NEW VARIATIONAL METHODS IN COMPUTER VISION

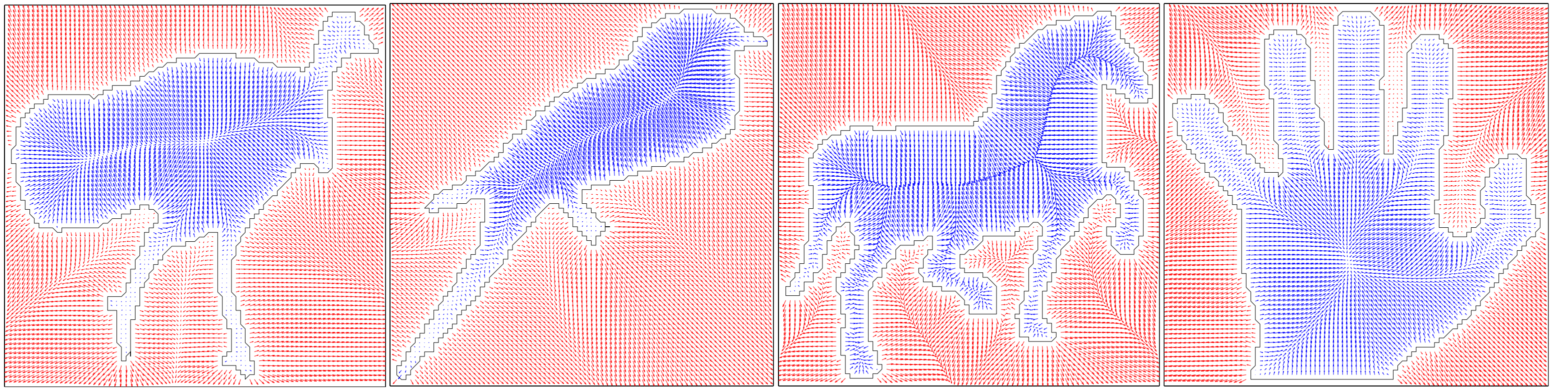
CV School, UCLA

Anand Rangarajan, Dept. of CISE, Univ. of Florida

Roadmap

- Distance transforms
- Calculus of variations
- Lagrangians and Hamiltonians
- Nonlinear Hamilton-Jacobi equation
- Linear Schrödinger wave equation
- Approximating the eikonal via linear solvers
- The method of stationary phase
- Distance transform gradient density

Distance transforms

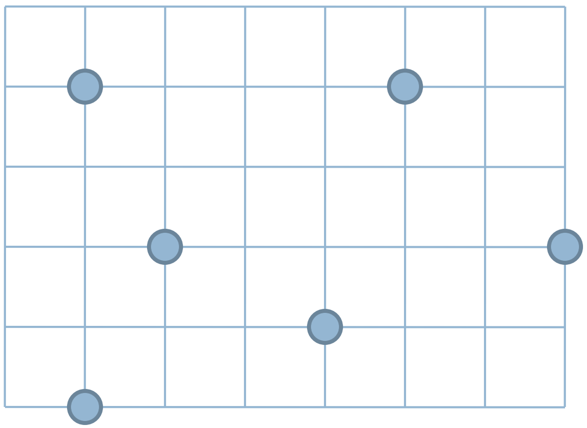


- Distance transforms: Ubiquitous shape representation
- Level sets, fast marching methods etc. standard tropes
- Relationship to classical physics (Hamilton-Jacobi theory) well established

Euclidean distance functions

- Given a point-set $\{Y_k\}_{k=1}^{K}$ and grid points *X*, the Euclidean distance function is

$$S(X) = \min_k \| X - Y_k \|$$



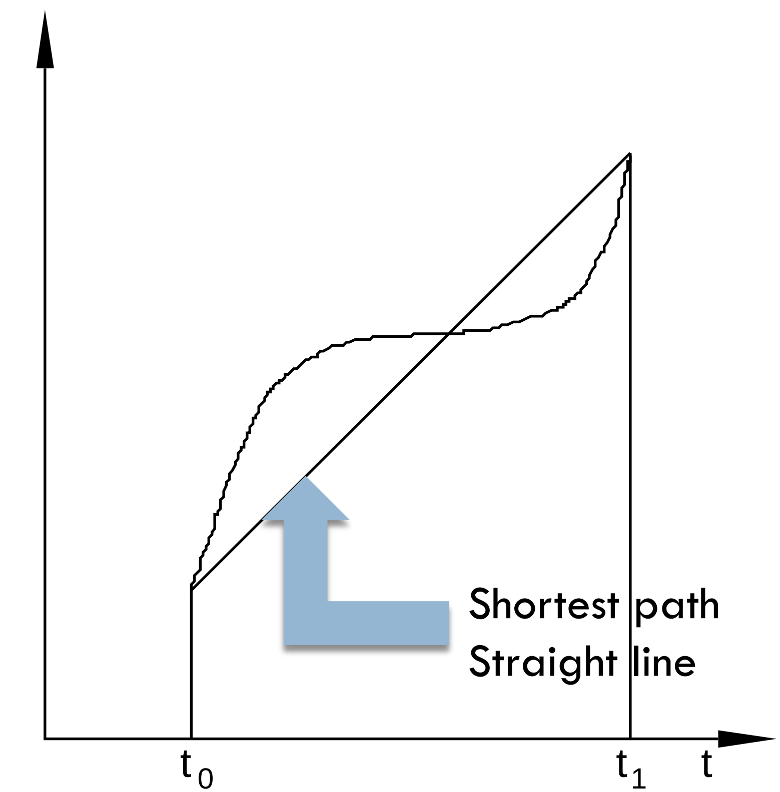
Calculus of Variations

- Consider the following variational problem

$$I[q] = \int_{t_0}^{t_1} L(q, \frac{dq}{dt}, t)\, dt$$

- The Lagrangian *L* is defined as

$$L = \frac{1}{2}\left(\left(\frac{dq_1}{dt}\right)^2 + \left(\frac{dq_2}{dt}\right)^2\right)$$



Lagrangians and Hamilton-Jacobi

- What is the difference between

$$I[q] = \int_{t_0}^{t_1} L(q, \frac{dq}{dt}, t)\, dt \text{ and } S(q(t)) = \int_{t_0}^{t} L(q, \frac{dq}{dt}, t)\, dt ?$$

- Former can be evaluated for any curve, latter only for optimal curve.
- Former has fixed endpoints, latter has variable endpoints.
- Latter leads to Hamilton-Jacobi equation.

The Hamilton-Jacobi equation

- Two variable endpoint problems:

$$S(q(t)) = \int_{t_0}^{t} L(q, \frac{dq}{dt}, t)\,dt \qquad S(q(t+\Delta t)) = \int_{t_0}^{t+\Delta t} L(q, \frac{dq}{dt}, t)\,dt$$

- Both curves *q(t)* and *q(t+Δt)* optimal
- Rate of change of optimal value: $\frac{dS}{dt}$

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial q}\frac{dq}{dt} = L(q, \frac{dq}{dt}, t)$$

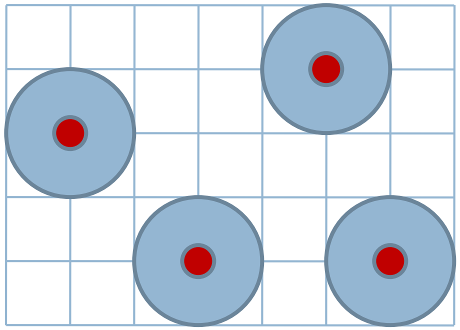
- For Euclidean distance function problem

$$\frac{\partial S}{\partial t} = -\frac{1}{2}\left[\left(\frac{\partial S}{\partial q_1}\right)^2 + \left(\frac{\partial S}{\partial q_2}\right)^2\right] = -\frac{1}{2} \Rightarrow \|\nabla S\| = 1$$

Nonlinear Hamilton-Jacobi (HJ)

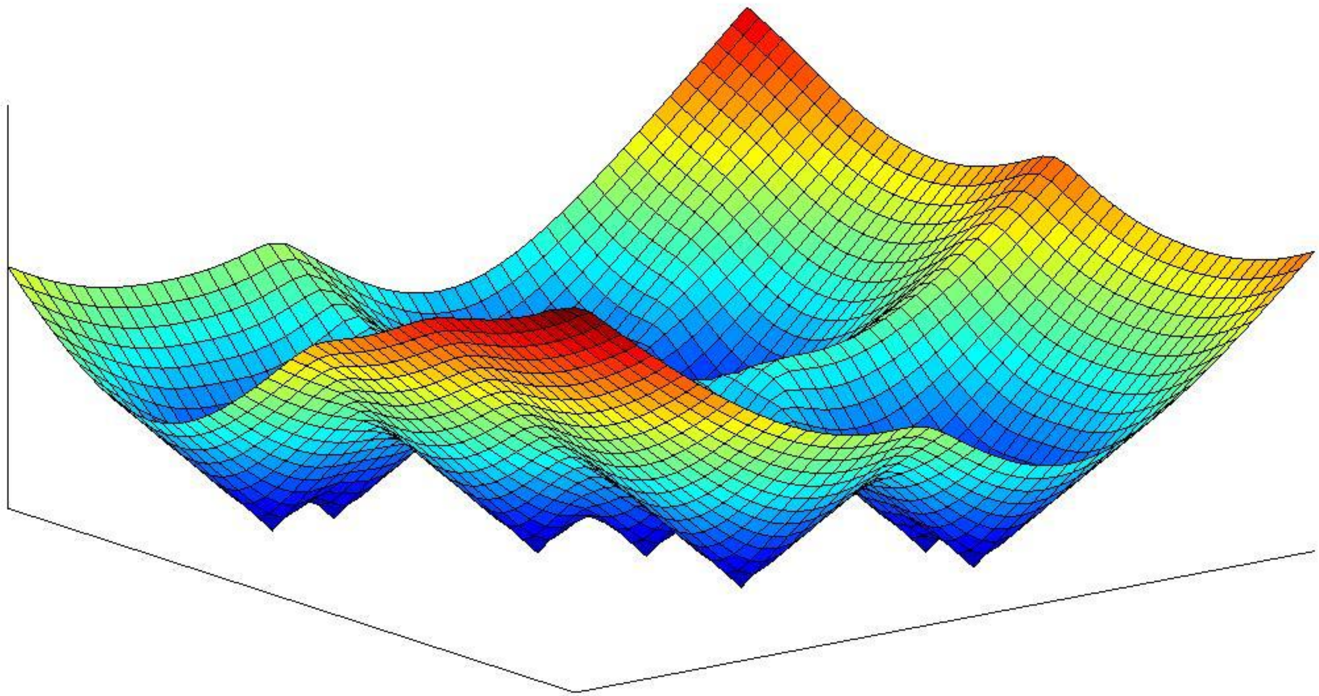
- Euclidean distance function formulated as HJ equation

$$\|\nabla S\| = 1$$

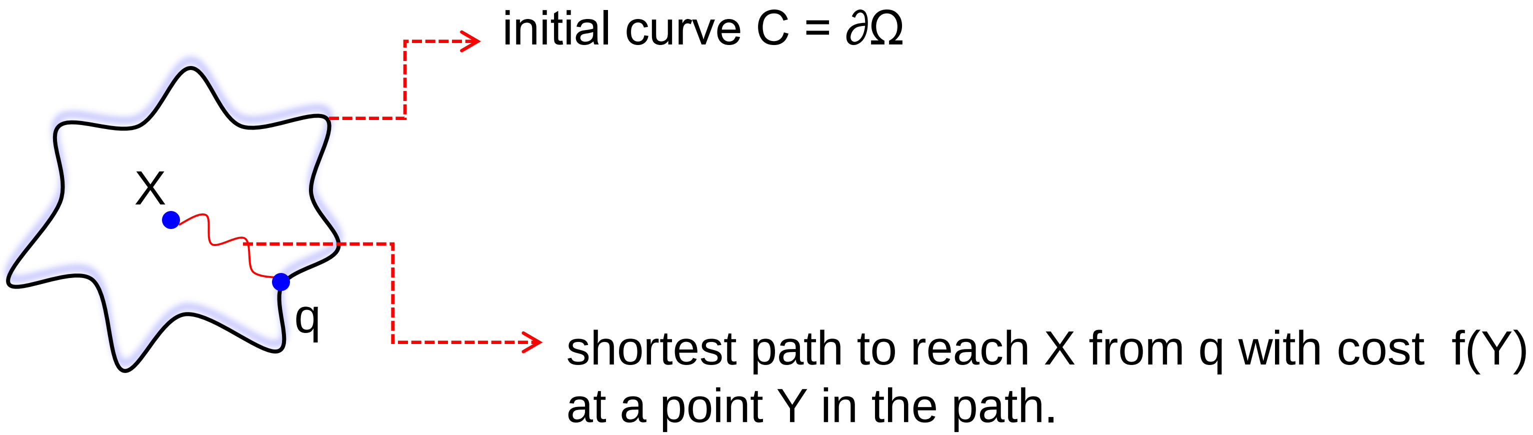


- Fast marching and fast sweeping - efficient solutions
- Zero level set is original shape
- Signed and unsigned distance functions
- Analytical solution unavailable

Visualizing the Distance Transform S



Parallel nature of Hamilton – Jacobi solution



$$S^*(X) = \min_{q \in C} dist(X, q)$$

Computed "simultaneously" for all points X inside the given domain Ω.

The Schrödinger Distance Transform

From Hamilton-Jacobi to Schrödinger

Schrödinger wave equation

- Famous wave equation for particles

$$i\hbar \frac{\partial \psi}{\partial t} = -\hbar^2 \nabla^2 \psi + V\psi$$

- Static Schrödinger equation for free particle

$$-\hbar^2 \nabla^2 \psi + \psi = 0$$

- Solve Schrödinger via Fast Fourier Transform (FFT)
- Quantization: Relationship between nonlinear Hamilton-Jacobi and linear Schrödinger.

Schrödinger and Hamilton-Jacobi

On Hamilton-Jacobi Theory as a Classical Root of Quantum Theory

J. Butterfield[1]

All Souls College
Oxford OX1 4AL

27 February 2003

Abstract

This paper gives a technically elementary treatment of some aspects of Hamilton-Jacobi theory, especially in relation to the calculus of variations. The second half of the paper describes the application to geometric optics, the optico-mechanical analogy and the transition to quantum mechanics. Finally, I report recent work of Holland providing a Hamiltonian formulation of the pilot-wave theory.

Schrödinger Distance Transform

- Forced version of Schrödinger equation

$$-\hbar^2\nabla^2\psi + \psi = \psi_0$$

- ψ_0 is peaked on shape, close to zero elsewhere
- Analytical solution in 2D

$$\psi(X) = \sum_{k=1}^{K} K_0\left(\frac{\|X - Y_k\|}{\hbar}\right)$$

- Schrödinger Distance Transform (SDT)

Modified Bessel function second kind

$$S(X) = -\hbar \log \psi(X) = -\hbar \log \sum_{k=1}^{K} K_0\left(\frac{\|X - Y_k\|}{\hbar}\right)$$

- Fast convolution solution via FFT

Comparison and Computation

Hamilton-Jacobi	Schrödinger
Non-linear	Linear
$\lVert \nabla S \rVert = 1$	$-\hbar^2 \nabla^2 y + y = y_0$
$S(X)=0$ on source	$\psi(X) \approx 1$ on source
Fast marching and Fast sweeping	Fast convolution via Fast Fourier Transform (FFT)
No smoothness control	Control over smoothness using $\hbar$

Linear approximation to the eikonal

- Approx. the eikonal similar to distance transforms.

$$\|\nabla S(X)\| = f(X),\ X \in \Omega$$

- Linear Schrodinger (inhomog. screened Poisson).

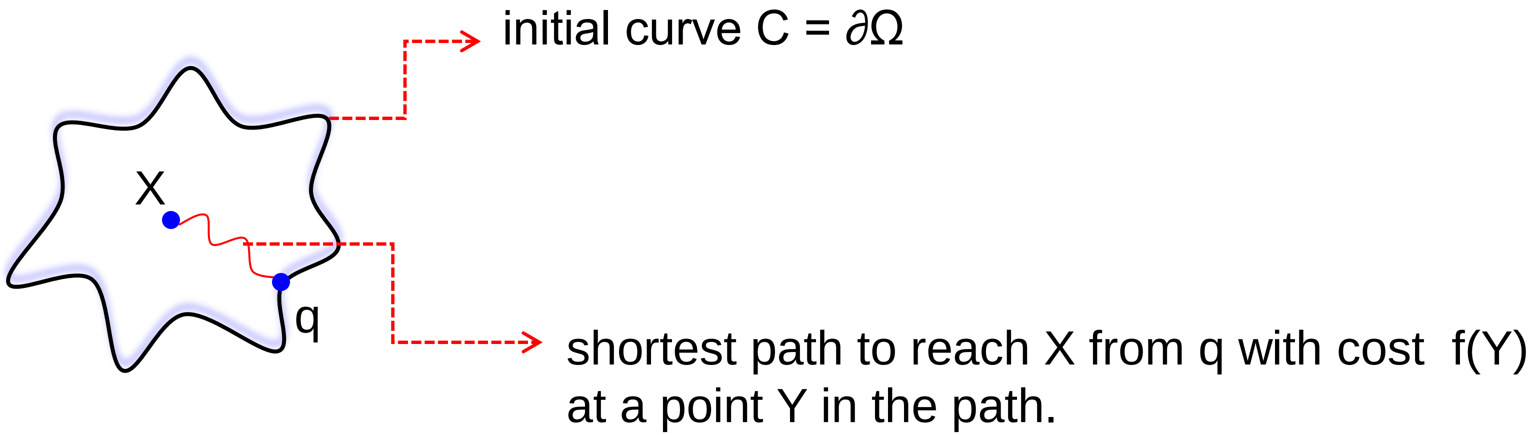
$$-\hbar^2 \nabla^2 \varphi + f^2 \varphi = f_0$$

- Use relation:

$$f(X) = \exp\left(-\frac{S(X)}{\hbar}\right)$$

- Discretize and solve sparse linear system.

Parallel nature of Hamilton – Jacobi solution

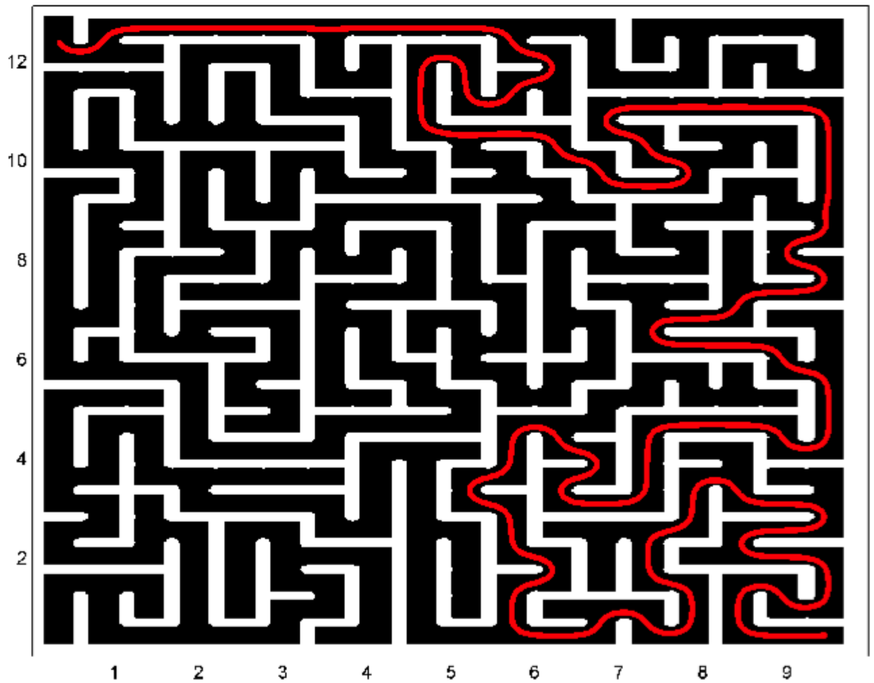


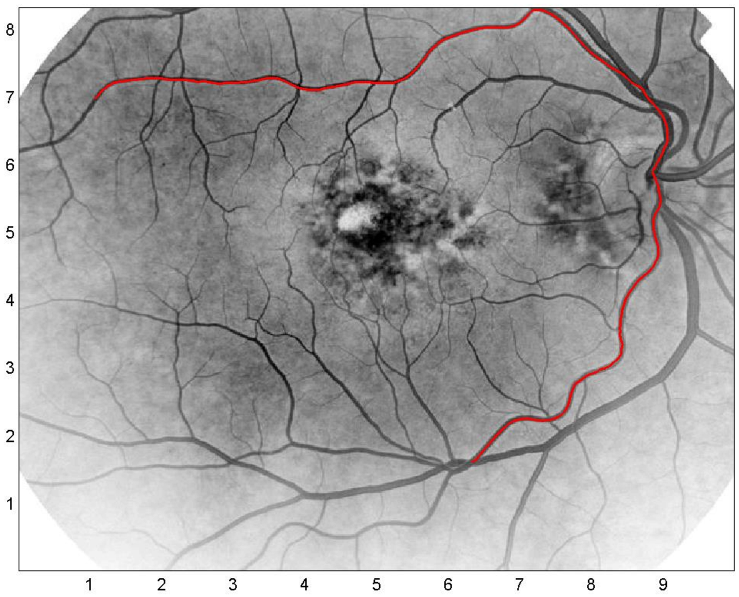
$$S^*(X) = \min_{q \in C} dist(X, q)$$

Computed "simultaneously" for all points X inside the given domain Ω.

Solve linear Schrödinger instead of nonlinear Hamilton-Jacobi

Showcase





The linear differential eq. ecosystem

- Crane *et al.* (Geodesic Heat '12)
- Dimitrov and Zucker (linear diff. eq. '05)
- Gilboa, Sochen, Zeevi (complex diff. eq. '04)
- Ronen Basri and collaborators (Poisson '05)
- Rangarajan, Gurumoorthy, Peter *et al.* (Schrödinger '10)
- Sibel Tari and collaborators (screened Poisson '97!)
- Luminita Vese and collaborators (nonlocal Ambrosio-Tortorelli etc.)

Gradient Density Estimation

Moving from space to frequency

Distance transform gradient density

- Distance transform gradients are unit vectors since $\|\nabla S\| = 1$
- Gradient density – related to HOG – is one dimensional and defined on orientations
- Detail wave function approach to gradient density computation
- Gradient density related to Fourier transform of normalized wave function

HOGging the Distance Transform

- Complex Wave Rep. (CWR) of Distance Transform

$$y(X) = \exp\left\{ i \frac{S(X)}{\hbar} \right\}$$

- Fourier Transform (FT) of CWR

$$F(u) = \text{Fourier Transform}\{y(X)\}$$

- Normalized power spectrum = HOG

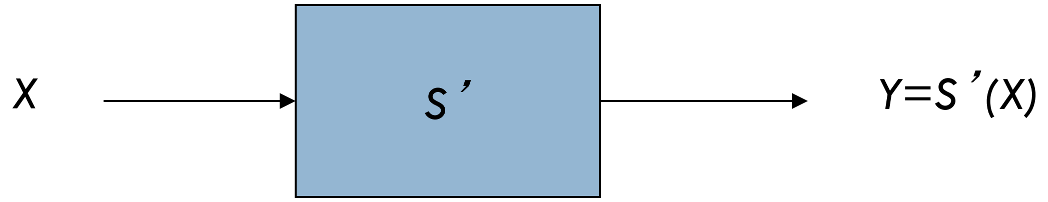
$$P(u) = F(u)\overline{F(u\ )}$$

- Spatial frequencies are gradient histogram bins

$$\nabla S = hu$$

1D Derivative Density Example

- Let *X* be a uniformly distributed random variable on
  $\Omega = [a,b]$.
- Define a random variable $Y = S'(X)$. S' behaves like the transformation function.
- The probability density of *Y* corresponds to the derivative density function of S'.



Derivative density

- The probability density function for the derivative (Y) is given by

$$Q(u_0) = \frac{1}{L} \sum_{S'(x_k)=u_0} \frac{1}{|S''(x_k)|}$$

- Summation is over the set of locations $x_k \in \Omega$ where $S'(x_k) = u_0$.

Stationary phase approximation

Gaussian integral

$$\int_a^b \exp\left(\frac{iS(x)}{\hbar}\right)\exp\left(\frac{-iux}{\hbar}\right)dx$$

$$\approx \exp\left(\frac{iS(x_0)}{\hbar}\right)\exp\left(\frac{-iux_0}{\hbar}\right)\int_a^b \exp\left(\frac{i}{2\hbar}(x-x_0)^2 S''(x_0)\right)dx$$

$$\approx \exp\left(\frac{iS(x_0)-iux_0}{\hbar}\right)\frac{\sqrt{2\pi\hbar}}{\sqrt{|S''(x_0)|}}\exp\left(\pm\frac{i\pi}{4}\right)$$

as $\hbar \to 0$

Integral peaked at $S'(x_0)=u$

Rigorously shown by F.W.J. Olver, Asymptotics and special functions, 1D

R. Wong, Asymptotic Approximations of Integrals, 2D and higher

Power spectrum of exp($iS/\hbar$)

Power spectrum $P(u_0) = F(u_0)\overline{F(u_0)}$

As $\hbar \to 0$ using stationary phase,

$$\approx \frac{1}{L}\sum_{k=1}^{N(u_0)} \frac{1}{|S''(x_k)|}$$

Required density term

$$+\frac{1}{L}\sum_{k=1}^{N(u_0)} \sum_{l=1;l\neq k}^{N(u_0)} \frac{\cos\left(\frac{1}{\hbar}\left[S(x_k)-S(x_l)-u_0(x_k-x_l)\right]+\theta(x_k,x_l)\right)}{\sqrt{|S''(x_k)|}\sqrt{|S''(x_l)|}}$$

$$\theta(x_k,x_l) = 0, \pi/2, -\pi/2$$

Cross terms (CT) killed by integration

Interval measures match

- So,

$$\lim_{\hbar \to 0} \int_{u_0}^{u_0+\alpha} P(u)du = \frac{1}{L} \sum_{k=1}^{N(u_0)} \int_{u_0}^{u_0+\alpha} \frac{1}{|S''(x_k(u))} du$$

- Hence,

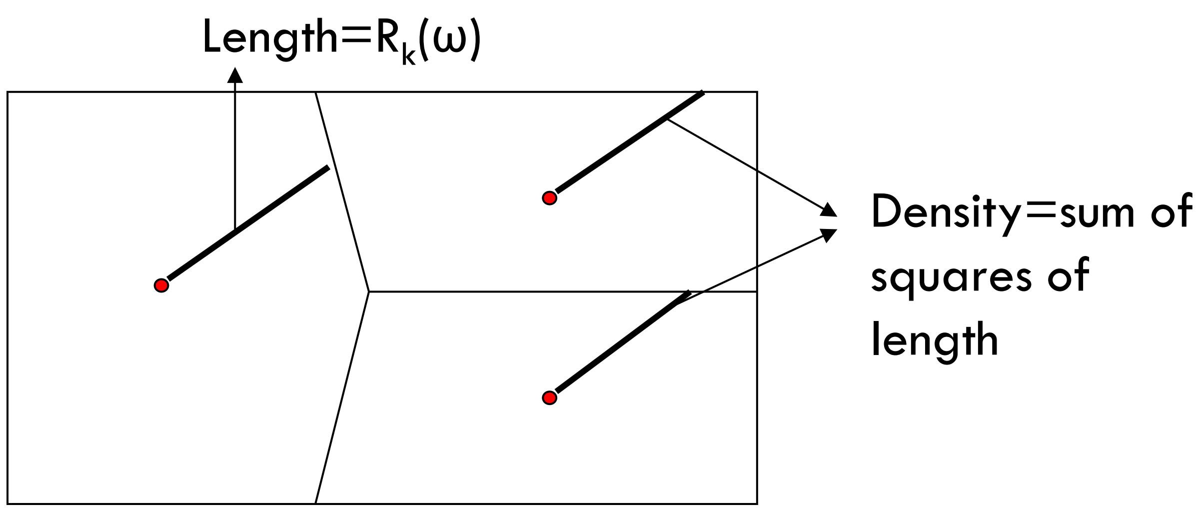
$$\lim_{\alpha \to 0} \frac{1}{\alpha} \lim_{\hbar \to 0} \int_{u_0}^{u_0+\alpha} P(u)du = \frac{1}{L} \sum_{k=1}^{N(u_0)} \frac{1}{|S''(x_k)|}$$

Limit and integration order cannot be swapped

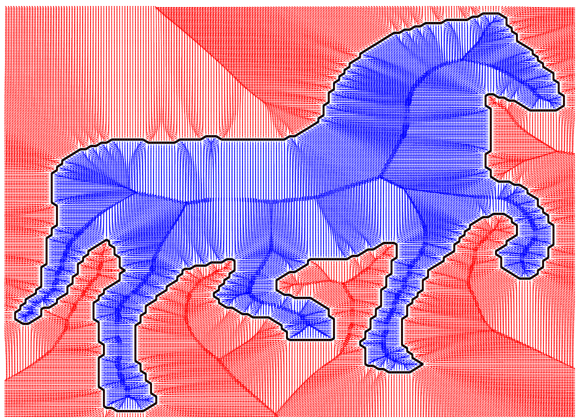
Gradient density

Distance transform gradient density

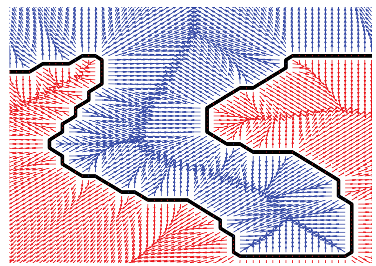
- Distribution function: $$W\left(\theta \le \omega \le \theta + \Delta\theta\right) = \frac{1}{L}\sum_{k=1}^{K}\int_{\theta}^{\theta+\Delta\theta} R_k^2(\omega)\,d\omega$$
- Density function: $$Q(\omega) = \frac{1}{L}\sum_{k=1}^{K} R_k^2(\omega)$$



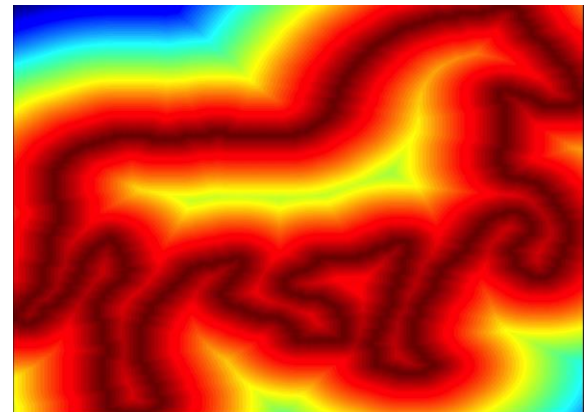
From CWR to HOG



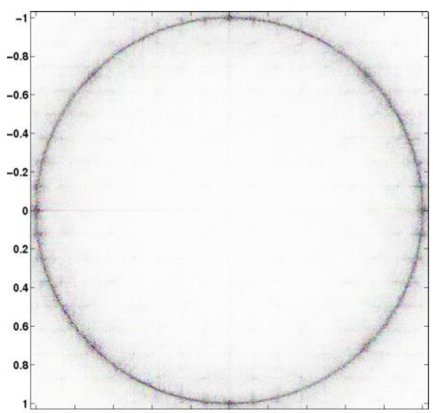
$S(X)$ and ∇S



Zoomed portion

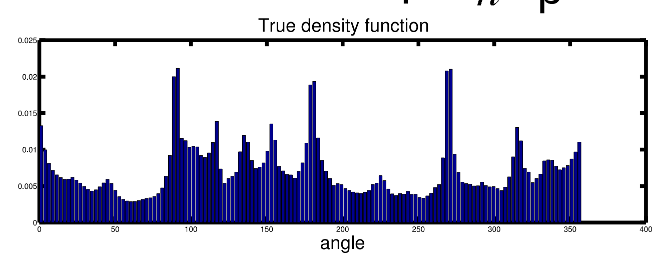


$CWR: \psi(X) = \exp\left\{ i \frac{S(X)}{\hbar} \right\}$

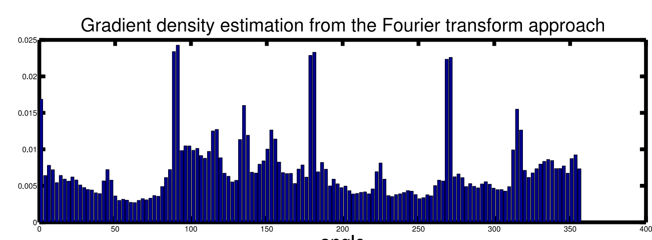


FFT of $\psi(X) : \Psi(u,v)$

HOG



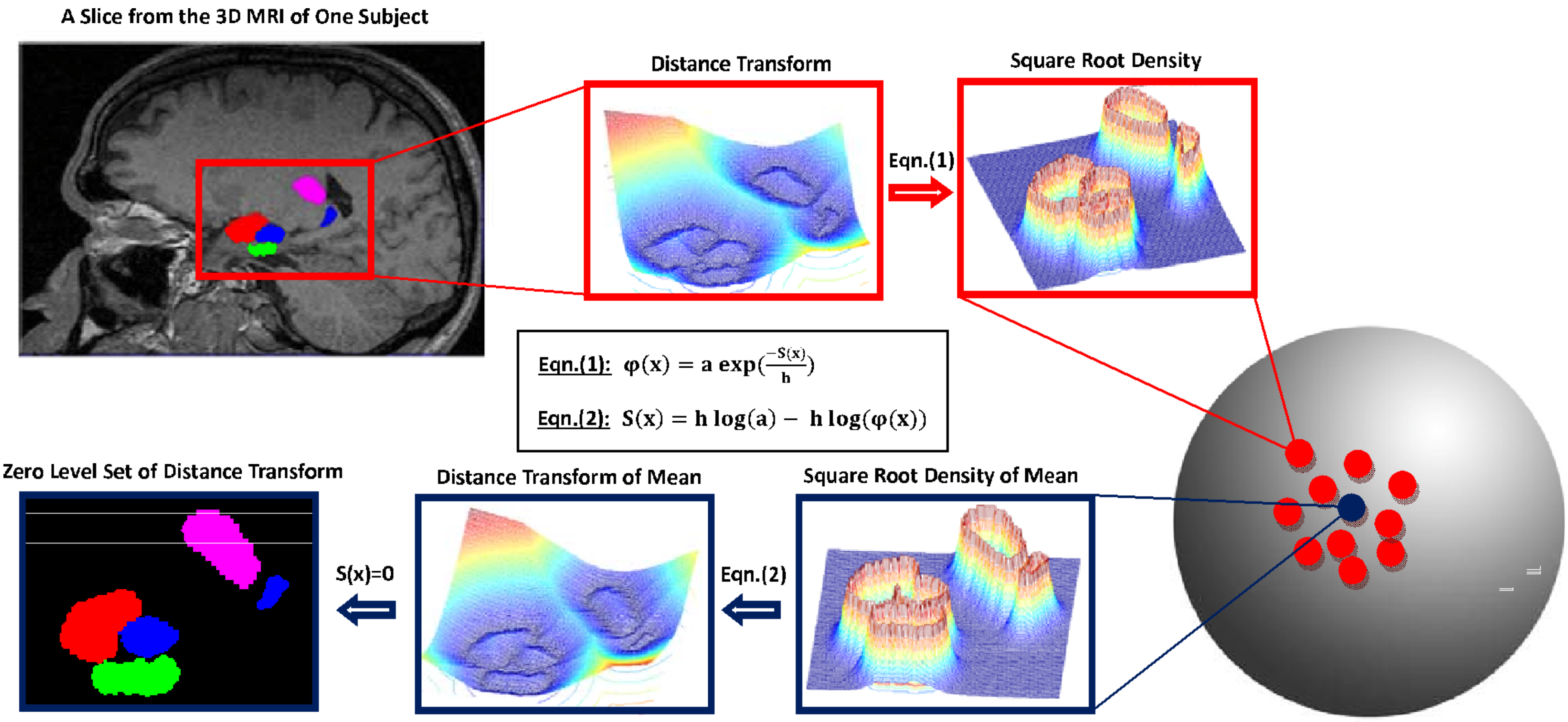
$c\left|\Psi(u,v)\right|^2$



Atlas computation

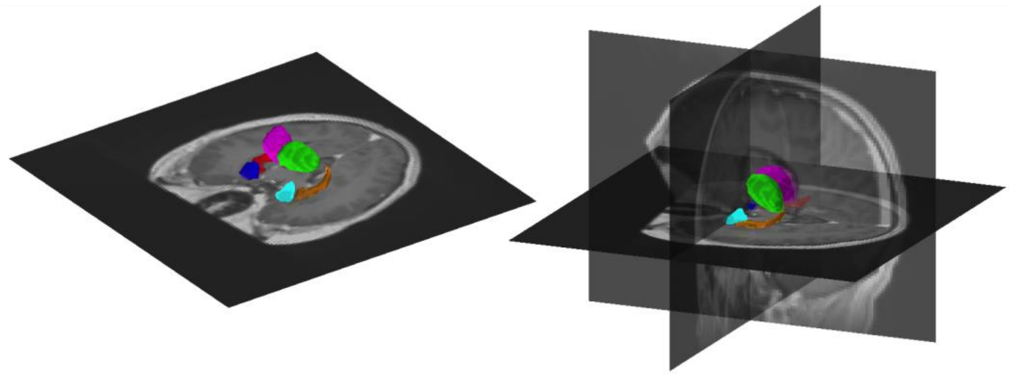
From Schrödinger distance transforms to square-root densities

Atlas Construction

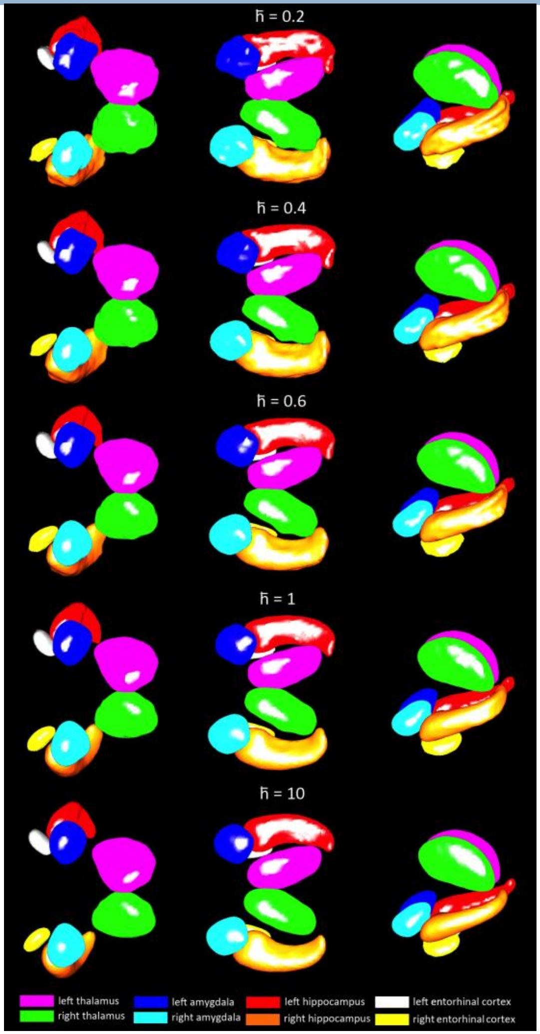


Note: The color for different label is only for visualization purpose.

Shape Complex Atlas



Neuroanatomical structures



Smoother atlas with increasing $\hbar$

Summary

- From calculus of variations to Hamilton-Jacobi.
- From Hamilton-Jacobi to Schrödinger.
- Schrödinger Distance Transform (SDT) by solving linear differential equation instead of nonlinear Hamilton-Jacobi.
- Linear solver ecosystem for the eikonal.
- Normalized power spectrum of $\exp(iS/\hbar)$ converges to distance transform gradient density as $\hbar$ tends to zero. (Interval measures match.)

Acknowledgments

- Collaborators
  - Ting Chen (Ventana), Shape atlases
  - Karthik Gurumoorthy (GE, Bangalore), SDT, HOG
  - Adrian Peter (FIT), eikonal solver
  - Manu Sethi (CISE, UF), SDT, HOG
  - Baba Vemuri (CISE, UF), Shape atlases
- Supported by NSF

Legendre transformation to obtain the Hamiltonian

- By applying Legendre transformation to the Lagrangian i.e. defining

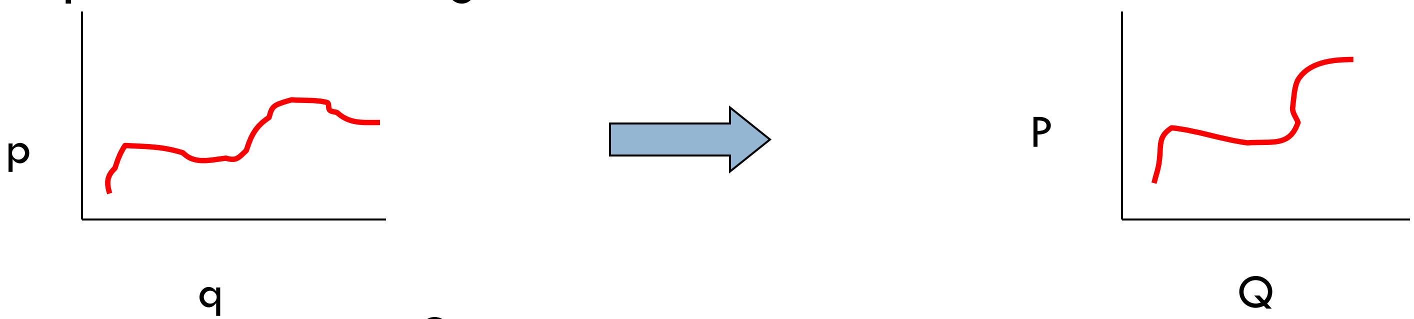
$$p_i = \frac{\partial L}{\partial \frac{dq_i}{dt}} = f^2(q_1, q_2)\frac{dq_i}{dt}$$

and writing $\frac{dq_i}{dt} = \frac{dq_i}{dt}(q, p, t)$ we get the Hamiltonian to be

$$H(q, p) = \sum p_i \frac{dq_i}{dt} - L = \frac{1}{2f^2}\left((p_1)^2 + (p_2)^2\right)$$

Canonical transformation to obtain the Hamilton-Jacobi equation

- The Hamilton-Jacobi equation is obtained via a canonical transformation of the Hamiltonian.
- In classical mechanics, a canonical transformation is defined as a change of variables which leaves the form of the Hamilton equations unchanged.



$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\dot{Q}_i = \frac{\partial K}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial K}{\partial Q_i}$$

Type 2 Canonical transformation

- For a type 2 canonical transformation, we have

$$\sum p_i \frac{dq_i}{dt} - H = \sum P_i \frac{dQ_i}{dt} - K(Q_1, Q_2, P_1, P_2) + \frac{dF}{dt}$$

where

$$F = -\sum Q_i P_i + S(q, P, t)$$

$$\frac{dF}{dt} = -\sum\left(\frac{dQ_i}{dt} P_i + Q_i \frac{dP_i}{dt}\right) + \frac{\partial S}{\partial t} + \sum\left(\frac{\partial S}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial S}{\partial P_i}\frac{dP_i}{dt}\right)$$

Hamilton-Jacobi formulation contd.

- Equating and canceling out terms, we get

$$p_i = \frac{\partial S}{\partial q_i}$$

$$Q_i = \frac{\partial S}{\partial P_i}$$

$$K = H + \frac{\partial S}{\partial t}$$

Hamilton-Jacobi equation

- When we pick a particular type 2 canonical transformation where in K=0, we get

$$\frac{\partial S}{\partial t} + H\left(q_1, q_2, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}\right) = 0$$

- Substituting $p_i = \frac{\partial S}{\partial q_i}$ in, $H = \frac{1}{2f^2}\left((p_1)^2 + (p_2)^2\right)$

$$\frac{\partial S}{\partial t} + \frac{\|\nabla S\|^2}{2f^2} = 0$$

Hamilton-Jacobi formulation contd.

- Since the Hamiltonian *H* is independent of time, by separation of variables

  S(X,t)=S*(X)-Et,

  S* satisfies the relation

$$\frac{1}{2f^2}\left[\left(\frac{\partial S^*}{\partial q_1}\right)^2+\left(\frac{\partial S^*}{\partial q_2}\right)^2\right]=E.$$

- Setting E to be ½ ,we get

$$\left\|\nabla S*\right\|^2 = f^2.$$

Modeling Fluctuating DF (1)

$$E(\omega) = \sum_{(i,j)\in\Omega} E_{Reg}(\omega_{i,j}) + \lambda \sum_{(i,j)\in\partial\Omega} E_{Bdy}(\omega_{i,j})$$

with $\omega(\mathbf{x}) = 0$ for $\mathbf{x} = (x,y) \in \partial\Omega$

$$E_{Reg} = \underbrace{\sum_{(i,j)\in\Omega} E^{G}_{Reg}(\omega_{i,j})}_{E^{G}_{Reg}\,:\,\text{global}} + \beta \underbrace{\sum_{(i,j)\in\Omega} E^{L}_{Reg}(\omega_{i,j})}_{E^{L}_{Reg}\,:\,\text{local}}$$

$$\arg\min_{v_\rho} \iint_\Omega \left[\rho \underbrace{|\nabla v_\rho(\mathbf{x})|^2}_{\text{local interaction}} + \frac{1}{\rho} \underbrace{(v_\rho(\mathbf{x}) - 1)^2}_{\text{boundary/interior separation}} \right] \mathrm{d}x\, \mathrm{d}y$$

$$\arg\min_{\omega} \iint_\Omega \sqrt{O(|\Omega|)}\,|\nabla\omega(\mathbf{x})|^2 + \frac{1}{\sqrt{O(|\Omega|)}} (\omega(\mathbf{x}) - t(\mathbf{x}))^2$$

Screened Poisson
(Tari, Shah and Pien 1996)

$$\left(\triangle - \frac{1}{O(|\Omega|)} \right) \omega(x,y) = \frac{1}{O(|\Omega|)}\, t(x,y)$$

$$\frac{\mathrm{d}\omega_{i,j}(\tau)}{\mathrm{d}\tau} = -\frac{\partial E}{\partial \omega_{i,j}}$$
$$= \mathbb{L}_*(\omega_{i,j}) - \frac{1}{O(|\Omega|)} \sum_{(k,l)\in\Omega} \omega_{k,l} - \frac{1}{O(|\Omega|)} \omega_{i,j} + \bar{t}_{i,j}$$

$$\left(\triangle - \frac{1}{O(|\Omega|)} \right) \omega(x,y) - \iint \omega(\alpha,\beta)\, \mathrm{d}\alpha\, \mathrm{d}\beta = -\frac{1}{O(|\Omega|)}\, t(x,y)$$