Inverse modelling using optimization to solve imaging tasks

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Stability of the minimizers of $\mathcal{F}_v$

$$\mathcal{F}_v(u) = \| Au - v \|_2^2 + \beta \Phi(u)$$
$$\Phi(u) = \sum_i \varphi(\| G_i u \|_2)$$

\( u \in \mathbb{R}^p \)
\( v \in \mathbb{R}^q \)

\{\( G_i \)\} linear operators \( \mathbb{R}^p \to \mathbb{R}^s, s \geq 1 \)

\( \varphi'(0^+) > 0 \Rightarrow \Phi \) is nonsmooth on \( \bigcup_i \{ u : G_i u = 0 \} \)

Systematically: \( \ker A \cap \ker G = \{0\} \)
\( G \doteq \begin{bmatrix} G_1 \\ G_2 \\ \vdots \end{bmatrix} \)

Question 2 Why?

\( \mathcal{F}_v \) nonconvex \( \Rightarrow \) there may be many local minima
• $N = \{(s,t) : t = \pm \arctan(s)\}$

• $N$ is closed in $\mathbb{R}^2$ and its Lebesgue measure in $\mathbb{R}^2$ is $L^2(N) = 0$

• $(x,y) = \text{random } \mathbb{R}^2$

**Question 3** What is the chance that $(x,y) \in N$?
Assumptions: $\varphi$ is piecewise $C^{m \geq 2}$, edge-preserving, possibly non-convex, $\text{rank}(A) = p$

- There is a closed $N \subset \mathbb{R}^q$ with $L^q(N) = 0$ such that $\forall v \in \mathbb{R}^q \setminus N$, every (local) minimizer $\hat{u}$ of $F_v$ is given by $\hat{u} = U(v)$ where $U$ is a $C^{m-1}$ (local) minimizer function.

**Question 4** Why knowledge on local minimizers is important?

**Question 5** Compare $\hat{u}$ and $U(v + \varepsilon)$ where $\varepsilon \in \mathbb{R}^q$ is small enough.

- $\exists \hat{N} \subset \mathbb{R}^q$ with $L^q(\hat{N}) = 0$ such that $\forall v \in \mathbb{R}^q \setminus \hat{N}$, $F_v$ has a unique global minimizer

**Question 6** What can happen if $v \in \hat{N}$?

- $\exists$ open subset of $\mathbb{R}^q \setminus \hat{N}$, dense in $\mathbb{R}^q$, where the global minimizer function $\hat{U}$ is $C^{m-1}$.

**Question 7** If $F_v$ is strictly convex, determine $N$ and $\hat{N}$.
Nonasymptotic bounds on minimizers

**Assumption:** \( \varphi \) is piecewise \( C^1 \)

- \( \varphi \) is strictly increasing **or** \( \text{rank}(A) = p \)

\[ \hat{u} \text{ is a (local) minimizer of } \mathcal{F}_v \quad \Rightarrow \quad \| A\hat{u} \| \leq \| v \| \]

- \( \| \varphi' \|_\infty < \infty \) (**\( \varphi \) is edge-preserving**) and \( \text{rank}(A) = q \leq p \)

\[ \hat{u} \text{ is a (local) minimizer of } \mathcal{F}_v \quad \Rightarrow \quad \| v - A\hat{u} \|_\infty \leq \frac{\beta}{2} \| \varphi' \|_\infty \| (AA^*)^{-1} A \|_\infty \| G \|_1 \]

\( \| \varphi' \|_\infty = 1 \) and **1st order differences:**

\[ \begin{cases} 
\text{signal} & \Rightarrow \quad \| v - \hat{u} \|_\infty \leq \beta \\
\text{image} & \Rightarrow \quad \| v - \hat{u} \|_\infty \leq 2\beta 
\end{cases} \]

**Question 8** If \( v = u_o + n \) for \( n \) Gaussian noise, is it possible to clean \( v \) from this noise by minimizing \( \mathcal{F}_v \)?
Non-Smooth Energies, Side Derivatives, Subdifferential

Rademacher’s theorem: If $\mathcal{F}_v : \mathbb{R}^p \to \mathbb{R}$ is Lipschitz continuous, then $\mathcal{F}_v$ is differentiable (in the usual sense) almost everywhere in $\mathbb{R}^p$.

A **kink** is a point $u$ where $\nabla \mathcal{F}_v(u)$ is not defined (in the usual sense).

Example: $\mathcal{F}_v(u) = \frac{1}{2}(u-v)^2 + \beta |u|$ for $\beta = 1 > 0$ and $u, v \in \mathbb{R}$

\[ \hat{u} = \begin{cases} 
  v + \beta & \text{if } v < -\beta \\
  0 & \text{if } |v| \leq \beta \\
  v - \beta & \text{if } v > \beta 
\end{cases} \]

**Question 9** What is drawn on the second row?

**Question 10** Give a condition for $\mathcal{F}_v$ to have a minimum at $\hat{u}$. 
3 Minimizers under Non-Smooth Regularization

\[ \mathcal{F}_v(u) = \Psi(u, v) + \beta \sum_{i=1}^{r} \varphi(\|G_i u\|), \quad \Psi \in C^{m>2}, \varphi \in C^m(\mathbb{R}^*_+), \quad 0 < \varphi'(0^+) \leq \infty \]

\[ \varphi(t) \begin{array}{c|c|c|c|c} t^\alpha, \alpha \in (0, 1) \hspace{1cm} \frac{\alpha t}{\alpha t + 1} \hspace{1cm} \ln(\alpha t + 1) \hspace{1cm} 1 - \alpha^t \end{array} \alpha \in (0, 1) \hspace{1cm} (\cdots), \hspace{1cm} \alpha > 0 \]

\[ \varphi(t) = t \text{ and } G_i u \approx (\nabla u)_i \Rightarrow \mathcal{F}_v(u) = TV(u) \text{ (total variation)} \quad [\text{Rudin, Osher, Fatemi 92}] \]
Let \( \hat{u} \) be a (local) minimizer of \( \mathcal{F}_v \). Set \( \hat{h} \doteq \{i : G_i \hat{u} = 0\} \)

Then \( \exists \, O \subset \mathbb{R}^q \text{ open, } \exists \, U \in C^{m-1} \) (local) minimizer function so that

\[
v' \in O, \quad \hat{u}' = U(v') \Rightarrow G_i \hat{u}' = 0, \quad \forall \, i \in \hat{h}
\]

Data \( v \) yield (local) minimizers \( \hat{u} \) of \( \mathcal{F}_v \) such that \( G_i \hat{u} = 0 \) for a set of indexes \( \hat{h} \)

\[
G_i = \nabla_i \Rightarrow \hat{u}[i] = \hat{u}[j] \text{ for many neighbors } (i, j) \text{ (the "stair-casing" effect)}
\]

\[
G_i u = u[i] \Rightarrow \text{many samples } \hat{u}[i] = 0 - \text{highly used in Compressed Sensing}
\]

**Question 11** What happens if \( \{G_i\} \) yield second-order differences?

**Question 12** Describe the prior that \( \hat{u} \) satisfies for a general \( \{G_i\} \).

Property **fails** if \( \mathcal{F}_v \) is smooth, except for \( v \in N \) where \( N \) is closed and \( \mathbb{L}^q(N) = 0 \).
\[ \mathcal{F}_v(u) = ||u - v||^2 + \beta \sum \varphi(|u[i] - u[i - 1]|) \]

\[ \varphi(t) = \sqrt{\alpha + t^2}, \quad \varphi'(0) = 0 \quad \text{(smooth at 0)} \]

\[ \varphi(t) = (t + \alpha \text{sign}(t))^2, \quad \varphi'(0^+) = 2\alpha \]

\[ \varphi(t) = |t|, \quad \varphi'(0^+) = 1 \]

\[ \varphi(t) = \alpha |t|/(1 + \alpha |t|), \quad \varphi'(0^+) = \alpha \]
Let $u_o \in \mathbb{R}$ and $\text{pdf}(u_o) = \frac{1}{2}e^{-|u_o|}$ (Laplacian distribution)

**Question 13** Give $\text{Pr}(u_o = 0)$.

Let $v = u_o + n$ where $\text{pdf}(n) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{n^2}{2\sigma^2}}$ (centered Gaussian distribution)

The corresponding MAP energy to recover $u_o$ from $v$ reads as

$$\mathcal{F}_v(u) = \frac{1}{2}(u - v)^2 + \beta |u| \quad \text{for} \quad \beta = \frac{1}{\sigma^2}$$

**Question 14** Give the minimizer function $\mathcal{U}$ for $\mathcal{F}_v$.

Useful reminder on p. 21.

**Question 15** Determine the set $\{\nu \in \mathbb{R} : \mathcal{U}(\nu) = 0\}$. Comment the result.
TV energy: $F_v(u) = \|Au - v\|^2 + \beta \sum_{i=1}^{r} \varphi(\|G_i u\|)$ for $\varphi(t) = t$ and $G_i$ discrete gradient at pixel $i$.

Minimization of \( F_v(u) = \|u - v\|^2_2 + \beta \text{TV}(u) \), \( \beta = 100 \) and \( \beta = 180 \)
Here  \( \varphi(t) = \begin{cases} 
0 & \text{if } t = 0 \\
1 & \text{if } t \neq 0 
\end{cases} \)

**Question 16**  Compute the global minimizer of \( F_v(u) = (u - v)^2 + \beta \varphi(u) \) for \( u, v \in \mathbb{R} \) and \( \beta > 0 \), according to the value of \( v \).

**Question 17**  Are there any values of \( v \) so that \( F_v \) has more than one global minimizer?

Consider \( F_v(u) = \|u - v\|_2^2 + \beta \sum_{i=1}^{p} \varphi(u[i]) \) for \( \beta > 0 \) and \( u, v \in \mathbb{R}_p \).

The global minimizer function \( U : \mathbb{R}^p \rightarrow \mathbb{R}^p \) for \( F_v \) has \( p \) components which depend on \( v \).

**Question 18**  Compute each component \( U_i \)

**Question 19**  Let \( h \subset \{1, \cdots, p\} \). Determine the subset \( O_h \subset \mathbb{R}^p \) such that

- if \( v \in O_h \) then the global minimizer \( \hat{u} \) of \( F_v \) satisfies \( \hat{u}[i] = 0 \), \( \forall i \in h \)
- and \( \hat{u}[i] \neq 0 \) if \( i \notin h \).

Note that  \( \sum_{i=1}^{p} \varphi(u[i]) = \#\{i : u[i] \neq 0\} = \ell_0(u) \)
4 Minimizers relevant to non-smooth data-fidelity

**General case**  

\[ \mathcal{F}_v(u) = \sum_i \psi(|a_iu - v[i]|) + \beta \Phi(u), \quad \Phi \in C^m, \; \psi \in C^m(\mathbb{R}^*_+), \; \psi'(0^+) > 0 \]

Let \( \hat{u} \) be a (local) minimizer of \( \mathcal{F}_v \). Set \( \hat{h} = \{ i : a_i\hat{u} = v[i] \} \).

Then \( \exists \; O \subset \mathbb{R}^q \text{ open}, \; \exists \; U \in C^{m-1} \) (local) minimizer function so that

\[ v' \in O, \; \hat{u}' = U(v') \Rightarrow \begin{cases} a_i \hat{u}' = v[i], & i \in \hat{h} \\ a_i \hat{u}' \neq v[i], & i \in \hat{h}^c \end{cases} \]

(Local) minimizers \( \hat{u} \) of \( \mathcal{F}_v \) achieve an exact fit to (noisy) data \( a_i \hat{u} = v[i] \) for a certain number of indexes \( i \).

Property fails if \( \mathcal{F} \) is smooth, except for \( v \in N \) where \( N \) is closed and \( \mathbb{L}^q(N) = 0 \).
Question 20  Suggest cases when you would like that your minimizer obeys this property.

Question 21  Propose some choices for $\psi$. Explain.

Question 22  Compute the minimizer of $F_v(u) = |u - v| + \beta u^2$ for $u, v \in \mathbb{R}$ and $\beta > 0$.

Question 23  Explain the relationship between the properties of the minimizer when $\varphi'(0^+) > 0$ and when $\psi'(0^+) > 0$.
Original $u_o$

Data $v = u_o + \text{outliers}$

Restoration $\hat{u}$ for $\beta = 0.14$

Residuals $v - \hat{u}$

$$\mathcal{F}_v(u) = \sum_i |u[i] - v[i]| + \beta \sum_{j \in \mathcal{N}_i} |u[i] - u[j]|^{1.1}$$
Restoration \( \hat{u} \) for \( \beta = 0.25 \)

Residuals \( v - \hat{u} \)

\[
\mathcal{F}_v(u) = \sum_i |u[i] - v[i]| + \beta \sum_{j \in \mathcal{N}_i} |u[i] - u[j]|^{1.1}
\]

Restoration \( \hat{u} \) for \( \beta = 0.2 \)

Residuals \( v - \hat{u} \)

TV-like energy: \( \mathcal{F}_v(u) = \sum_i (u[i] - v[i])^2 + \beta \sum_{j \in \mathcal{N}_i} |u[i] - u[j]| \)
Detection and cleaning of outliers using $\ell_1$ data-fidelity

\[
\mathcal{F}_v(u) = \sum_{i=1}^{p} |u[i] - v[i]| + \frac{\beta}{2} \sum_{i=1}^{p} \sum_{j \in \mathcal{N}_i} \varphi(|u[i] - u[j]|)
\]

$\varphi$: smooth, convex, edge-preserving

Assumptions: \[
\begin{cases}
\text{data } v \text{ contain uncorrupted samples } v[i] \\
v[i] \text{ is outlier if } |v[i] - v[j]| \gg 0, \forall j \in \mathcal{N}_i
\end{cases}
\]

$v \in \mathbb{R}^p \Rightarrow \hat{u} = \arg \min_u \mathcal{F}_v(u) \quad \begin{cases}
v[i] \text{ is regular if } i \in \hat{h} \\
v[i] \text{ is outlier if } i \in \hat{h}^c
\end{cases}
\hat{h} = \{i : \hat{u}[i] = v[i]\}

Outlier detector: \[v \rightarrow \hat{h}^c(v) = \{i : \hat{u}[i] \neq v[i]\}\]

Smoothing: \[\hat{u}[i] \text{ for } i \in \hat{h}^c = \text{estimate of the outlier}\]

Justification based on the properties of $\hat{u}$
Original image $u_o$

10% random-valued noise

Median ($\|\hat{u} - u_o\|_2 = 4155$)

Recursive CWM ($\|\hat{u} - u_o\|_2 = 3566$)

PWM ($\|\hat{u} - u_o\|_2 = 3984$)

Proposed ($\|\hat{u} - u_o\|_2 = 2934$)
Recovery of frame coefficients using $\ell_1$ data-fitting

- Data: $v = u_o + \text{noise}$
- Frame coefficients: $y = Wv = Wu_o + \text{noise}$
  
  $\tilde{W} =$ left inverse of $W$

- Hard thresholding $y_T[i] = \begin{cases} 
0 & \text{if } |y[i]| \leq T \\
  y[i] & \text{if } |y[i]| > T
\end{cases}$

  keeps relevant information if $T$ small

- $\tilde{u} = \tilde{W}y_T$ — Gibbs oscillations and wavelet-shaped artifacts

- Hybrid energy methods—combine fitting to $y_T$ with prior $\Phi(u)$
  
  [Bobichon, Bijaoui 97], [Coifman, Sowa 00], [Durand, Froment 03]...
Desiderata: $\mathcal{F}_y$ convex and

<table>
<thead>
<tr>
<th>Keep $\hat{x}[i] = y_T[i]$</th>
<th>Restore $\hat{x}[i] \neq y_T[i]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>significant coefs: $y[i] \approx (Wu_o)[i]$</td>
<td>outliers: $</td>
</tr>
<tr>
<td>thresholded coefs: $(Wu_o)[i] \approx 0$</td>
<td>edge coefs: $</td>
</tr>
</tbody>
</table>

Then:

$$\minimize \quad \mathcal{F}_y(x) = \sum_i \lambda_i |(x - y_T)[i]| + \int_{\Omega} \varphi(|\nabla \tilde{W} x|) \quad \Rightarrow \quad \hat{x}$$

$$\hat{u} = \tilde{W} \hat{x} \quad \text{for} \quad \tilde{W} \quad \text{left inverse,} \quad \varphi \quad \text{edge-preserving}$$

**Question 24** Explain why the minimizers of $\mathcal{F}_y$ fulfill the desiderata.

**Question 25** Any open questions?
Original and data

Sure-shrink method

Hard thresholding

Total variation

The proposed method

Magnitude of coefficients

Restored signal (—), original signal (−−).
Fast 2-stage restoration under impulse noise  
[R.Chan, Nikolova et al. 04, 05, 08]

1. Approximate the outlier-detection stage by rank-order filter (e.g. adaptive or center-weighted median)  
   Corrupted pixels $\hat{h}^c = \{ i : \hat{v}[i] \neq v[i] \}$ where $\hat{v}=$Rank-Order Filter ($v$)  
   $\Rightarrow$ improve speed and accuracy

2. Restore $\hat{u}$ (denoise, deblur) using an edge-preserving energy method  
   subject to $a_i \hat{u} = v[i]$ for all $i \in \hat{h}$
50% RV noise | ACWMF | DPVM | Our method
---|---|---|---
70% SP noise (6.7dB) | Adapt.med. (25.8dB) | Our method (29.3dB) | Original Lena
L. Bar, A. Brook, N. Sochen and N. Kiryati,
“Deblurring of Color Images Corrupted by Impulsive Noise”,
IEEE Trans. on Image Processing, 2007

\[ F_v(u) = \|Au - v\|_1 + \beta \Phi(u) \]
One-step real-time dejittering of digital video

- Image \( u \in \mathbb{R}^{m \times n} \), rows \( u_i \), its pixels \( u_i[j] \)
- Data \( v_i[j] = u_i[j + d_i] \), \( d_i \) integer, \( |d_i| \leq M \), typically \( M \leq 20 \).
- Restore \( \hat{u} \equiv \text{restore } \hat{d}_i, 1 \leq i \leq m \)

The gray-values of the columns of natural images can be seen as large pieces of 2\(^{nd}\) (or 3\(^{rd}\)) order polynomials which is false for their jittered versions.
Each column \( \hat{u}_i \) is restored using

\[
\hat{d}_i = \arg \min_{|d_i| \leq N} \mathcal{F}(d_i)
\]

\[
\mathcal{F}(d_i) = \sum_{j=N+1}^{c-N} |v_i[j + d_i] - 2\hat{u}_{i-1}[j] + \hat{u}_{i-2}[j]|^\alpha, \quad \alpha \in \{0.5, 1\}, \quad N > M
\]

Question 26 Explain why the minimizers of \( \mathcal{F} \) can solve the problem as stated.

Question 27 What changes if \( \alpha = 1 \) or if \( \alpha = 0.5 \)?

Question 28 Is it easy to solve the numerical problem?

A Monte-Carlo experiment shows that in almost all cases, \( \alpha = 0.5 \) is the best choice.
Jitter \{-15,\ldots,15\} \quad \alpha = 1, \ \alpha = 0.5 \quad \text{Original image}
Jitter Image Bayesian TV Bake & Shake

Jitter Jittered Image Bayesian TV Bake & Shake

Original Our: \( \alpha = 0.5 \) Our: Error \( u_o - \hat{u} \)

[Kokaram98, Laborelli03, Shen04, Kang06, Scherzer11]
Comparison with Smooth Energies

\[ F_v(u) = \Psi(u, v) + \beta \Phi(u), \quad F \in C^{m \geq 2} + \text{easy assumptions.} \]

If \( h \neq \emptyset \) \( \Rightarrow \)

\[ \{ v \in \mathbb{R}^q : F_v - \text{minimum at } \hat{u}, \ G_i \hat{u} = 0, \ \forall i \in h \} \]

\[ \{ v \in \mathbb{R}^q : F_v - \text{minimum at } \hat{u}, \ \langle a_i, \hat{u} \rangle = v_i, \ \forall i \in h \} \]

\( \text{closed and negligible in } \mathbb{R}^q \)

For \( F_v \) smooth, the chance that noisy data \( v \) yield a minimizer \( \hat{u} \) of \( F_v \) which for some \( i \) satisfies exactly \( G_i \hat{u} = 0 \) or \( \langle a_i, \hat{u} \rangle = v_i \) is negligible

Nearly all \( v \in \mathbb{R}^q \) lead to \( \hat{u} = U(v) \) satisfying \( G_i \hat{u} \neq 0, \ \forall i \) and \( \langle a_i, \hat{u} \rangle \neq v_i, \ \forall i \)

Question 29  What are the consequences if one approximates a nonsmooth energy by a smooth energy?
Let $u \in \mathbb{R}^p$ and $v \in \mathbb{R}^q$.

Consider that $A \in \mathbb{R}^{q \times p}$ and $G \in \mathbb{R}^{r \times p}$ satisfy $\ker(A) \cap \ker(G) = \{0\}$.

$$ \mathcal{F}_v(u) = \|Au - v\|_2^2 + \beta\|Gu\|_2^2 \quad \text{for} \quad \beta > 0 $$

**Question 30** Calculate $\nabla \mathcal{F}_v(u)$.

**Question 31** Determine the minimizer function $\mathcal{U}$.

Let $G_i \in \mathbb{R}^{1 \times p}$ denote the $i$th row of $G$.

**Question 32** Characterize the set $\mathcal{K} = \{\nu \in \mathbb{R}^p : G_i \mathcal{U}(\nu) = 0\}$.

Let $a_i \in \mathbb{R}^{1 \times p}$ denote the $i$th row of $A$.

**Question 33** Characterize the set $\mathcal{L} = \{\nu \in \mathbb{R}^p : a_i \mathcal{U}(\nu) = \nu[i]\}$. 
5 Nonconvex Regularization: Why Edges are Sharp?  

\[ \mathcal{F}_v(u) = \|Au - v\|^2 + \beta \sum_{i \in J} \phi(\|G_i u\|) \quad J = \{1, \ldots, r\} \]

**Standard assumptions on \( \phi \):** \( C^2 \) on \( \mathbb{R}_+ \) and \( \lim_{t \to \infty} \phi''(t) = 0 \), as well as:

\[ \phi'(0) = 0 \ (\Phi \text{ is smooth}) \]

\[ \phi'(0^+) > 0 \ (\Phi \text{ is nonsmooth}) \]

\[ \phi''(t) > 0 \quad \text{increase, } \leq 0 \]

\[ \phi''(t) \quad \text{increase, } \leq 0 \]
Sharp edge property

There exist $\theta_0 > 0$ and $\theta_1 > \theta_0$ such that any (local) minimizer $\hat{u}$ of $\mathcal{F}_v$ satisfies

$$\text{either } \|G_i \hat{u}\| \leq \theta_0 \text{ or } \|G_i \hat{u}\| \geq \theta_1 \quad \forall i \in J$$

\[
\begin{align*}
\hat{h}_0 &= \{ i : \|G_i \hat{u}\| \leq \theta_0 \} \quad \text{homogeneous regions} \\
\hat{h}_1 &= \{ i : \|G_i \hat{u}\| \geq \theta_1 \} \quad \text{edges}
\end{align*}
\]

When $\beta$ increases, then $\theta_0$ decreases and $\theta_1$ increases.

In particular

$$\varphi'(0^+) > 0 \Rightarrow \theta_0 = 0 \quad \text{fully segmented image} \quad (G_i \hat{u} = 0, \ \forall i \in \hat{h}_0)$$

**Question 34** Explain the prior model involved in $\mathcal{F}_v$ when $\varphi$ is nonconvex with $\varphi'(0) = 0$ and with $\varphi'(0^+) > 0$. 
**Image Reconstruction in Emission Tomography**

- **Original phantom**
- **Emission tomography simulated data**

\[ \varphi \text{ is smooth (Huber function)} \]

\[ \varphi(t) = \frac{t}{\alpha + t} \text{ (non-smooth, non-convex)} \]

Reconstructions using

\[ \mathcal{F}_v(u) = \Psi(u, v) + \beta \sum_{j \in \mathcal{N}_i} \varphi(|u[i] - u[j]|), \quad \Psi = \text{smooth, convex} \]
\[ F_v(u) = (u - v)^2 + \beta \varphi(u) \quad u, v \in \mathbb{R} \quad \beta > 0 \]

- **Assumption:** \( \beta > -\frac{2}{\min_{t \in \mathbb{R}} \varphi''(t)} \) (if \( \varphi'(0^+) > 0 \) then \( \min_{t \in \mathbb{R}} \varphi''(t) = \varphi''(0^+) \)).

**Question 35** Determine the sign of \( \beta \), i.e. \( > 0 \) or \( < 0 \).

- \( \mathcal{C}_\beta = \left\{ t \in (0, +\infty) : \varphi''(t) < -\frac{2}{\beta} \right\} \)

- Recall: \( F_v \) has a (local) minimum at a \( \hat{u} \) where \( F_v \) is twice differentiable if and only if

\[ F'_v(\hat{u}) = 0 \quad \text{and} \quad F''_v(\hat{u}) \geq 0 \]

**Question 36** Show that \( \forall v \in \mathbb{R} \), if \( \hat{u} \) is a (local) minimizer of \( F_v \), then \( |\hat{u}| \notin \mathcal{C}_\beta \).
Comparison with Convex Edge-Preserving Regularization

Data \( v = u_o + n \)

\( \varphi(t) = |t| \)

\( \varphi(t) = \frac{\alpha|t|}{1 + \alpha|t|} \)

original data

\( \varphi(t) = |t|^{1.4} \)

\( \varphi(t) = \min\{\alpha t^2, 1\} \)

Question 37 Why edges are sharper when \( \varphi \) is nonconvex?
Each blue curve curve: $u \rightarrow \mathcal{F}_v(u)$ for $v \in \{0, 2, \cdots\}$

**Question 38** How to describe the global minimizer when $v$ increases?
6. Nonsmooth data-fidelity and regularization

Consequence of §3 and §4: if \( \Phi \) and \( \Psi \) non-smooth, \[ \begin{cases} 
G_i \hat{u} = 0 & \text{for } i \in \hat{h}_\varphi \neq \emptyset \\
\alpha_i \hat{u} = v[i] & \text{for } i \in \hat{h}_\psi \neq \emptyset 
\end{cases} \]

The \( L_1 \)-TV energy


\[ \mathcal{F}_v(u) = \| u - 1_{\Omega} \|_1 + \beta \int_{\mathbb{R}^d} \| \nabla u(x) \|_2 \, dx \text{ where } 1_{\Omega}(x) = \begin{cases} 1 & \text{if } x \in \Omega \\ 0 & \text{else} \end{cases} \]

- \( \exists \hat{u} = 1_{\Sigma} \) (\( \Omega \) convex \( \Rightarrow \Sigma \subset \Omega \) and \( \hat{u} \) unique for almost every \( \beta > 0 \))
- Contrast invariance: if \( \hat{u} \) minimizes for \( v = 1_{\Omega} \) then \( c\hat{u} \) minimizes \( \mathcal{F}_{cv} \)
  - the contrast of image features is more important than their shapes

- Critical values \( \beta^* \)
  \[ \begin{cases} 
\beta < \beta^* & \Rightarrow \text{ objects in } \hat{u} \text{ with good contrast} \\
\beta > \beta^* & \Rightarrow \text{ they suddenly disappear} 
\end{cases} \]

\( \Rightarrow \) data-driven scale selection
Binary images by L1 – TV

Classical approach to find a binary image \( \hat{u} = \mathbb{1}_\Sigma \) from binary data \( \mathbb{1}_\Omega, \ \Omega \subset \mathbb{R}^2 \)

\[
\hat{\Sigma} = \arg \min_\Sigma \left\{ \| \mathbb{1}_\Sigma - \mathbb{1}_\Omega \|^2_2 + \beta \text{TV}(\mathbb{1}_\Sigma) \right\}
\]

nonconvex problem \((\star)\)

usual techniques (curve evolution, level-sets) fail

\[
\hat{\Sigma} \text{ solves } (\star) \iff \hat{u} = \mathbb{1}_\hat{\Sigma} \text{ minimizes } \| u - \mathbb{1}_\Omega \|_1 + \beta \text{TV}(u) \quad \text{(convex)}
\]