Tutorial on Surface Ricci Flow, Theory, Algorithm and Application

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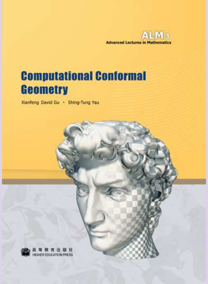
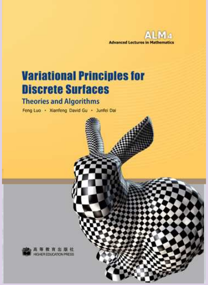
Collaborators

The work is collaborated with Shing-Tung Yau, Feng Luo, Ronald Lok Ming Lui and many other mathematicians, computer scientists and medical doctors.

Books

The theory, algorithms and sample code can be found in the following books.





You can find them in the book store.

"Ricci Flow for Shape Analysis and Surface Registration - Theories, Algorithms and Applications", Springer, 2013.

Lecture Notes

1. Detailed lecture notes can be found at:

   http://www.cs.sunysb.edu/˜gu/lectures/index.html

2. Binary code and demos can be found at:

   http://www.cs.sunysb.edu/˜gu/software/index.html

3. Source code and data sets:

   http://www.cs.sunysb.edu/˜gu/software/index.html

Conformal Mapping

Definition (Conformal Mapping)

Suppose $(S_1, \mathbf{g}_1)$ and $(S_2, \mathbf{g}_2)$ are two surfaces with Riemannian metrics. A conformal mapping $\phi : S_1 \to S_2$ is a diffeomorphism, such that

$$\phi^* \mathbf{g}_2 = e^{2\lambda} \mathbf{g}_1.$$

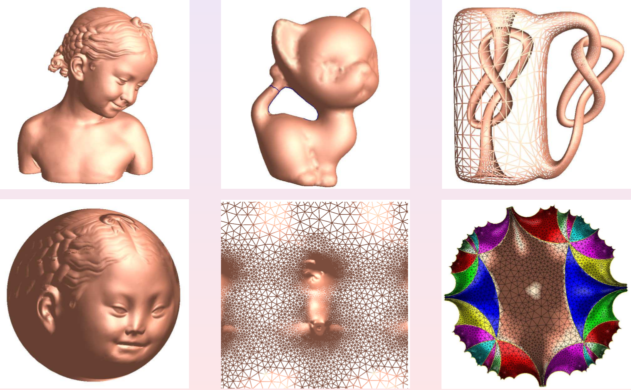
Conformal Mapping

Properties

Conformal mappings preserve infinitesimal circles, and preserve angles.

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ conformal to $\mathbf{g}$ which has constant Gauss curvature.

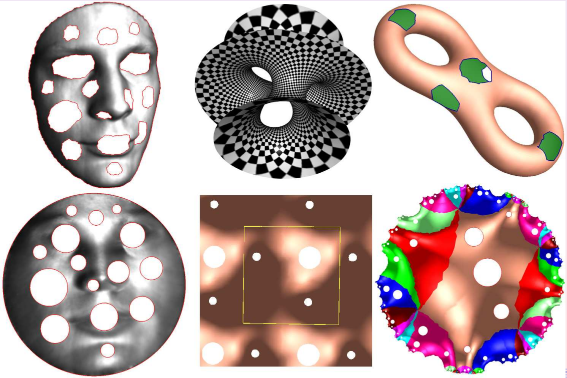


Spherical Euclidean Hyperbolic

Uniformization

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ be a compact 2-dimensional Riemannian manifold with finite number of boundary components. Then there is a metric $\tilde{\mathbf{g}}$ conformal to $\mathbf{g}$ which has constant Gauss curvature, and constant geodesic curvature.



Yamabe Problem

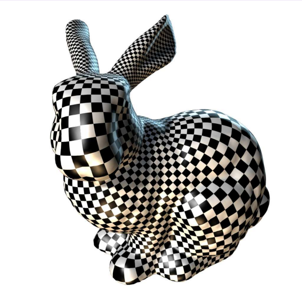
Isothermal Coordinates

Relation between conformal structure and Riemannian metric

Isothermal Coordinates

A surface M with a Riemannian metric $\mathbf{g}$, a local coordinate system (u,v) is an isothermal coordinate system, if

$$\mathbf{g} = e^{2\lambda(u,v)}(du^2 + dv^2).$$



Gaussian Curvature

Suppose $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$K = -\Delta_{\mathbf{g}}\lambda = -\frac{1}{e^{2\lambda}}\Delta\lambda,$$

where

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}$$

Conformal Metric Deformation

Definition

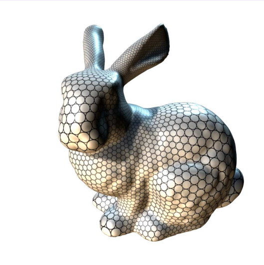
Suppose M is a surface with a Riemannian metric,

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

Suppose $\lambda : \Sigma \to \mathbb{R}$ is a function defined on the surface, then $e^{2\lambda}\mathbf{g}$ is also a Riemannian metric on Σ and called a conformal metric. λ is called the conformal factor.

$$\mathbf{g} \to e^{2\lambda}\mathbf{g}$$

Conformal metric deformation.



Angles are invariant measured by conformal metrics.

Curvature and Metric Relations

Yamabi Equation

Suppose $\bar{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ is a conformal metric on the surface, then the Gaussian curvature on interior points are

$$\bar{K} = e^{-2\lambda}(-\Delta_{\mathbf{g}}\lambda + K),$$

geodesic curvature on the boundary

$$\bar{k_g} = e^{-\lambda}(-\partial_n\lambda + k_g).$$

Surface Ricci Flow

Key Idea

Because

$$K = -\Delta_{\mathbf{g}}\lambda,$$

Let

$$\frac{d\lambda}{dt} = -K,$$

then

$$\frac{dK}{dt} = \Delta_{\mathbf{g}}K + 2K^2$$

Nonlinear heat equation! When $t \to \infty$, $K(\infty) \to$ *constant*.

Surface Ricci Flow

Definition (Hamilton's Normalized Surface Ricci Flow)

A closed surface S with a Riemannian metric $\mathbf{g}$, the Ricci flow on it is defined as

$$\frac{dg_{ij}}{dt} = \frac{4\pi\chi(S)}{A(0)} - 2Kg_{ij}.$$

where $\chi(S)$ is the Euler characteristic number of the surface S, $A(0)$ is the total area at time 0. The total area of the surface is preserved during the normalized Ricci flow flow. The Ricci flow will converge to a metric such that the Gaussian curvature is constant every where,

$$K(\infty) \equiv \frac{2\pi\chi(S)}{A(0)}.$$

Surface Ricci Flow

Furthermore, the normalized surface Ricci flow

$$\frac{dg_{ij}}{dt} = \frac{4\pi\chi(S)}{A(0)} - 2Kg_{ij}.$$

is conformal,

$$\mathbf{g}(t) = e^{2u(t)}\mathbf{g}(0),$$

where $u(t) : S \to \mathbb{R}$ is a the conformal factor function, and the normalized Surface Ricci flow can be written as

$$\frac{du(p,t)}{dt} = \frac{2\pi\chi(S)}{A(0)} - K(p,t),$$

for every point $p \in S$.

Theorem (Hamilton 1982)

For a closed surface of non-positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Theorem (Bennett Chow)

For a closed surface of positive Euler characteristic, if the total area of the surface is preserved during the flow, the Ricci flow will converge to a metric such that the Gaussian curvature is constant (equals to $\bar{K}$) every where.

Summary

Surface Ricci Flow

- Conformal metric deformation

$$\mathbf{g} \to e^{2u}\mathbf{g}$$

- Curvature Change - heat diffusion

$$\frac{dK}{dt} = \Delta_{\mathbf{g}} K + 2K^2$$

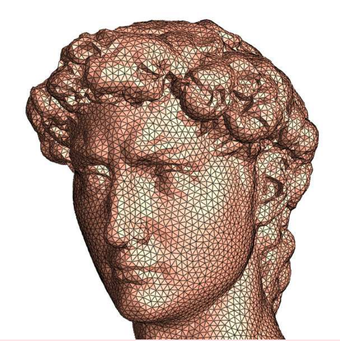
- Ricci flow

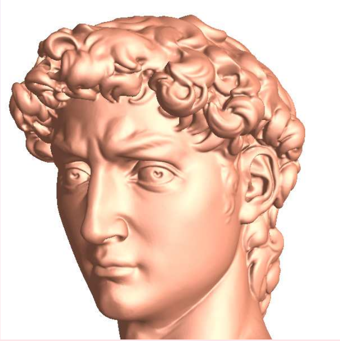
$$\frac{du}{dt} = \bar{K} - K.$$

Discrete Surface Ricci Flow

Generic Surface Model - Triangular Mesh

- Surfaces are represented as polyhedron triangular meshes.
- Isometric gluing of triangles in $\mathbb{E}^2$.
- Isometric gluing of triangles in $\mathbb{H}^2, \mathbb{S}^2$.





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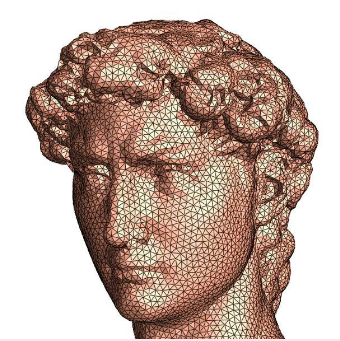

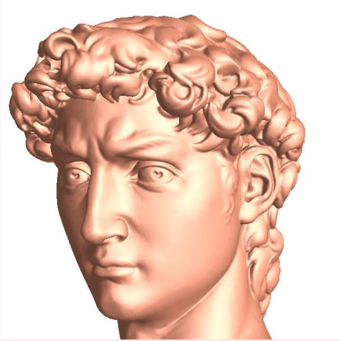

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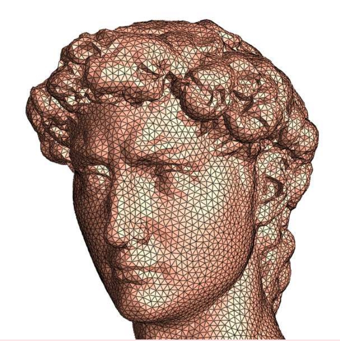
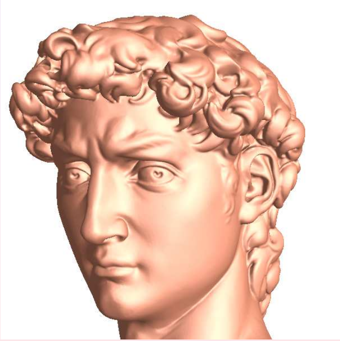




Discrete Generalization

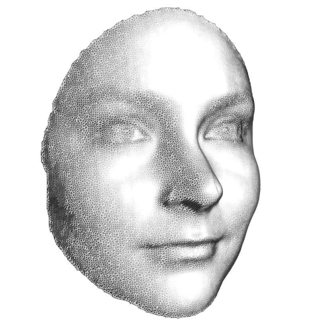
Concepts

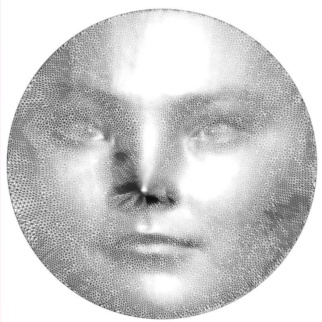
1. Discrete Riemannian Metric
2. Discrete Curvature
3. Discrete Conformal Metric Deformation

Definition (Discrete Metric)

A Discrete Metric on a triangular mesh is a function defined on the vertices, $l : E = \{all\ edges\} \to \mathbb{R}^+$, satisfies triangular inequality.

A mesh has infinite metrics.





Discrete Curvature

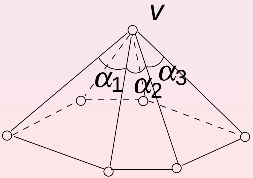
Definition (Discrete Curvature)

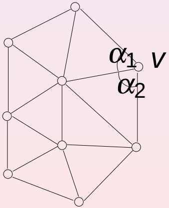
Discrete curvature: $K : V = \{vertices\} \to \mathbb{R}^1$.

$$K(v) = 2\pi - \sum_i \alpha_i, v \notin \partial M; K(v) = \pi - \sum_i \alpha_i, v \in \partial M$$

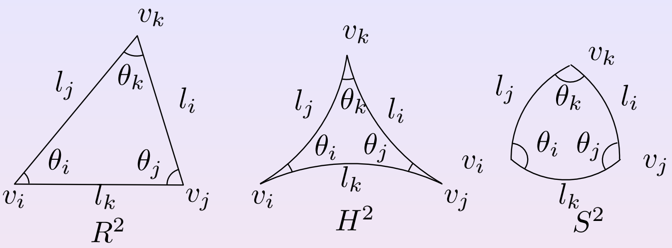
Theorem (Discrete Gauss-Bonnet theorem)

$$\sum_{v \notin \partial M} K(v) + \sum_{v \in \partial M} K(v) = 2\pi\chi(M).$$





Discrete Metrics Determines the Curvatures



cosine laws

$$\cos l_i = \frac{\cos\theta_i + \cos\theta_j\cos\theta_k}{\sin\theta_j\sin\theta_k} \quad (1)$$

$$\cosh l_i = \frac{\cosh\theta_i + \cosh\theta_j\cosh\theta_k}{\sinh\theta_j\sinh\theta_k} \quad (2)$$

$$1 = \frac{\cos\theta_i + \cos\theta_j\cos\theta_k}{\sin\theta_j\sin\theta_k} \quad (3)$$

Discrete Conformal Metric Deformation

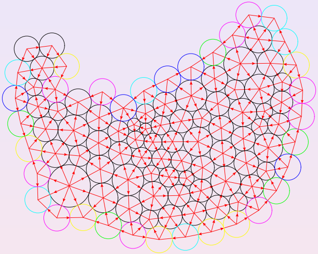
Conformal maps Properties

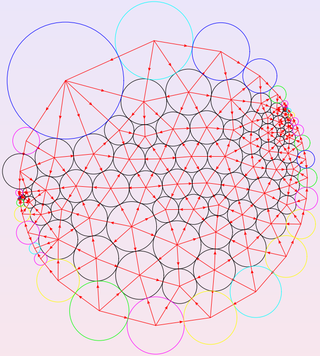
- transform infinitesimal circles to infinitesimal circles.
- preserve the intersection angles among circles.

Idea - Approximate conformal metric deformation

Replace infinitesimal circles by circles with finite radii.

Discrete Conformal Metric Deformation vs CP





Circle Packing Metric

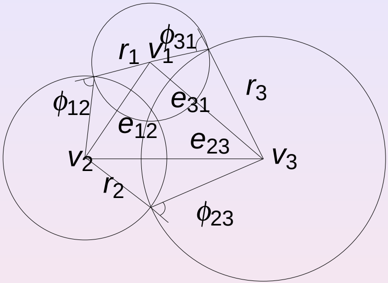
CP Metric

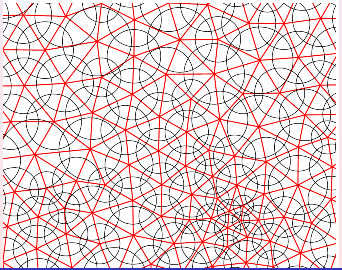
We associate each vertex v_i with a circle with radius γ_i. On edge e_{ij}, the two circles intersect at the angle of Φ_{ij}. The edge lengths are

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j \cos\Phi_{ij}$$

CP Metric (Σ, Γ, Φ), Σ triangulation,

$$\Gamma = \{\gamma_i | \forall v_i\}, \Phi = \{\phi_{ij} | \forall e_{ij}\}$$





Discrete Conformal Factor

Conformal Factor

Defined on each vertex $\mathbf{u} : V \to \mathbb{R}$,

$$u_i = \begin{cases} \log \gamma_i & \mathbb{R}^2 \\ \log \tanh \frac{\gamma_i}{2} & \mathbb{H}^2 \\ \log \tan \frac{\gamma_i}{2} & \mathbb{S}^2 \end{cases}$$

Properties

- Symmetry

$$\frac{\partial K_i}{\partial u_j} = \frac{\partial K_j}{\partial u_i}$$

- Discrete Laplace Equation

$$d\mathbf{K} = \Delta d\mathbf{u},$$

Δ is a discrete Lapalce-Beltrami operator.

Unified Framework of Discrete Curvature Flow

Analogy

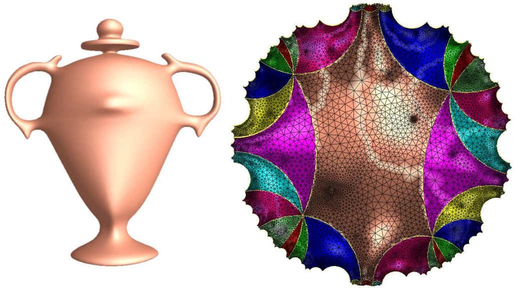
- Curvature flow
$$\frac{du}{dt} = \bar{K} - K,$$
- Energy
$$E(\mathbf{u}) = \int \sum_i (\bar{K}_i - K_i) du_i,$$
- Hessian of E denoted as Δ,
$$d\mathbf{K} = \Delta d\mathbf{u}.$$

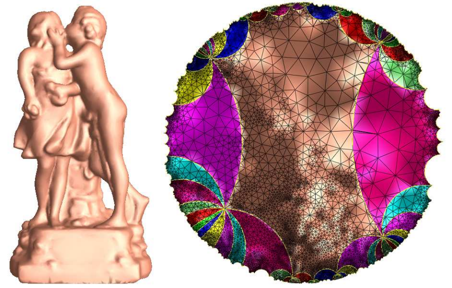
Criteria for Discretization

Key Points

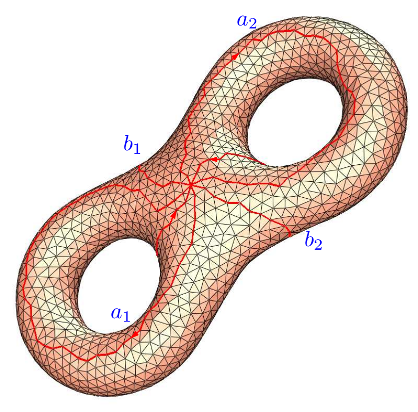
- Convexity of the energy $E(\mathbf{u})$
- Convexity of the metric space (**u**-space)
- Admissible curvature space (**K**-space)
- Preserving or reflecting richer structures
- Conformality

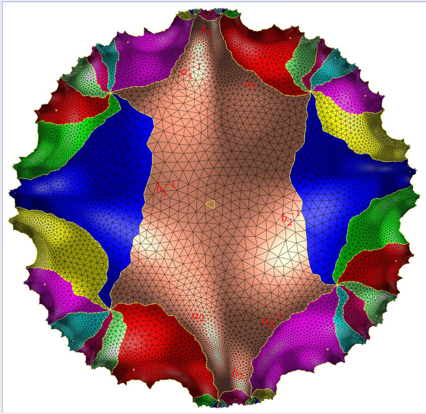
Hyperbolic Ricci Flow





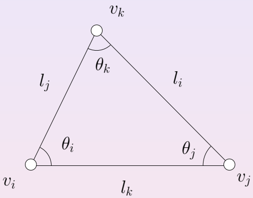
Hyperbolic Yamabe Flow





Convergence and Uniqueness of discrete Ricci flow

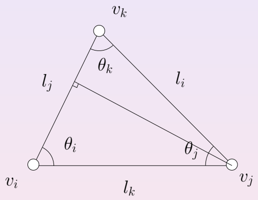
Cosine law



$$A = l_j l_k \sin\theta_i$$

$$\begin{aligned} 2l_j l_k \cos\theta_i &= l_j^2 + l_k^2 - l_i^2 \\ -2l_j l_k \sin\theta_i \frac{d\theta_i}{dl_i} &= -2l_i \\ \frac{d\theta_i}{dl_i} &= \frac{l_i}{A} \end{aligned}$$

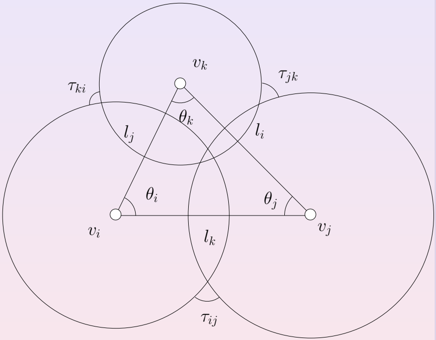
Cosine law



$$l_j = l_i \cos\theta_k + l_k \cos\theta_i$$

$$\begin{aligned}
2l_jl_k\cos\theta_i &= l_j^2 + l_k^2 - l_i^2 \\
2l_j &= 2l_k\cos\theta_i - \\
& \quad 2l_jl_k\sin\theta_i \frac{d\,\theta_i}{dl_j} \\
\frac{d\,\theta_i}{dl_j} &= \frac{l_k\cos\theta_i - l_j}{A} \\
&= -\frac{l_i\cos\theta_k}{A} \\
&= -\frac{d\,\theta_i}{dl_i}\cos\theta_k
\end{aligned}$$

Cosine law



$$l_k^2 = r_i^2 + r_j^2 + 2\cos\tau_{ij} r_i r_j$$

$$\begin{aligned}
l_i^2 &= r_j^2 + r_k^2 + 2 r_j r_k \cos\tau_{jk} \\
2 l_i \frac{dl_i}{dr_j} &= 2 r_j + 2 r_k \cos\tau_{jk} \\
\frac{dl_i}{dr_j} &= \frac{2 r_j^2 + 2 r_j r_k \cos\tau_{jk}}{2 l_i r_j} \\
&= \frac{r_j^2 + r_k^2 + 2 r_j r_k \cos\tau_{jk} + r_j^2}{2 l_i r_j} \\
&= \frac{l_i^2 + r_j^2 - r_k^2}{2 l_i r_j}
\end{aligned}$$

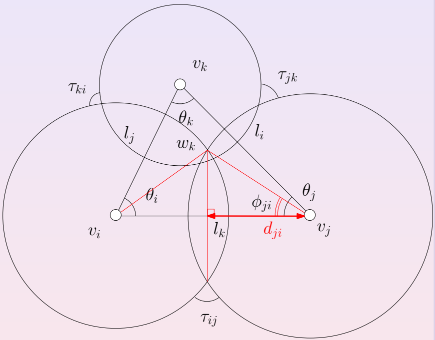
Cosine law

Let $u_i = \log r_i$, then

$$\begin{pmatrix} d\theta_1 \\ d\theta_2 \\ d\theta_3 \end{pmatrix} = \frac{-1}{A} \begin{pmatrix} l_1 & 0 & 0 \\ 0 & l_2 & 0 \\ 0 & 0 & l_3 \end{pmatrix} \begin{pmatrix} -1 & \cos\theta_3 & \cos\theta_2 \\ \cos\theta_3 & -1 & \cos\theta_1 \\ \cos\theta_2 & \cos\theta_1 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{l_1^2+r_2^2-r_3^2}{2l_1r_2} & \frac{l_1^2+r_3^2-r_2^2}{2l_1r_3} \\ \frac{l_2^2+r_1^2-r_3^2}{2l_2r_1} & 0 & \frac{l_2^2+r_3^2-r_1^2}{2l_2r_3} \\ \frac{l_3^2+r_1^2-r_2^2}{2l_3r_1} & \frac{l_3^2+r_2^2-r_1^2}{2l_3r_2} & 0 \end{pmatrix} \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix} \begin{pmatrix} du_1 \\ du_2 \\ du_3 \end{pmatrix}$$

Cosine law



$$l_k^2 = r_i^2 + r_j^2 + 2\cos\tau_{ij} r_i r_j$$

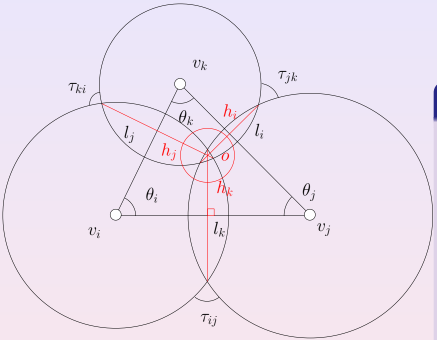
$$2l_k \frac{dl_k}{dr_j} = 2r_j + 2r_i \cos\tau_{ij}$$

$$r_j \frac{dl_k}{dr_j} = \frac{2r_j^2 + 2r_i r_j \cos\tau_{ij}}{2l_k} = \frac{l_k^2 + r_j^2 - r_i^2}{2l_k}$$

In triangle $[v_i, v_j, w_k]$,

$$\frac{dl_k}{du_j} = 2\frac{l_k r_j \cos\phi_{ji}}{2l_k} = r_j \cos\phi_{ji} = d_{ji}$$

Cosine law



There is a unique circle orthogonal to three circles (v_i, r_i), the center is o, the distance from o to edge $[v_i, v_j]$ is h_k.

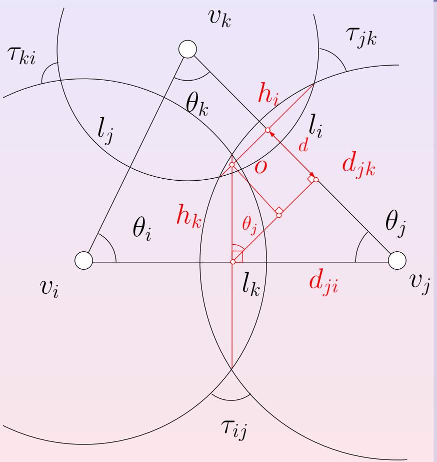
Theorem

$$\frac{d\theta_i}{du_j} = \frac{d\theta_j}{du_i} = \frac{h_k}{l_k}$$

$$\frac{d\theta_j}{du_k} = \frac{d\theta_k}{du_j} = \frac{h_i}{l_i}$$

$$\frac{d\theta_k}{du_i} = \frac{d\theta_i}{du_k} = \frac{h_j}{l_j}$$

Cosine law



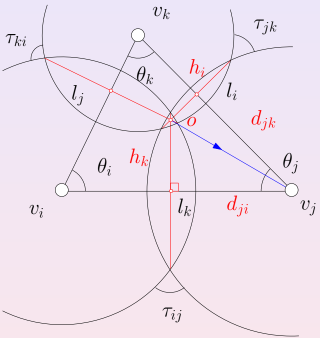
$$\frac{d\theta_i}{du_j} = \frac{h_k}{l_k}$$

Proof.

$$
\begin{aligned}
\frac{\partial\theta_i}{\partial u_j} &= \frac{\partial\theta_i}{\partial l_i}\frac{\partial l_i}{\partial u_j} + \frac{\partial\theta_i}{\partial l_k}\frac{\partial l_k}{\partial u_j} \\
&= \frac{\partial\theta_i}{\partial l_i}\left(\frac{\partial l_i}{\partial u_j} - \frac{\partial l_k}{\partial u_j}\cos\theta_j\right) \\
&= \frac{l_i}{A}(d_{jk} - d_{ji}\cos\theta_j) \\
&= \frac{dl_i}{l_i l_k \sin\theta_j} \\
&= \frac{h_k \sin\theta_j}{l_k \sin\theta_j} \\
&= \frac{h_k}{l_k}
\end{aligned}
$$

□

Cosine law



$$\frac{\partial \langle v_j - v_i, v_j - v_i\rangle}{\partial u_j} = 2\langle \frac{\partial v_j}{\partial u_j}, v_j - v_i\rangle$$

$$\frac{\partial l_k^2}{\partial u_j} = 2\langle \frac{\partial v_j}{\partial u_j}, v_j - v_i\rangle$$

$$\frac{\partial l_k}{\partial u_j} = \langle \frac{\partial v_j}{\partial u_j}, \frac{v_j - v_i}{l_k}\rangle$$

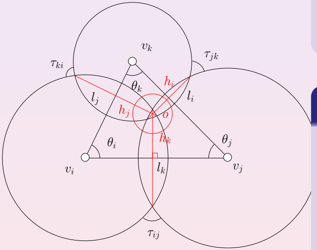
$$d_{ji} = \langle \frac{\partial v_j}{\partial u_j}, \frac{v_j - v_i}{l_k}\rangle$$

Similarly

$$d_{jk} = \langle \frac{\partial v_j}{\partial u_j}, \frac{v_j - v_k}{l_i}\rangle$$

$$\frac{\partial v_j}{\partial u_j} = v_j - o$$

So $\frac{\partial v_j}{\partial u_j} = v_j - 0.$



Lemma

For any three obtuse angles $\tau_{ij}, \tau_{jk}, \tau_{ki} \in [0, \frac{\pi}{2})$ and any three positive numbers r_1, r_2 and r_3, there is a configuration of 3 circles in Euclidean geometry, unique upto isometry, having radii r_i and meeting in angles τ_{ij}.

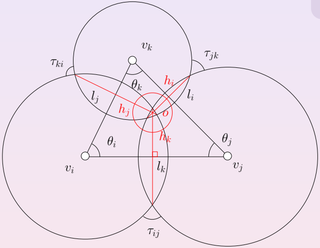
Proof.

$$max\{r_i^2, r_j^2\} < r_i^2 + r_j^2 + 2r_i r_j \cos\tau_{ij} \leq (r_i + r_j)^2$$

$$max\{r_i^2, r_j^2\} < l_k \leq r_i + r_j$$

so

$$l_k \leq r_i + r_j < l_i + l_j.$$



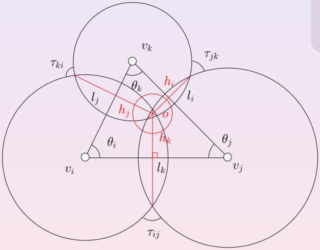
$$\omega = \theta_i du_i + \theta_j du_j + \theta_k du_j$$

Lemma

ω is closed 1-form in $\Omega := \{(u_1, u_2, u_3) \in \mathbb{R}^3\}$.

Because $\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i}$, so

$$\begin{aligned} d\omega &= \left(\frac{\partial \theta_i}{\partial u_j} - \frac{\partial \theta_j}{\partial u_i}\right) du_j \wedge du_i + \\ &\left(\frac{\partial \theta_j}{\partial u_k} - \frac{\partial \theta_k}{\partial u_j}\right) du_k \wedge du_j + \\ &\left(\frac{\partial \theta_k}{\partial u_i} - \frac{\partial \theta_i}{\partial u_k}\right) du_i \wedge du_k \\ &= 0. \end{aligned}$$



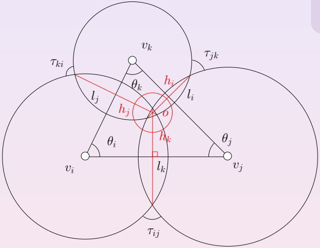
Lemma

The Ricci energy $E(u_1,u_2,u_3)$ is well defined.

Because $\Omega = \mathbb{R}^3$ is convex, closed 1-form is exact, therefore $E(u_1,u_2,u_3)$ is well defined.

$$E(u_1,u_2,u_3) = \int_{(0,0,0)}^{(u_1,u_2,u_3)} \omega$$

Discrete Ricci Energy



$$E(u_1,u_2,u_3) = \int_{(0,0,0)}^{(u_1,u_2,u_3)} \omega$$

Lemma

The Ricci energy $E(u_1,u_2,u_3)$ is strictly concave on the subspace $u_1+u_2+u_3=0$.

The gradient $\nabla E = (\theta_1,\theta_2,\theta_3)$, the Hessian matrix is

$$H = \begin{pmatrix} \frac{\partial\theta_1}{\partial u_1} & \frac{\partial\theta_1}{\partial u_2} & \frac{\partial\theta_1}{\partial u_3} \\ \frac{\partial\theta_2}{\partial u_1} & \frac{\partial\theta_2}{\partial u_2} & \frac{\partial\theta_2}{\partial u_3} \\ \frac{\partial\theta_3}{\partial u_1} & \frac{\partial\theta_3}{\partial u_2} & \frac{\partial\theta_3}{\partial u_3} \end{pmatrix}$$

because of $\theta_1+\theta_2+\theta_3=\pi$,

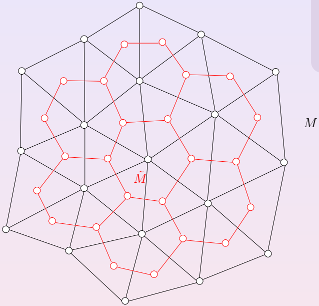
$$\frac{\partial\theta_i}{\partial u_i} = -\frac{\partial\theta_i}{\partial u_j} - \frac{\partial\theta_i}{\partial u_k} = -\frac{\partial\theta_j}{\partial u_i} - \frac{\partial\theta_k}{\partial u_i}$$

Proof.

$$H = -\begin{pmatrix} \frac{h_3}{l_3}+\frac{h_2}{l_2} & -\frac{h_3}{l_3} & -\frac{h_2}{l_2} \\ -\frac{h_3}{l_3} & \frac{h_3}{l_3}+\frac{h_1}{l_1} & -\frac{h_1}{l_1} \\ -\frac{h_2}{l_2} & -\frac{h_1}{l_1} & \frac{h_2}{l_2}+\frac{h_1}{l_1} \end{pmatrix}$$

$-H$ is diagonal dominant, it has null space $(1,1,1)$, on the subspace $u_1+u_2+u_3=0$, it is strictly negative definite. Therefore the discrete Ricci energy $E(u_1,u_2,u_3)$ is strictly concave. □

Discrete Ricci Energy



$$\omega = \sum_{v_i \in M} K_i du_i$$

$$E(\mathbf{u}) = \int_{\mathbf{0}}^{\mathbf{u}} \omega.$$

Lemma

The Ricci energy $E(\mathbf{u})$ is strictly convex on the subspace $\sum_{v_i \in M} u_i = 0$.

The gradient $\nabla E = (K_1, K_2, \cdots, K_n)$. The Ricci energy

$$E(\mathbf{u}) = 2\pi \sum_{v_i \in M} u_i - \sum_{[v_i, v_j, v_k] \in M} E_{ijk}(u_i, u_j, u_k)$$

where E_{ijk} is the ricci energy defined on the face $[v_i, v_j, v_k]$. The linear term won't affect the convexity of the energy. The null space of the Hessian is $(1, 1, \cdots, 1)$. In the subspace $\sum u_i = 0$, the energy is strictly convex.

Uniqueness

Lemma

Suppose $\Omega \subset \mathbb{R}^n$ is a convex domain, $f : \Omega \to \mathbb{R}$ is a strictly convex function, then the map

$$\mathbf{x} \to \nabla f(\mathbf{x})$$

is one-to-one.

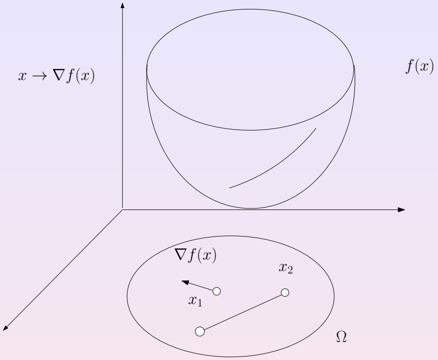
Proof.

Suppose $x_1 \neq x_2$, $\nabla f(x_1) = \nabla f(x_2)$. Because Ω is convex, the line segment $(1-t)x_1 + tx_2$ is contained in Ω. construct a convex function $g(t) = f((1-t)x_1 + tx_2)$, then $g'(t)$ is monotonous. But

$$g'(0) = \langle \nabla f(x_1), x_2 - x_1 \rangle = \langle \nabla f(x_2), x_2 - x_1 \rangle = g'(1),$$

contradiction. □

Uniqueness



Lemma

Suppose $\Omega \subset \mathbb{R}^n$ is a convex domain, $f : \Omega \to \mathbb{R}$ is a strictly convex function, then the map

$$\mathbf{x} \to \nabla f(\mathbf{x})$$

Uniqueness

Theorem

Suppose M is a mesh, with circle packing metric, all edge intersection angles are non-obtuse. Given the target curvature $(K_1, K_2, \cdots, K_n)$, $\sum_i K_i = 2\pi\chi(M)$. If the solution $(u_1, u_2, \cdots, u_n) \in \Omega(M), \sum_i u_i = 0$ exists, then it is unique.

Proof.

The discrete Ricci energy E on $\Omega \cap \{\sum_i u_i = 0\}$ is convex,

$$\nabla E(u_1, u_2, \cdots, u_n) = (K_1, K_2, \cdots K_n).$$

Use previous lemma. □

Thurston's Circle Packing Metric

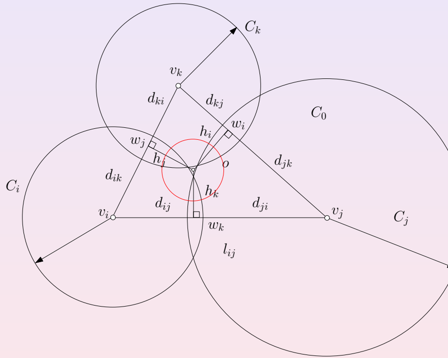
CP Metric

We associate each vertex v_i with a circle with radius γ_i. On edge e_{ij}, the two circles intersect at the angle of Φ_{ij}. The edge lengths are

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j\eta_{ij}$$

CP Metric (Σ, Γ, η), Σ triangulation,

$$\Gamma = \{\gamma_i | \forall v_i\}, \eta = \{\eta_{ij} < 1 | \forall e_{ij}\}$$



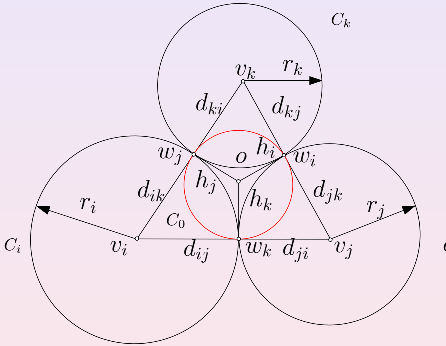
Tangential Circle Packing Metric

Tangential CP Metric

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\gamma_i\gamma_j,$$

equivalently

$$\eta_{ij} \equiv 1.$$



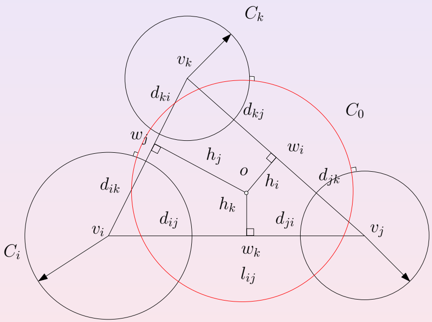
Inversive Distance Circle Packing Metric

Tangential CP Metric

$$l_{ij}^2 = \gamma_i^2 + \gamma_j^2 + 2\eta_{ij}\gamma_i\gamma_j,$$

equivalently

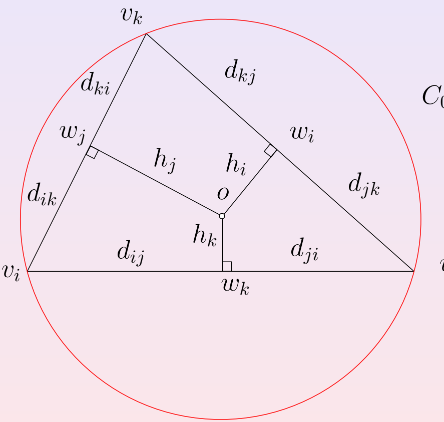
$$\eta_{ij} > 1.$$



Yamabe Flow

Yamabe Flow

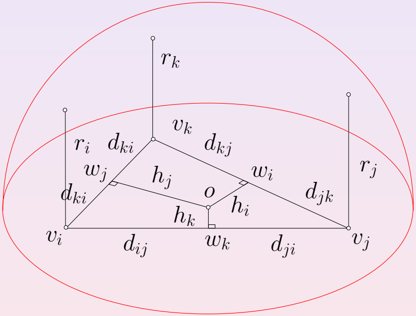
$$l_{ij}^2 = \eta_{ij}\gamma_i\gamma_j,$$



Imaginary Radius Circle Packing Metric

Imaginary Radius Circle Packing Metric

$$l_{ij}^2 = -\gamma_i^2 - \gamma_j^2 + 2\eta_{ij}\gamma_i\gamma_j,$$

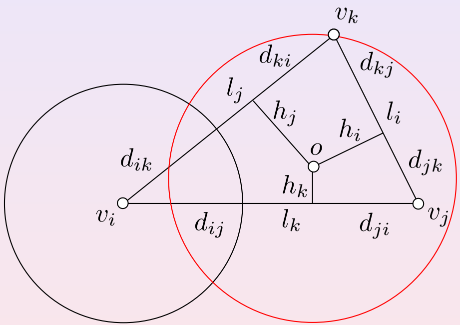


Mixed Type

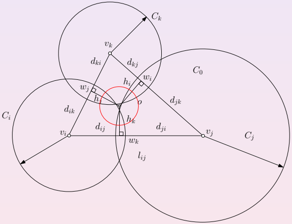
Mixed Circle Packing

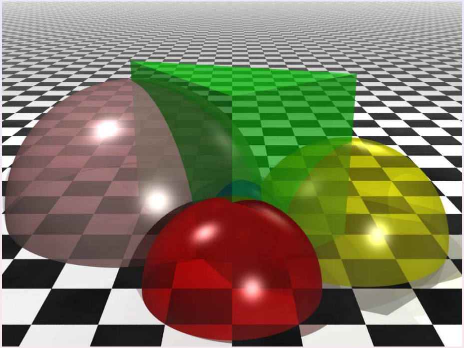
$$l_{ij}^2 = \alpha_i \gamma_i^2 + \alpha_j \gamma_j^2 + 2\eta_{ij}\gamma_i\gamma_j,$$

$$(\alpha_i, \alpha_j, \alpha_k) = (+1, -1, 0)$$

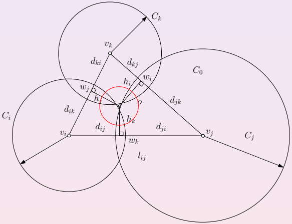


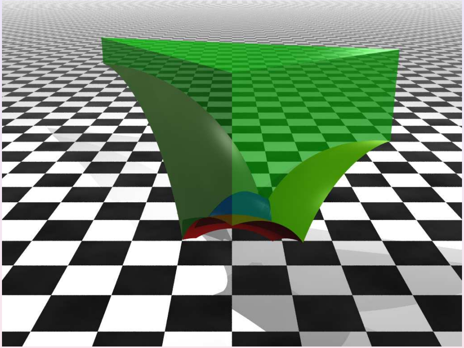
Geometric Interpretation - Discrete Entropy Energy



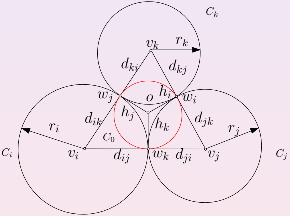


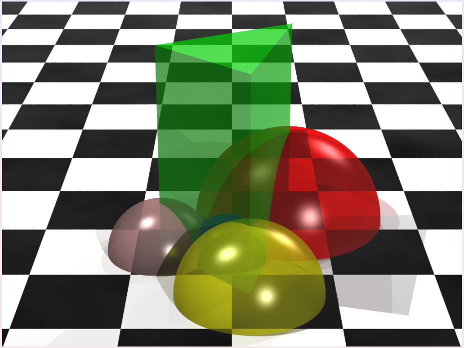
Geometric Interpretation - Discrete Entropy Energy



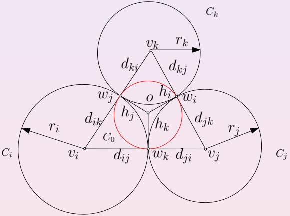


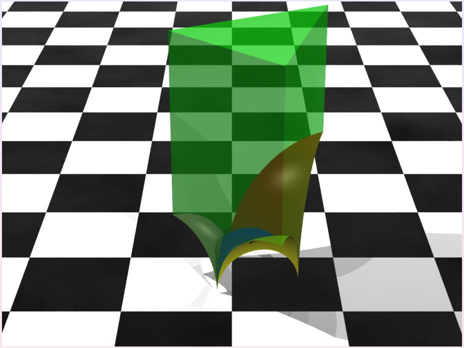
Geometric Interpretation - Discrete Entropy Energy



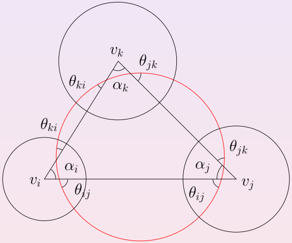


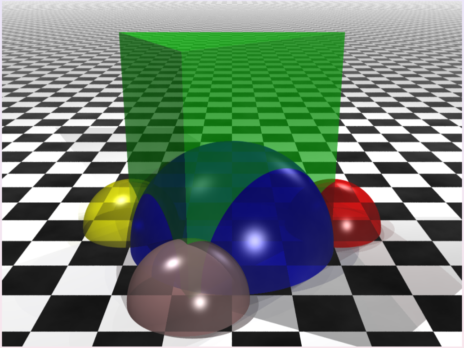
Geometric Interpretation - Discrete Entropy Energy

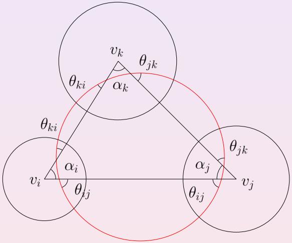


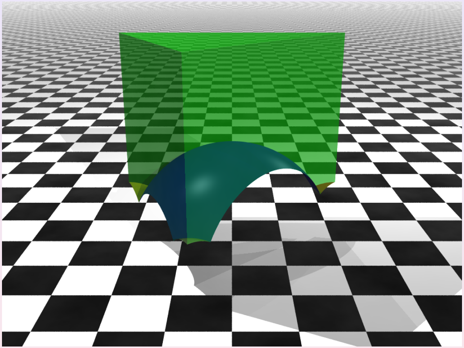


Geometric Interpretation - Discrete Entropy Energy

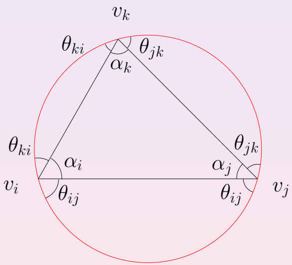


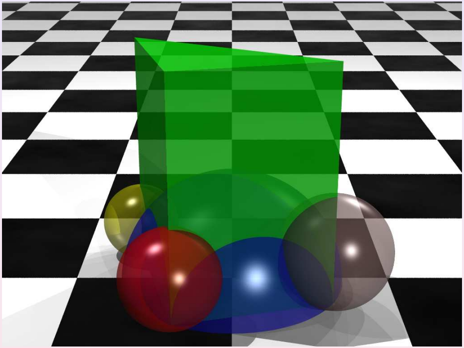



v_k
θ_{jk}
θ_{ki}
α_k
θ_{ki}
θ_{jk}
α_i
α_j
v_i
θ_{ij}
θ_{ij}
v_j

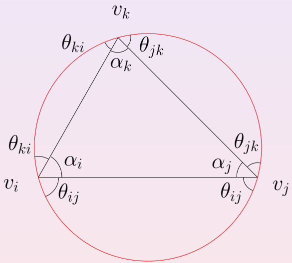


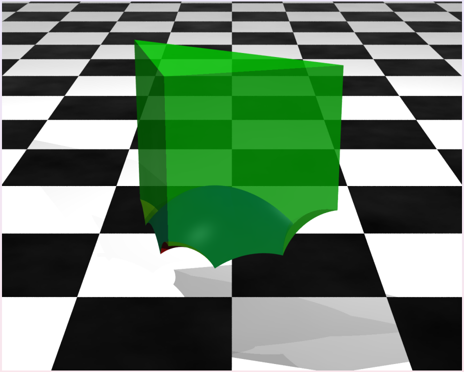
Geometric Interpretation - Discrete Entropy Energy



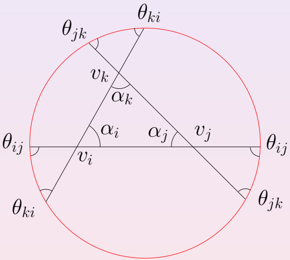


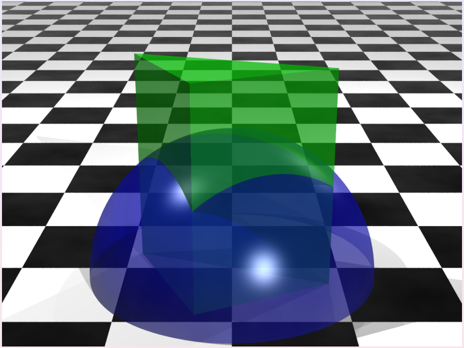
Geometric Interpretation - Discrete Entropy Energy



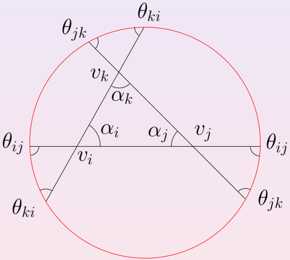


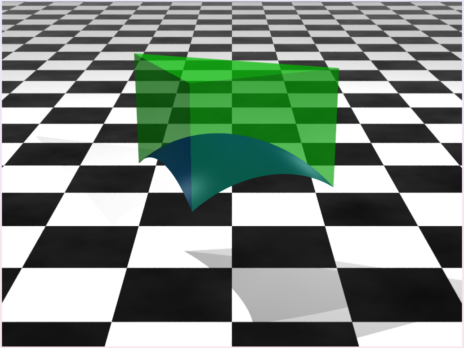
Geometric Interpretation - Discrete Entropy Energy





Geometric Interpretation - Discrete Entropy Energy





Ricci Flow Algorithm

Ricci Flow Algorithm

Use Newton's method to minimize the discrete Ricci energy,

$$E(u_1, u_2, \cdots, u_n) = \sum_{i=1}^{n} \bar{K}_i u_i - \int_{\mathbf{0}}^{\mathbf{u}} \sum_{i=1}^{n} K_i du_i,$$

The gradient of the energy is

$$\nabla E(\mathbf{u}) = \bar{\mathbf{K}} - \mathbf{K}(\mathbf{u}),$$

The Hessian matrix is given by $H = (h_{ij})$,

$$h_{ij} = \begin{cases} -w_{ij} & [v_i, v_j] \in M \\ \sum_k w_{ik} & i = j \\ 0 & otherwise \end{cases}$$

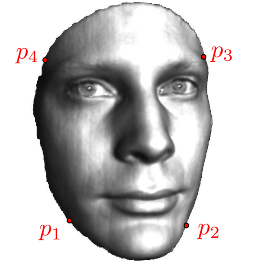
Ricci Flow Algorithm

1. Compute initial circle packing metric, determine the circle radii γ_i for each vertex v_i and intersection angles φ_{ij} for each edge $[v_i, v_j]$.
2. Determine the target curvature $\bar{K}_i$ for each vertex,
3. Compute the discrete metric (edge length)
4. Compute the discrete curvature K_i
5. Compute the power circle of each face, compute the Hessian matrix H
6. Solve linear system

$$\bar{\mathbf{K}} - \mathbf{K} = H\delta\mathbf{u}$$

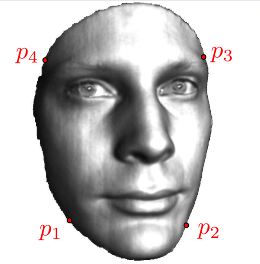
7. Update the vertex radii $\gamma_i + = \delta u_i$
8. Repeat step 2 through 7, until the curvature is close enough to the target curvature.

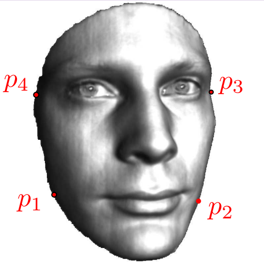
Topological Quadrilateral



Target curvatures at the corners: $\frac{\pi}{2}$, and 0 everywhere else.

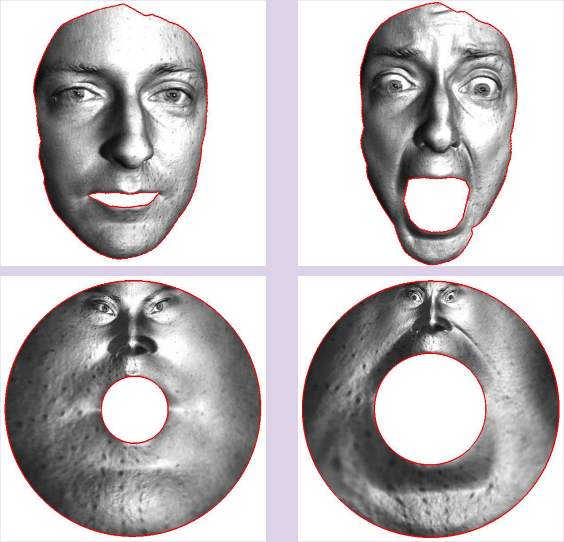
Topological Quadrilateral





Topological Annulus

Target curvature to be zeros everywhere, composed with e^z

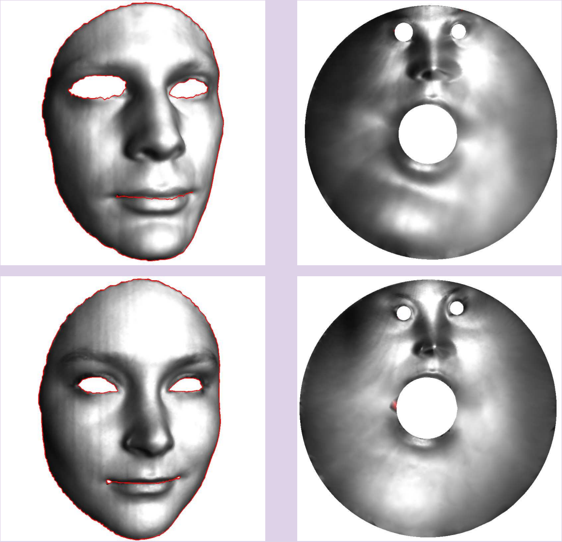


Topological Disk

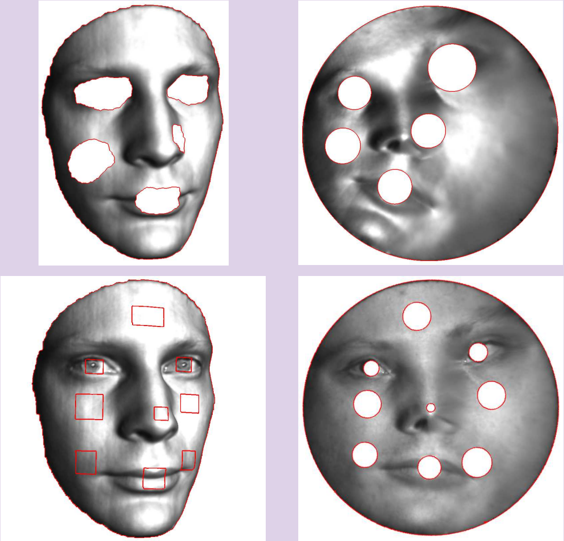
Punch a hole to an annulus

Multiply Connected Domains

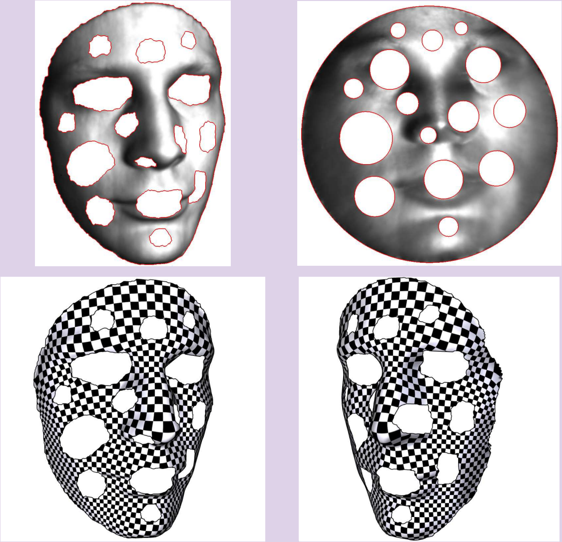
Zero interior curvature, constant boundary curvature



Total curvature for inner boundary is -2π

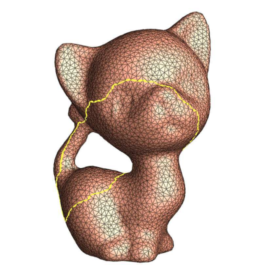


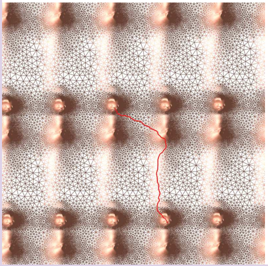
Multiply Connected Domains



Topological Torus

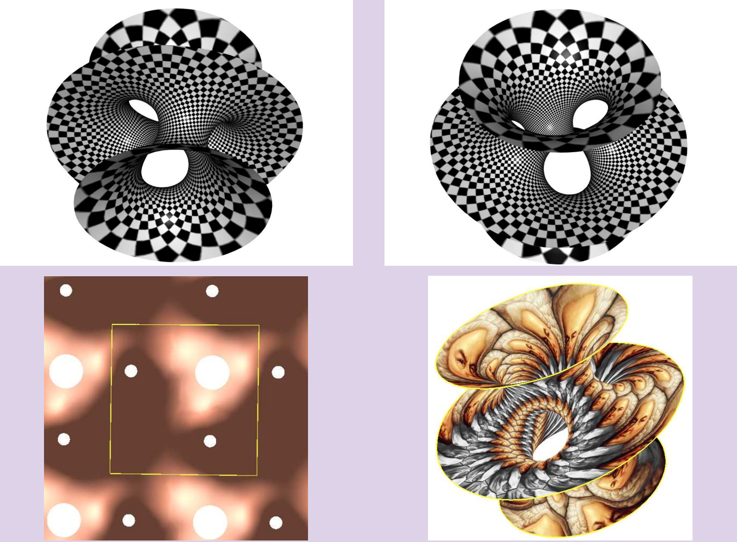
Zero target curvature everywhere





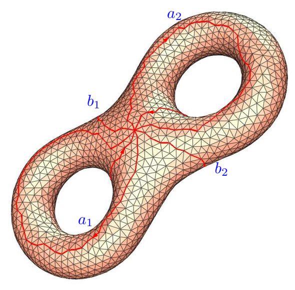
Topological Torus with holes

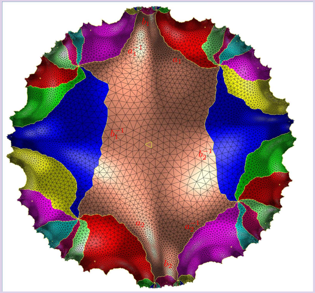
constant boundary curvature with total -2π



High genus surfaces

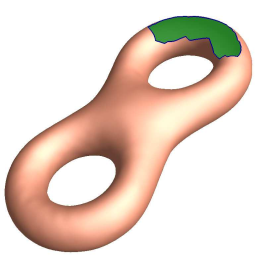
Zero Target Curvature Everywhere using hyperbolic Ricci flow

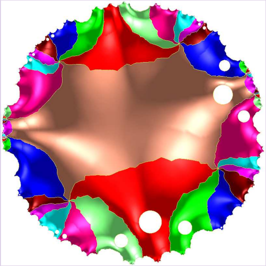




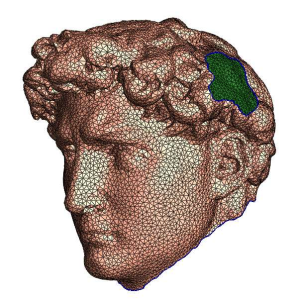
High Genus Surface with holes

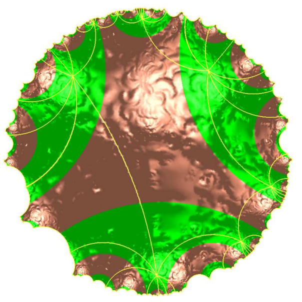
Zero interior curvature, constant boundary curvature





Topological Pants

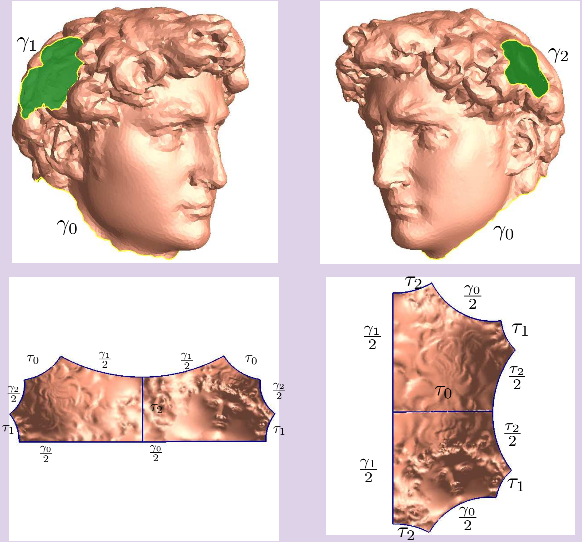




Genus 0 surface with 3 boundaries. The double covered surface is of genus 2. The boundaries are mapped to hyperbolic lines.

Conformal Module

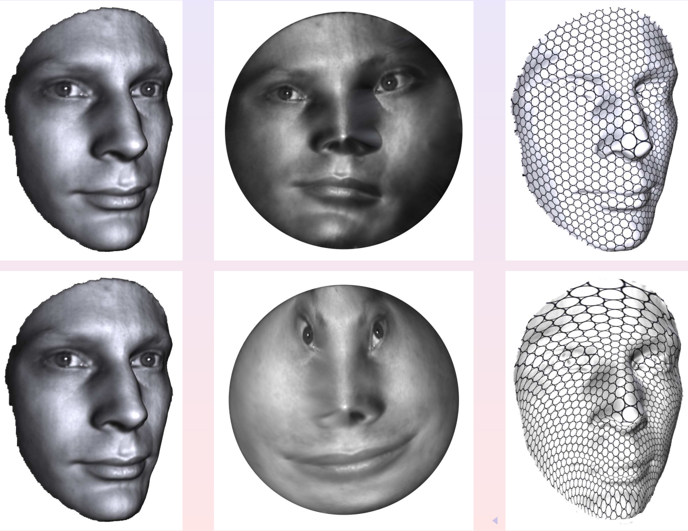
Topological Pants - 3D



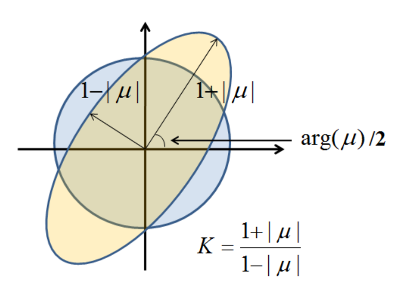
Quasi-Conformal Maps

Quasi-Conformal Map

Most homeomorphisms are quasi-conformal, which maps infinitesimal circles to ellipses.



Beltrami-Equation



Beltrami Coefficient

Let $\phi : S_1 \to S_2$ be the map, z, w are isothermal coordinates of S_1, S_2, Beltrami equation is defined as $\|\mu\|_\infty < 1$

$$\frac{\partial \phi}{\partial \bar{z}} = \mu(z) \frac{\partial \phi}{\partial z}$$

Solving Beltrami Equation

The problem of computing Quasi-conformal map is converted to compute a conformal map.

Solveing Beltrami Equation

Given metric surfaces $(S_1, \mathbf{g}_1)$ and $(S_2, \mathbf{g}_2)$, let z, w be isothermal coordinates of S_1, S_2,$w = \phi(z)$.

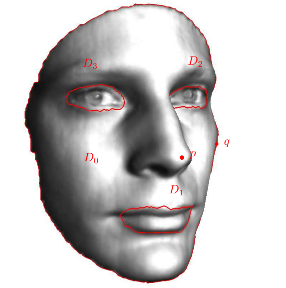
$$\mathbf{g}_1 = e^{2u_1} dz d\bar{z} \quad (4)$$

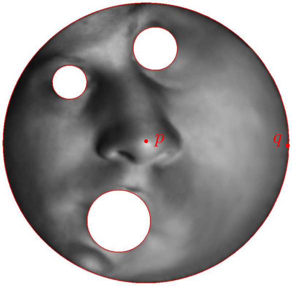
$$\mathbf{g}_2 = e^{2u_2} dw d\bar{w}, \quad (5)$$

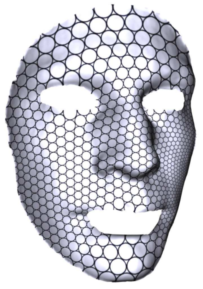
Then

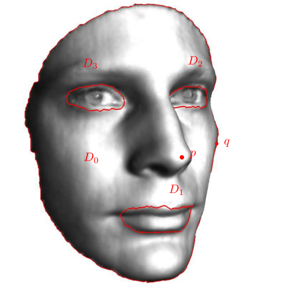
- $\phi : (S_1, \mathbf{g}_1) \to (S_2, \mathbf{g}_2)$, quasi-conformal with Beltrami coefficient μ.
- $\phi : (S_1, \phi^* \mathbf{g}_2) \to (S_2, \mathbf{g}_2)$ is isometric
- $\phi^* \mathbf{g}_2 = e^{u_2} |dw|^2 = e^{u_2} |dz + \mu d\bar{z}|^2$.
- $\phi : (S_1, |dz + \mu d\bar{z}|^2) \to (S_2, \mathbf{g}_2)$ is conformal.

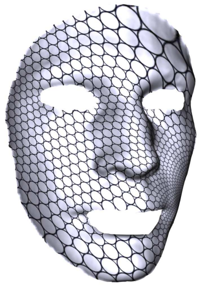
Quasi-Conformal Map Examples



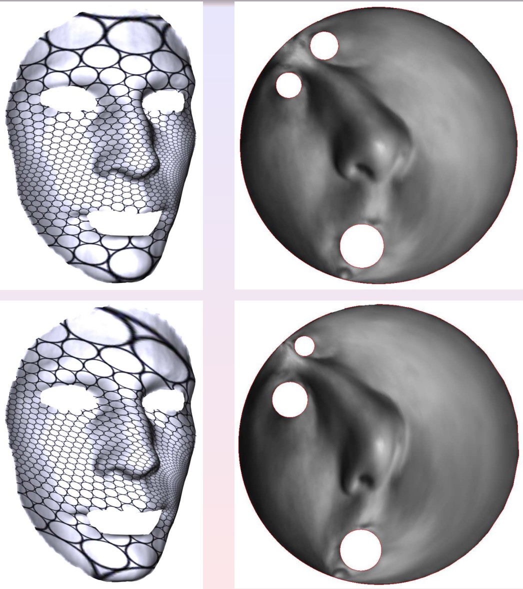








Quasi-Conformal Map Examples



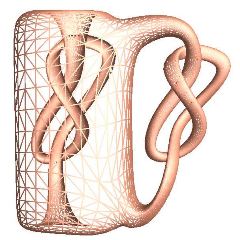
Solving Beltrami Equation

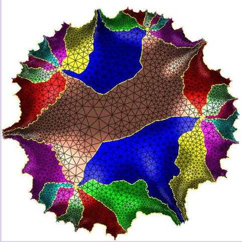
Summary

- Theory of discrete surface Ricci flow
- Algorithm for discrete surface Ricci flow
- Application for surface uniformization

Thanks

For more information, please email to gu@cs.sunysb.edu.





Thank you!