NEW VARIATIONAL METHODS IN COMPUTER VISION

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Roadmap

- Distance transforms
- Calculus of variations
- Lagrangians and Hamiltonians
- Nonlinear Hamilton-Jacobi equation
- Linear Schrödinger wave equation
- Approximating the eikonal via linear solvers
- The method of stationary phase
- Distance transform gradient density
Distance transforms

- Distance transforms: Ubiquitous shape representation
- Level sets, fast marching methods etc. standard tropes
- Relationship to classical physics (Hamilton-Jacobi theory) well established
Euclidean distance functions

- Given a point-set $\{Y_k\}_{k=1}^K$ and grid points $X$, the Euclidean distance function is

$$S(X) = \min_k \| X - Y_k \|$$
Calculus of Variations

Consider the following variational problem

\[ I[q] = \int_{t_0}^{t_1} L(q, \frac{dq}{dt}, t) \, dt \]

The Lagrangian \( L \) is defined as

\[ L = \frac{1}{2} \left( \frac{dq_1}{dt} \right)^2 + \frac{1}{2} \left( \frac{dq_2}{dt} \right)^2 + \ldots \]

Shortest path

Straight line
What is the difference between

\[ I[q] = \int_{t_0}^{t_1} L(q, \frac{dq}{dt}, t) \, dt \quad \text{and} \quad S(q(t)) = \int_{t_0}^{t} L(q, \frac{dq}{dt}, t) \, dt \]

- Former can be evaluated for any curve, latter only for optimal curve.
- Former has fixed endpoints, latter has variable endpoints.
- Latter leads to Hamilton-Jacobi equation.
The Hamilton-Jacobi equation

- Two variable endpoint problems:
  
  \[ S(q(t)) = \int_{t_0}^{t} L(q, \frac{dq}{dt}, t) \, dt \quad \text{and} \quad S(q(t + \Delta t)) = \int_{t_0}^{t+\Delta t} L(q, \frac{dq}{dt}, t) \, dt \]

- Both curves \( q(t) \) and \( q(t+\Delta t) \) optimal

- Rate of change of optimal value:
  
  \[ \frac{dS}{dt} = \frac{S}{t} + \frac{S}{q} \frac{dq}{dt} = L(q, \frac{dq}{dt}, t) \]

- For Euclidean distance function problem
  
  \[ \frac{\partial S}{\partial t} = \frac{1}{2} \left[ \left( \frac{\partial S}{\partial q_1} \right)^2 + \left( \frac{\partial S}{\partial q_2} \right)^2 \right] = \frac{1}{2} \Rightarrow \|\nabla S\| = 1 \]
Nonlinear Hamilton-Jacobi (HJ)

- Euclidean distance function formulated as HJ equation
  \[ \| \nabla S \| = 1 \]

- Fast marching and fast sweeping - efficient solutions
- Zero level set is original shape
- Signed and unsigned distance functions
- Analytical solution unavailable

From Calculus of Variations to Hamilton-Jacobi
Visualizing the Distance Transform $S$
Parallel nature of Hamilton – Jacobi solution

Initial curve \( C = \partial \Omega \)

Shortest path to reach \( X \) from \( q \) with cost \( f(Y) \) at a point \( Y \) in the path.

\[
S^*(X) = \min_{q \in C} \text{dist}(X, q)
\]

Computed “simultaneously” for all points \( X \) inside the given domain \( \Omega \).
The Schrödinger Distance Transform

From Hamilton-Jacobi to Schrödinger
Schrödinger wave equation

- Famous wave equation for particles
  \[ i\hbar \frac{\partial \psi}{\partial t} = -\hbar^2 \nabla^2 \psi + V\psi \]

- Static Schrödinger equation for free particle
  \[ -\hbar^2 \nabla^2 \psi + \psi = 0 \]

- Solve Schrödinger via Fast Fourier Transform (FFT)

- **Quantization**: Relationship between nonlinear Hamilton-Jacobi and linear Schrödinger.
On Hamilton-Jacobi Theory as a Classical Root of Quantum Theory

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Abstract

This paper gives a technically elementary treatment of some aspects of Hamilton-Jacobi theory, especially in relation to the calculus of variations. The second half of the paper describes the application to geometric optics, the optico-mechanical analogy and the transition to quantum mechanics. Finally, I report recent work of Holland providing a Hamiltonian formulation of the pilot-wave theory.
Schrödinger Distance Transform

- Forced version of Schrödinger equation
  \[ \tilde{\hbar}^2 \nabla^2 + = 0 \]
  \(0\) is peaked on shape, close to zero elsewhere

- Analytical solution in 2D
  \[ (X) = \sum_{k=1}^{K} K_0 \frac{\|X - Y_k\|}{\tilde{\hbar}} \]

- Schrödinger Distance Transform (SDT)
  Modified Bessel function second kind
  \[ S(X) = \tilde{\hbar} \log (X) = \tilde{\hbar} \log \sum_{k=1}^{K} K_0 \frac{\|X - Y_k\|}{\tilde{\hbar}} \]

- Fast convolution solution via FFT
Comparison and Computation

<table>
<thead>
<tr>
<th>Hamilton-Jacobi</th>
<th>Schrödinger</th>
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<td>Non-linear</td>
<td>Linear</td>
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<tr>
<td>$|\nabla S| = 1$</td>
<td>$\hbar^2 \nabla^2 + = 0$</td>
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<tr>
<td>$S(X) = 0$ on source</td>
<td>$\psi(X) \approx 1$ on source</td>
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<td>Fast marching and Fast sweeping</td>
<td>Fast convolution via Fast Fourier Transform (FFT)</td>
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<td>No smoothness control</td>
<td>Control over smoothness using $\hbar$</td>
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</table>
Linear approximation to the eikonal

- Approx. the eikonal similar to distance transforms.

\[ \| \nabla S(X) \| = f(X), \quad X \in W \]

- Linear Schrodinger (inhomog. screened Poisson).

\[ \hbar^2 \nabla^2 + f^2 = 0 \]

- Use relation:

\[ (X) = \exp \left( \frac{S(X)}{\hbar} \right) \]

- Discretize and solve sparse linear system.
Parallel nature of Hamilton – Jacobi solution

initial curve $C = \partial \Omega$

shortest path to reach $X$ from $q$ with cost $f(Y)$ at a point $Y$ in the path.

$$S^*(X) = \min_{q \in C} \text{dist}(X, q)$$

Computed “simultaneously” for all points $X$ inside the given domain $\Omega$.

Solve linear Schrödinger instead of nonlinear Hamilton-Jacobi.
Showcase
The linear differential eq. ecosystem

- Crane et al. (Geodesic Heat ‘12)
- Dimitrov and Zucker (linear diff. eq. ‘05)
- Gilboa, Sochen, Zeevi (complex diff. eq. ‘04)
- Ronen Basri and collaborators (Poisson ‘05)
- Rangarajan, Gurumoorthy, Peter et al. (Schrödinger ‘10)
- Sibel Tari and collaborators (screened Poisson ‘97!)
- Luminita Vese and collaborators (nonlocal Ambrosio-Tortorelli etc.)
Gradient Density Estimation

Moving from space to frequency
Distance transform gradient density

- Distance transform gradients are unit vectors since
  \[ \| \nabla S \| = 1 \]
- Gradient density — related to HOG — is one dimensional and defined on orientations
- Detail wave function approach to gradient density computation
- Gradient density related to Fourier transform of normalized wave function
HOGging the Distance Transform

- Complex Wave Rep. (CWR) of Distance Transform
  \( (X) = \exp i \frac{S(X)}{\hbar} \)

- Fourier Transform (FT) of CWR
  \( F(u) = \text{Fourier Transform} \{ (X) \} \)

- Normalized power spectrum = HOG
  \( P(u) = \overline{F(u)F(u')} \)

- Spatial frequencies are gradient histogram bins
  \( \nabla S = hu \)
Let $X$ be a uniformly distributed random variable on $\Omega = [a,b]$.

Define a random variable $Y = S'(X)$. $S'$ behaves like the transformation function.

The probability density of $Y$ corresponds to the derivative density function of $S'$.
The probability density function for the derivative \((Y)\) is given by

\[
Q(u_0) = \frac{1}{L} \sum_{S'(x_k) = u_0} \frac{1}{|S'''(x_k)|}
\]

Summation is over the set of locations \(x_k \in \Omega\) where \(S'(x_k) = u_0\).
Stationary phase approximation

\[
\begin{align*}
\int_a^b \exp \left( \frac{iS(x)}{\hbar} \right) \div \exp \left( \frac{iux}{\hbar} \right) dx \\
\exp \left( \frac{iS(x_0)}{\hbar} \right) \div \exp \left( \frac{iux_0}{\hbar} \right) \int_a^b \exp \left( \frac{i}{2\hbar} (x-x_0)^2 S''(x_0) \right) dx \\
\exp \left( \frac{iS(x_0)}{\hbar} \right) \frac{iux_0}{\hbar} \div \frac{\sqrt{2}}{\hbar} \frac{1}{\sqrt{|S''(x_0)|}} \exp \left( \frac{\pm i}{4} \right) \div \text{as } \hbar \to 0
\end{align*}
\]

Integral peaked at \( S'(x_0) = u \)

Rigorously shown by F.W.J. Olver, Asymptotics and special functions, 1D

R. Wong, Asymptotic Approximations of Integrals, 2D and higher
Power spectrum of $\exp(iS/\hbar)$

Power spectrum

$$P(u_0) = F(u_0) \overline{F(u_0)}$$

As $\hbar \to 0$ using stationary phase,

$$\approx \frac{1}{L} \sum_{k=1}^{N(u_0)} \frac{1}{|S'''(x_k)|}$$

$$+ \frac{1}{L} \sum_{k=1}^{N(u_0)} \sum_{l=1; l \neq k}^{N(u_0)} \frac{\cos \left( \frac{1}{\hbar} \left[ S(x_k) - S(x_l) - u_0 (x_k - x_l) \right] + \theta(x_k, x_l) \right)}{\sqrt{|S'''(x_k)|} \sqrt{|S'''(x_l)|}}$$

$$\theta(x_k, x_l) = 0, \frac{\pi}{2}, -\frac{\pi}{2}$$

Required density term

Cross terms (CT) killed by integration
Interval measures match

- So,

\[ \lim_{h \to 0} \int_{u_0}^{u_0+\alpha} P(u) \, du = \frac{1}{L} \sum_{k=1}^{N(u_0)} \int_{u_0}^{u_0+\alpha} \frac{1}{|S''(x_k(u))|} \, du \]

- Hence,

\[ \lim_{\alpha \to 0} \lim_{h \to 0} \int_{u_0}^{u_0+\alpha} P(u) \, du = \frac{1}{L} \sum_{k=1}^{N(u_0)} \frac{1}{|S''(x_k)|} \]

Limit and integration order cannot be swapped

Gradient density
Distance transform gradient density

- Distribution function:
  \[ W(\theta \leq \omega \leq \theta + \Delta \theta) = \frac{1}{L} \sum_{k=1}^{K} \int_{\theta}^{\theta+\Delta\theta} R_k^2(\omega) d\omega \]

- Density function:
  \[ Q(\omega) = \frac{1}{L} \sum_{k=1}^{K} R_k^2(\omega) \]

Length = \( R_k(\omega) \)

Density = sum of squares of length
From CWR to HOG

\[ S(X) \text{ and } \nabla S \]

Zoomed portion

\[ CWR: \quad (X) = \exp \left( i \frac{S(X)}{\hbar} \right) \]

\[ \text{FFT of } (X): \quad (u, v) \]

\[ c \left| \langle u, v \rangle \right|^2 \]

HOG
From Schrödinger distance transforms to square-root densities
Atlas Construction

A Slice from the 3D MRI of One Subject

Distance Transform

Square Root Density

Eqn. (1): \( \varphi(x) = a \exp\left(-\frac{s(x)}{h}\right) \)

Eqn. (2): \( S(x) = h \log(a) - h \log(\varphi(x)) \)

Zero Level Set of Distance Transform

Distance Transform of Mean

Square Root Density of Mean

Note: The color for different label is only for visualization purpose.
Shape Complex Atlas

Neuroanatomical structures

Smoother atlas with increasing $\hat{n}$
Summary

- From calculus of variations to Hamilton-Jacobi.
- From Hamilton-Jacobi to Schrödinger.
- Schrödinger Distance Transform (SDT) by solving linear differential equation instead of nonlinear Hamilton-Jacobi.
- Linear solver ecosystem for the eikonal.
- Normalized power spectrum of $\exp(iS/\hbar)$ converges to distance transform gradient density as $\hbar$ tends to zero. (Interval measures match.)
Acknowledgments

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  - Baba Vemuri (CISE, UF), Shape atlases

- Supported by NSF
Legendre transformation to obtain the Hamiltonian

- By applying Legendre transformation to the Lagrangian i.e. defining

\[ p_i = \frac{\partial L}{\partial \dot{q}_i} = f^2(q_1, q_2) \frac{dq_i}{dt} \]

and writing \( \frac{dq_i}{dt} = \frac{dq_i}{dt}(q, p, t) \) we get the Hamiltonian to be

\[
H(q, p) = \sum p_i \frac{dq_i}{dt} - L = \frac{1}{2} f^2 \left( (p_1)^2 + (p_2)^2 \right)
\]
Canonical transformation to obtain the Hamilton-Jacobi equation

- The Hamilton-Jacobi equation is obtained via a canonical transformation of the Hamiltonian.
- In classical mechanics, a canonical transformation is defined as a change of variables which leaves the form of the Hamilton equations unchanged.

\[
\begin{align*}
\dot{q}_i &= \frac{\partial H}{\partial p_i}, \\
\dot{p}_i &= -\frac{\partial H}{\partial q_i}
\end{align*}
\]

\[
\begin{align*}
\dot{Q}_i &= \frac{\partial K}{\partial P_i}, \\
\dot{P}_i &= -\frac{\partial K}{\partial Q_i}
\end{align*}
\]
Type 2 Canonical transformation

For a type 2 canonical transformation, we have

\[ \sum p_i \frac{dq_i}{dt} - H = \sum P_i \frac{dQ_i}{dt} - K(Q_1, Q_2, P_1, P_2) + \frac{dF}{dt} \]

where

\[ F = -\sum Q_i P_i + S(q, P, t) \]

\[ \frac{dF}{dt} = -\sum \left( \frac{dQ_i}{dt} P_i + Q_i \frac{dP_i}{dt} \right) + \frac{\partial S}{\partial t} + \sum \left( \frac{\partial S}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial S}{\partial P_i} \frac{dP_i}{dt} \right) \]
Hamilton-Jacobi formulation contd.

- Equating and canceling out terms, we get

\[ p_i = \frac{\partial S}{\partial q_i} \]
\[ Q_i = \frac{\partial S}{\partial P_i} \]
\[ K = H + \frac{\partial S}{\partial t} \]
Hamilton-Jacobi equation

- When we pick a particular type 2 canonical transformation where $K=0$, we get

$$\frac{\partial S}{\partial t} + H\left(q_1, q_2, \frac{\partial S}{\partial q_1}, \frac{\partial S}{\partial q_2}\right) = 0$$

- Substituting

$$p_i = \frac{\partial S}{\partial q_i} \text{ in,} \quad H = \frac{1}{2f^2}\left((p_1)^2 + (p_2)^2\right)$$

$$\frac{\partial S}{\partial t} + \frac{\|\nabla S\|^2}{2f^2} = 0$$
Hamilton-Jacobi formulation contd.

- Since the Hamiltonian $H$ is independent of time, by separation of variables
  \[ S(X,t) = S^*(X) - Et, \]
  $S^*$ satisfies the relation
  \[
  \frac{1}{2f^2} \left[ \left( \frac{\partial S^*}{\partial q_1} \right)^2 + \left( \frac{\partial S^*}{\partial q_2} \right)^2 \right] = E.
  \]

- Setting $E$ to be $\frac{1}{2}$, we get
  \[
  \left\| \nabla S^* \right\|^2 = f^2.
  \]
Modeling Fluctuating DF (1)

\[ E(\omega) = \sum_{(i,j) \in \Omega} E_{\text{Reg}}(\omega_{i,j}) + \lambda \sum_{(i,j) \in \partial \Omega} E_{\text{Bdy}}(\omega_{i,j}) \]

with \( \omega(x) = 0 \) for \( x = (x, y) \in \partial \Omega \)

\[ E_{\text{Reg}} = \sum_{(i,j) \in \Omega} E_{\text{Reg}}^G(\omega_{i,j}) + \beta \sum_{(i,j) \in \Omega} E_{\text{Reg}}^L(\omega_{i,j}) \]

\( E_{\text{Reg}}^G : \text{global} \)
\( E_{\text{Reg}}^L : \text{local} \)

\[ \arg \min_{v_\rho} \int_{\Omega} \left[ \rho |\nabla v_\rho(x)|^2 + \frac{1}{\rho} \frac{(v_\rho(x) - 1)^2}{\sqrt{O(|\Omega|)}} \right] dx \, dy \]

(\( \rho \): local interaction, \( \partial \Omega \): boundary/interior separation)

\[ \arg \min_{\omega} \int_{\Omega} \sqrt{O(|\Omega|)} |\nabla \omega(x)|^2 + \frac{1}{\sqrt{O(|\Omega|)}} (\omega(x) - t(x))^2 \]

\( \Delta - \frac{1}{O(|\Omega|)} \) \( \omega(x, y) = \frac{1}{O(|\Omega|)} t(x, y) \)

\( \left( \Delta - \frac{1}{O(|\Omega|)} \right) \omega(x, y) = \int \int \omega(\alpha, \beta) \, d\alpha \, d\beta = -\frac{1}{O(|\Omega|)} t(x, y) \)