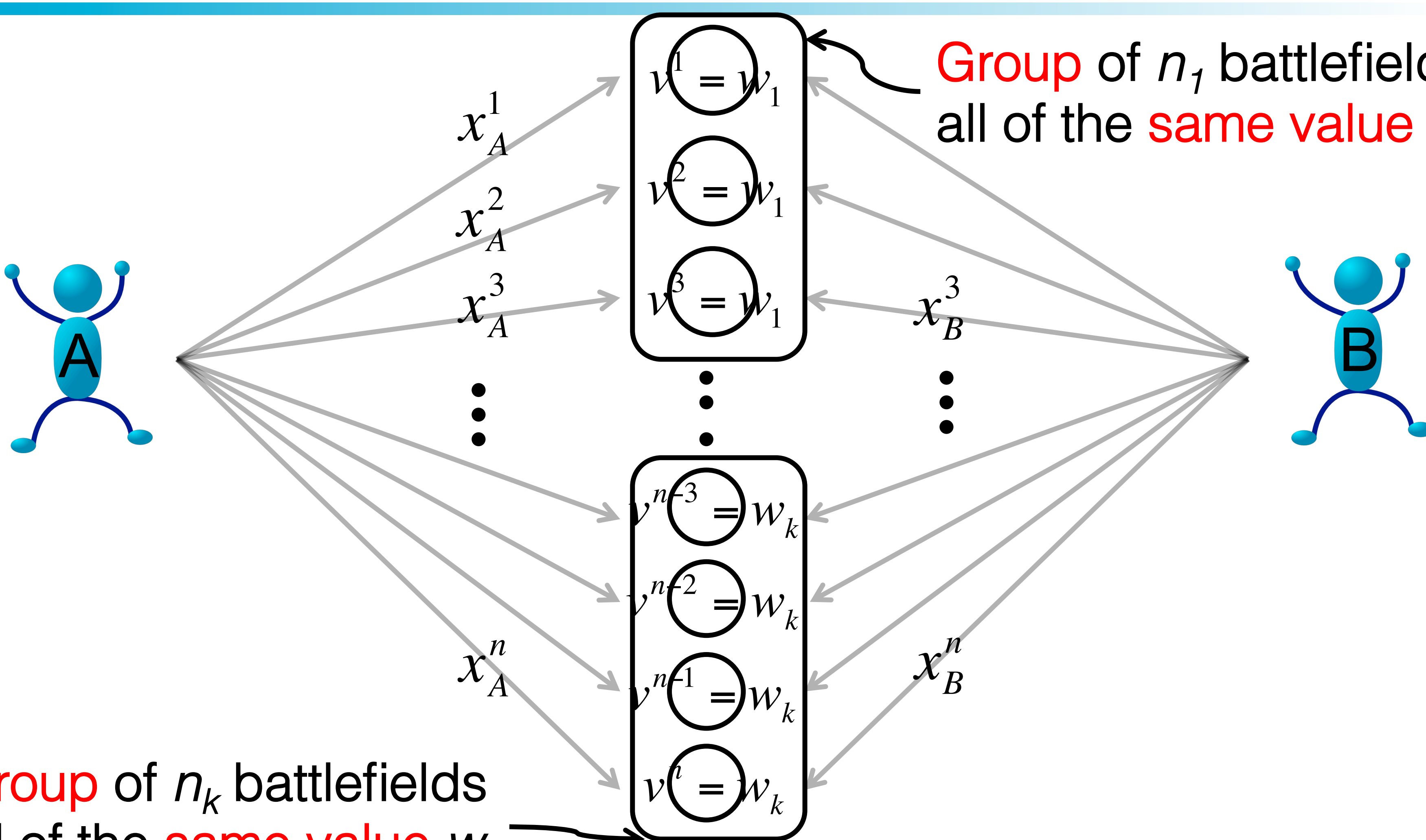
  - Unique equilibrium payoffs: $\Pi_A = \frac{X_A}{2X_B} V; \quad \Pi_B = \left(1 - \frac{X_A}{2X_B}\right) V$
  - Proof:
    - construct a joint distribution with these marginals (direct construction, not based on geometric arguments)
    - uniqueness from all-pay auctions results

Example of joint distribution

- Suppose $n = 3, \quad v^1 = v^2 = v^3, \quad X_A = X_B = 1$
- Need a distribution that respects budget constraints with marginals:
$$x_A^i \sim Uniform[0, 2/3], \; i = 1,2,3$$
- Joint distribution:
  – Reorder the battlefields randomly
  – Assign $x_A^1 \sim Uniform[0, 1/3], \quad x_A^2 = \frac{1}{3} + x_A^1, \quad x_A^3 = 1 - x_A^1 - x_A^2$
- Why does it work?
  – Take a given battlefield. In the previous procedure, it gets
    - with proba 1/3, $x_A^1 \sim Uniform[0, 1/3]$
    - with proba 1/3, $x_A^2 \sim Uniform[1/3, 2/3]$
    - with proba 1/3, $x_A^3 \sim Uniform[0, 2/3]$

  → correct marginal overall!

Heterogeneous battlefields / asymmetric players [Schwartz, L., Sastry, 2014]



Group of n_1 battlefields all of the same value w_1

Group of n_k battlefields all of the same value w_k

Theorem [Schwartz, L., Sastry, 2014]

- Assume that, for each group *j*, $\frac{2}{n_j} < \frac{X_A}{X_B} \leq 1$, then:
  - Unique equilibrium marginals:

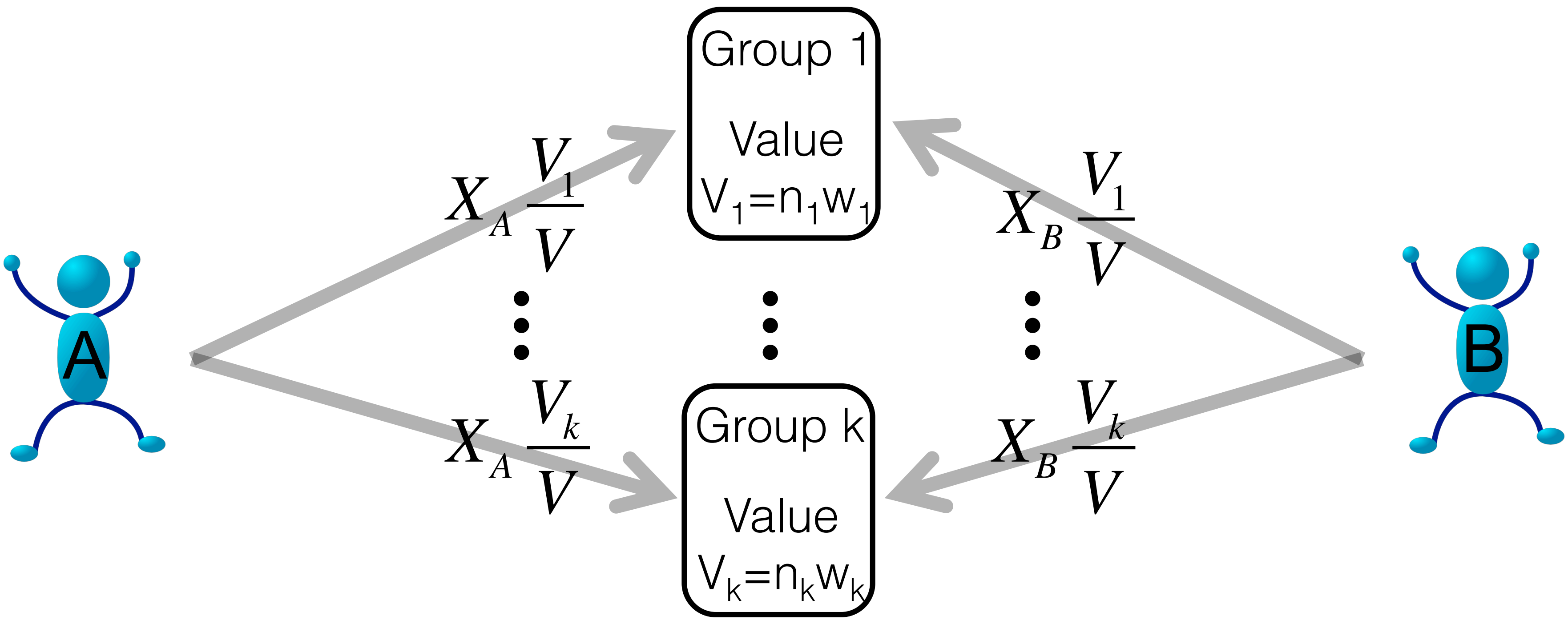
$$x_A^i \sim Uniform\left(\left[0, \frac{2v^i}{V} X_B\right]\right) \text{ with proba } \frac{X_A}{X_B};\ 0 \text{ with proba } \left(1 - \frac{X_A}{X_B}\right)$$

$$x_B^i \sim Uniform\left(\left[0, \frac{2v^i}{V} X_B\right]\right)$$

  - Unique equilibrium payoffs $\quad \text{A: } \frac{X_A}{2X_B} V; \quad \text{B: } \left(1 - \frac{X_A}{2X_B}\right) V$
  - There exists a valid joint distribution respecting budget constraints (proof by construction)

Joint distribution construction

- Step 1: allocate resources to groups of battlefields proportionally to total group value:



- Step 2: within groups, allocate resources as in Roberson '06

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Proofs & remarks

- Remark: possible thanks to assumption $\frac{2}{n_j} < \frac{X_A}{X_B} \leq 1$
- This joint distribution works!
  - It gives the correct marginals
  - It respects the resource budget constraint
- Uniqueness from all-pay auction results
- Requires $n_j \geq 3$ for all *j* and all groups in linear regime
- The joint distribution is not unique

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Variant (1): The General Lotto game

- Budget constraint imposed in expectation only

- Generic solution is much simpler
  - Find the marginals
  - Draw each battlefield's allocation independently

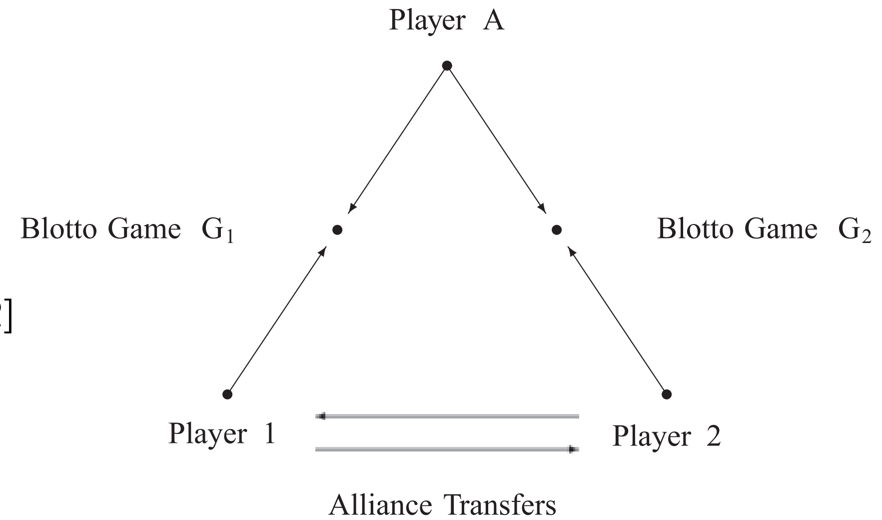
- The same applies for a very large number of battlefields

- Example: [Myerson 1993] in the context of politics

Variant (2): the coalitional colonel Blotto game [Kovenock, Roberson 2012]

- Two players can transfer resources and face a common adversary

Illustration from [Kovenock, Roberson 2012]



- There exists cases where a self-enforcing alliance without commitment will occur
  - Stronger player of coalition transfers resources to weaker player
  - Modifies the allocation of A to the two Blotto games
  - Improves the payoff of both players of the coalition

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Related game (1): The Gladiator game [Rinott, Scarsini, Yu, 2012]

- Rules of the game
  - Two teams of m and n gladiators
  - Each team coach allocates a finite amount of “strength” to its gladiators
  - Gladiators fight sequentially, the survivor fights at next round
  - The outcome of a fight is random, the probability of winning is proportional to the strength
    - If player resources are a and b, the probability of winning is a/(a+b)
  - The first player with no more gladiator loses all
- Result:
  - There exists a pure equilibrium
    - Stronger player allocates uniformly to all players
    - Weaker player allocates uniformly to a subset
  - ➢ The randomness is already in the payoff!

Related game (2): the FlipIt game [van Dijk, Juels, Oprea, Rivest, 2013]

- Game of timing
- Rules:
  - Each player chooses when to flip
  - Time is continuous, finite length T
  - Costs of flip for each player are known
  - Payoffs: the fraction of time the player "owns" the resource
- Results:
  - If both players flip periodically: characterization of equilibrium choice of period
  - If both players can choose between periodic and renewal flip: periodic dominates renewal
  - More general strategy: open problem!

Conclusion

- Summary:
  - The Colonel Blotto game (and related games) are beautiful and useful to think about resource allocation in a strategic setting!
  - We can solve the game with asymmetric players and heterogeneous battlefields, under minor restrictions
  - We provide an algorithm for allocating resources across battlefields
- Applications are numerous
  - Security, politics, advertisement, etc.
- Open questions:
  - Equilibrium for players with moderately asymmetric resources
  - Players with unequal valuations of the battlefields
  - Limit of large number of battlefields

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THANK YOU