Variational Methods and Partial Differential Equations

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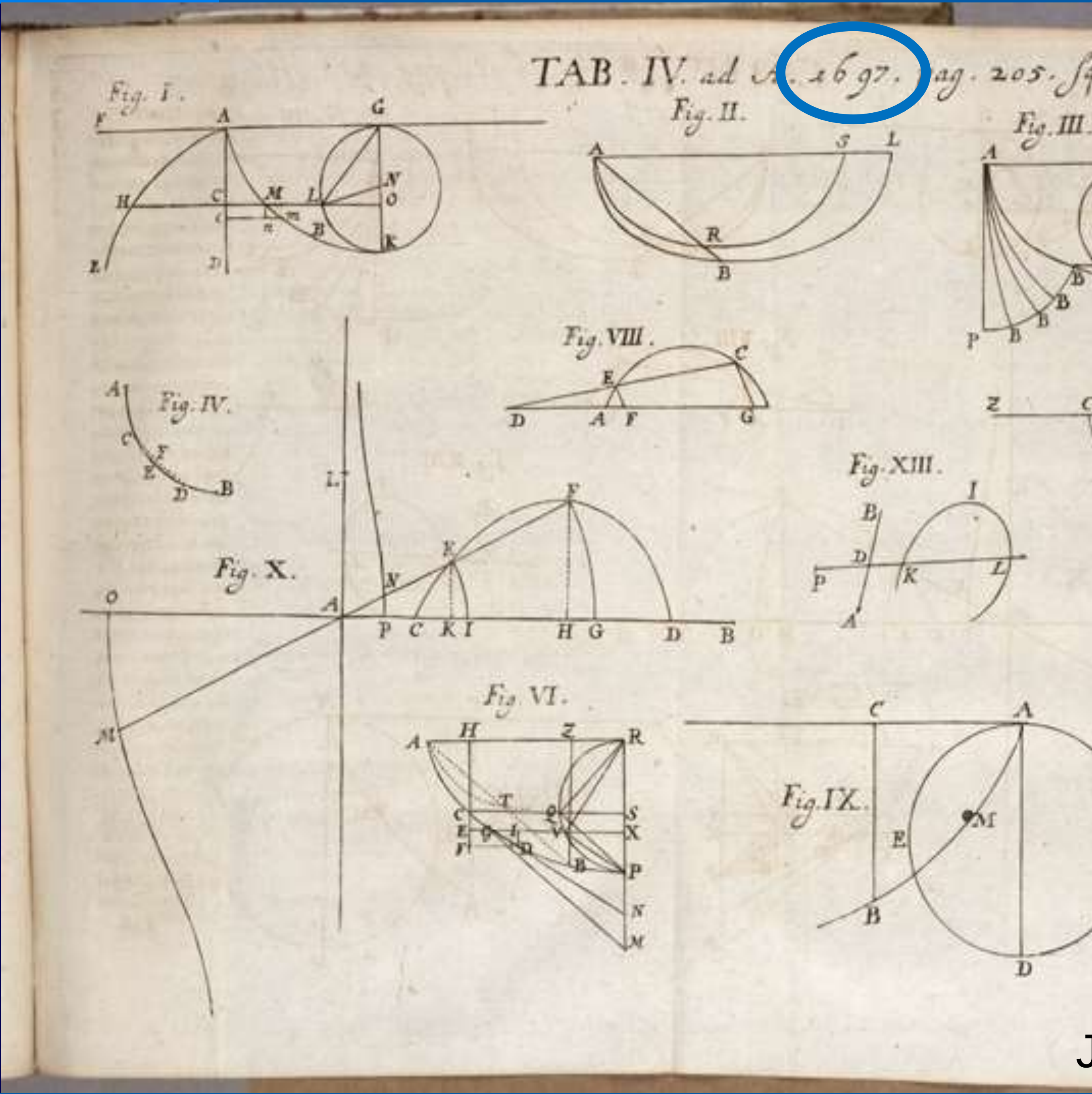
infinite-dimensional optimization

Which path is the fastest?

Bernoulli & The Brachistochrone



Johann Bernoulli (1667-1748)

Image segmentation:

Kass et al. '88, Mumford, Shah '89, Caselles et al. '95,

Kichenassamy et al. '95, Paragios, Deriche '99,

Chan, Vese '01, Tsai et al. '01, …

Multiview stereo reconstruction:

Faugeras, Keriven '98, Duan et al. '04, Yezzi, Soatto '03,

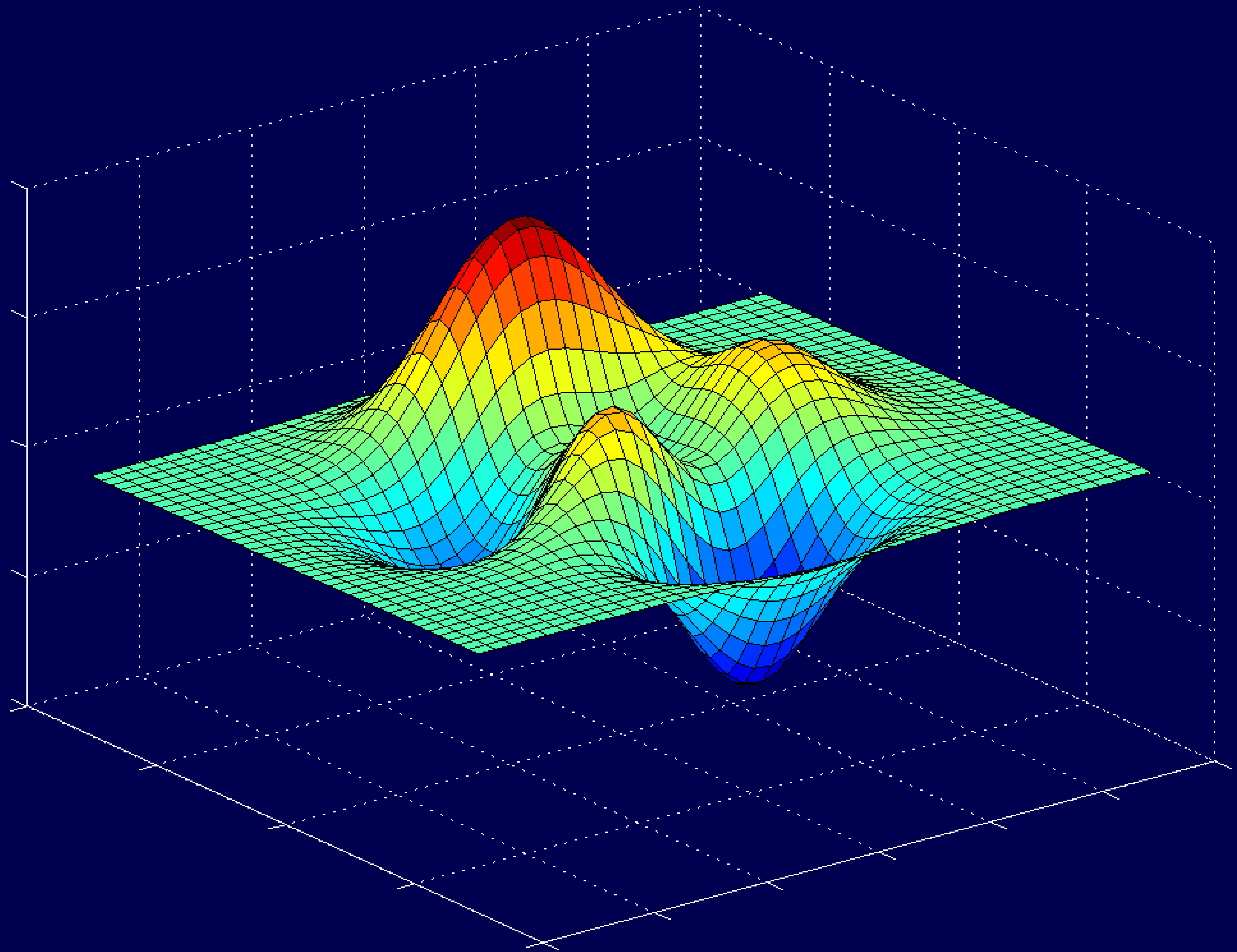
Labatut et al. '07, Kolev et al. IJCV '09…

Optical flow estimation:

Horn, Schunck '81, Nagel, Enkelmann '86, Black, Anandan '93,

Alvarez et al. '99, Brox et al. '04, Zach et al. '07, Sun et al. '08,

Wedel et al. '09, Werlberger et al. '10, …



Compute solutions via energy minimization

For $u : \Omega \to \mathbb{R},\ \Omega \subset \mathbb{R}^2,$ compute:

$$u_{den} = \arg\min_u \int_\Omega (u - f)^2\, dx + \lambda \int_\Omega |\nabla u|\, dx$$

input image $f : \Omega \to \mathbb{R}^3$

denoised $u_{den} : \Omega \to \mathbb{R}^3$

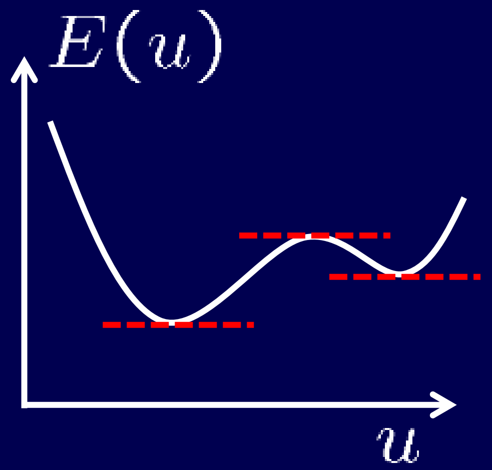
Rudin, Osher, Fatemi, Physica D '92

For $u : \Omega \to \mathbb{R},\ \Omega \subset \mathbb{R}^2,$ consider the functional

$$E(u) = \int_\Omega (u - f)^2 \, dx \; + \; \lambda \int_\Omega |\nabla u|^2 \, dx$$

Extremality principle as a necessary condition:

$$\boxed{\frac{dE}{du} = 0}$$

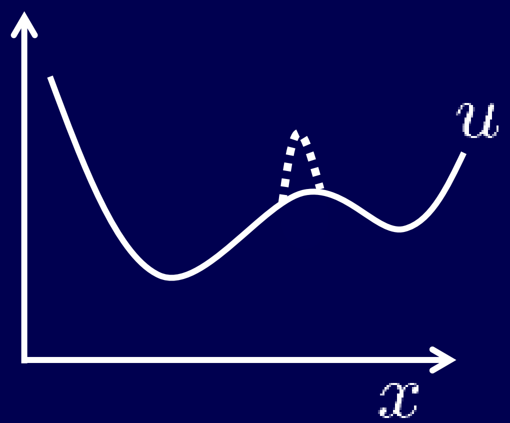


But: How to define the derivative with respect to a function?

The Gateaux Derivative

$$E(u) = \int_\Omega (u - f)^2 + \lambda\,|u'|^2\,dx \;=\; \int_\Omega \mathcal{L}(u, u')\,dx$$

Derivative in "direction" h :



$$\left.\frac{dE}{du}\right|_h = \lim_{\epsilon \to 0} \frac{E(u + \epsilon h) - E(u)}{\epsilon}$$

$$= \lim_{\epsilon \to 0} \frac{\int \mathcal{L}(u + \epsilon h, u' + \epsilon h') - \mathcal{L}(u, u')\,dx}{\epsilon}$$

Taylor expansion:

$$= \lim_{\epsilon \to 0} \frac{\int \cancel{\mathcal{L}(u, u')} + \cancel{\epsilon} h \frac{\partial \mathcal{L}}{\partial u} + \cancel{\epsilon} h' \frac{\partial \mathcal{L}}{\partial u'} - \cancel{\mathcal{L}(u, u')}\,dx}{\cancel{\epsilon}}$$

The Gateaux Derivative

$$\left.\frac{dE}{du}\right|_h = \int h \frac{\partial \mathcal{L}}{\partial u} + h' \frac{\partial \mathcal{L}}{\partial u'} \, dx$$

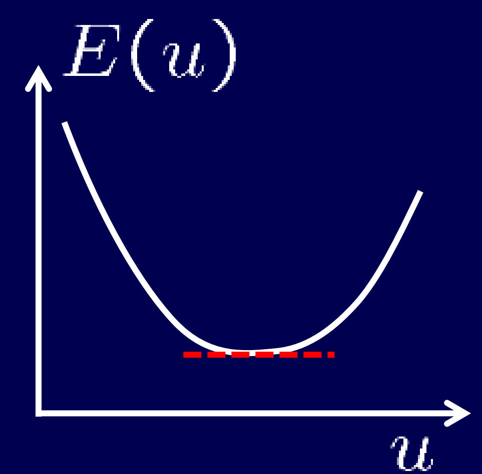
Integration by parts:

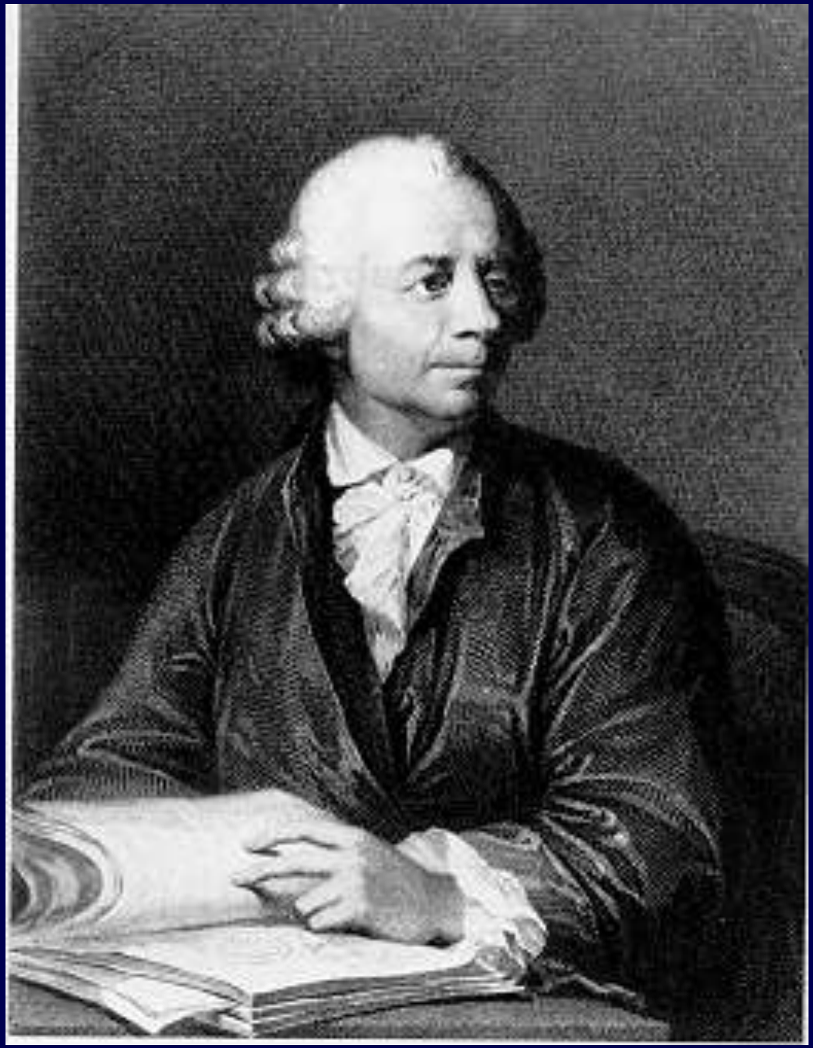
$$= \int h \left(\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial u'} \right) \right) dx \overset{!}{=} \int h \frac{dE}{du} \, dx$$

Necessary optimality condition:

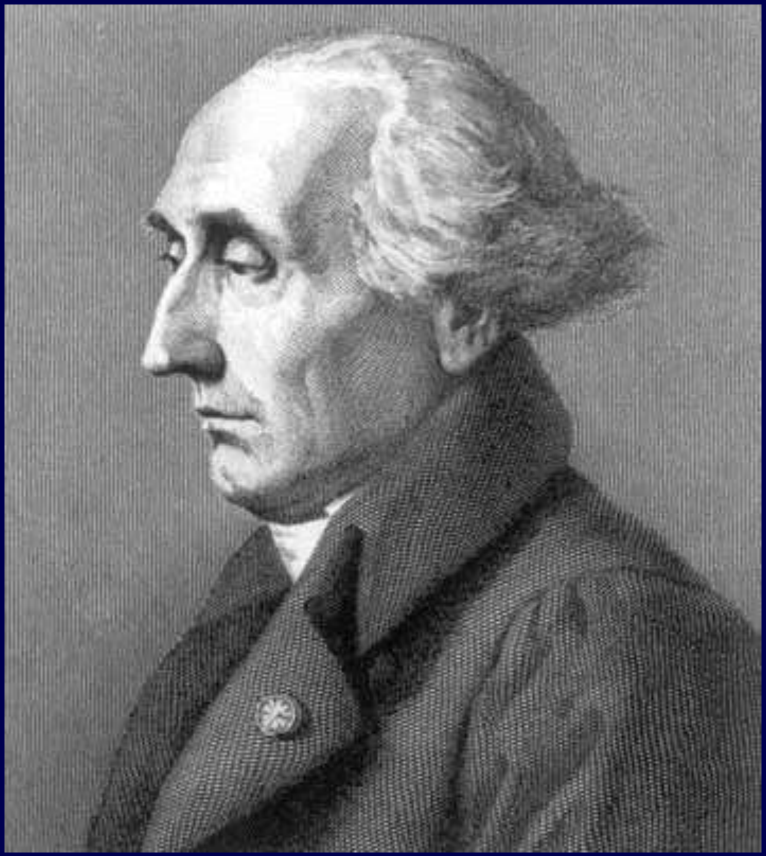
$$\boxed{\frac{dE}{du} = \frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial u'} \right) = 0}$$

Euler-Lagrange equation





Leonhard Euler
(1703-1783)

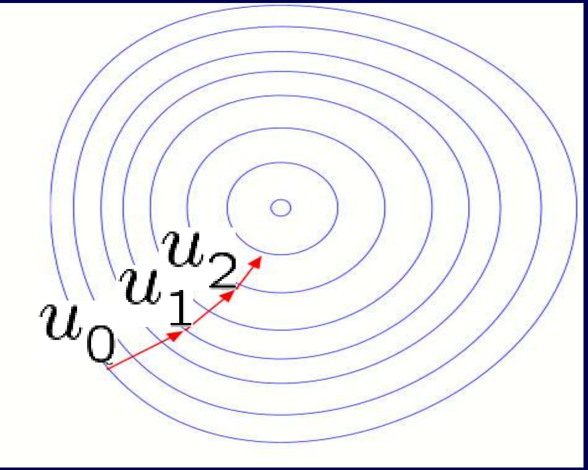


Joseph-Louis Lagrange
(1736 – 1813)

Iteratively walk "down-hill":

$$\begin{cases} u(x,0) = u_o(x) \\ \dfrac{\partial u}{\partial t} = -\dfrac{dE}{du} = -\dfrac{\partial \mathcal{L}}{\partial u} + \dfrac{d}{dx}\left(\dfrac{\partial \mathcal{L}}{\partial u'}\right) \end{cases}$$

direction of steepest descent



For the energy

$$E(u) = \frac{1}{2}\int_\Omega (u-f)^2 + \lambda\,|u'|^2\,dx$$

we obtain:

$$\boxed{\frac{\partial u}{\partial t} = f - u + \lambda u''}$$

diffusion equation

$$\int_\Omega |\nabla u|^2 \, dx \;\rightarrow\; \min$$

Slightly modify the regularization (*Rudin, Osher, Fatemi ‘92*):

$$E(u) = \frac{1}{2}\int_\Omega (u-f)^2\,dx \;+\; \frac{\lambda}{2}\boxed{\int_\Omega |\nabla u|^{\cancel{2}}\,dx}$$

total variation

Gradient descent:

$$\begin{cases} u(x,0) = u_o(x) \\ \dfrac{\partial u}{\partial t} = -\dfrac{dE}{du} = f - u + \lambda\,\mathrm{div}\,(g\,\nabla u)\,, \quad g = \dfrac{1}{|\nabla u|} \end{cases}$$

nonlinear diffusion

Perona, Malik ‘90,   Rudin et al. ’92,   Weickert ‘98

Linear vs. Nonlinear Diffusion

$$\int_\Omega |\nabla u|^2 \, dx \;\to\; \min$$

$$\int_\Omega |\nabla u| \, dx \;\to\; \min$$

$$u_{deb} = \arg\min_u \int_\Omega (b * u - f)^2 \, dx + \lambda \int_\Omega |\nabla u| \, dx$$

original image | blurred image f | deblurred image u_{deb}

Lions, Osher, Rudin, '92

$$u_{sr} = \arg\min_u \sum_i \|D_i u - f_i\| + \lambda \int_\Omega |\nabla u| \, dx$$

One of several input images f_i

Super-resolution estimate u_{sr}

Schoenemann, Cremers, IEEE TIP '12

$$\min_{u:\Omega\to\mathbb{R}^2} \int_\Omega \left| f_1(x) - f_2(x+u) \right| dx + J(u)$$

Input video

Optical flow field

Horn & Schunck ‘81, Zach et al. DAGM ’07, Wedel et al. ICCV ’09

$$\min_{u:\Omega\to\mathbb{R}^2} \int_\Omega \left| f_1(x) - f_2(x+u) \right| dx + J(u)$$

Input video

Optical flow field*

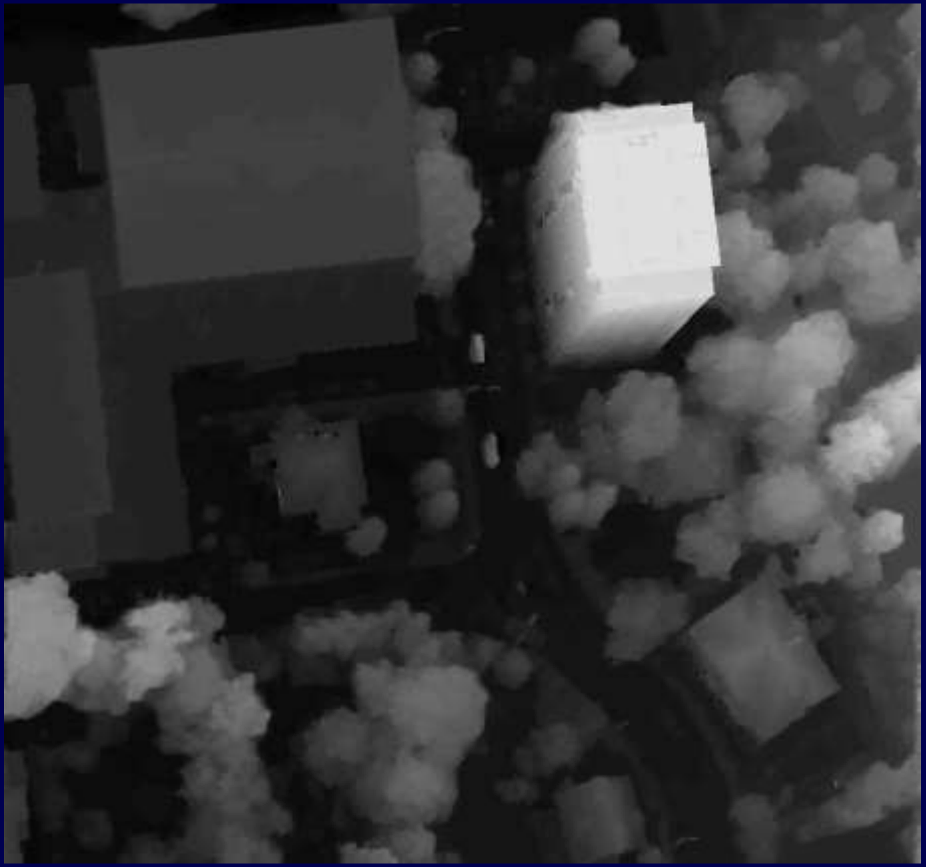
* 60 fps @ 640x480

Horn & Schunck '81, Zach et al. DAGM '07, Wedel et al. ICCV '09

Variational Stereo Reconstruction

$$\min_{u:\Omega\to\mathbb{R}} \int_\Omega \rho(x,u)\,dx + J(u)$$

one of two input images



depth reconstruction

Pock , Schoenemann, Graber, Bischof, Cremers ECCV '08

Variational Scene Flow

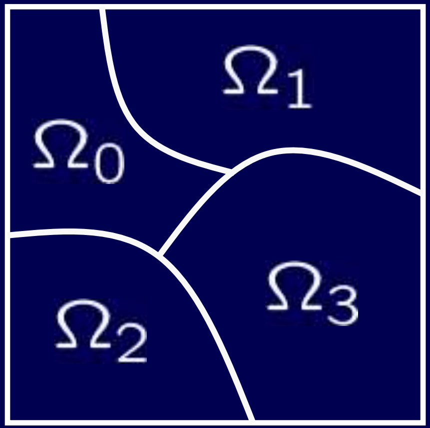
Wedel & Cremers, “Scene Flow”, Springer 2011

Variational Image Segmentation

Mumford, Shah '89, Chambolle, Cremers, Pock '12:

$$\min_{v_1,\dots,v_n:\Omega\to\{0,1\}} \sum_i \int_\Omega v_i\, f_i\, dx + \int_\Omega |\nabla v_i|\, dx$$

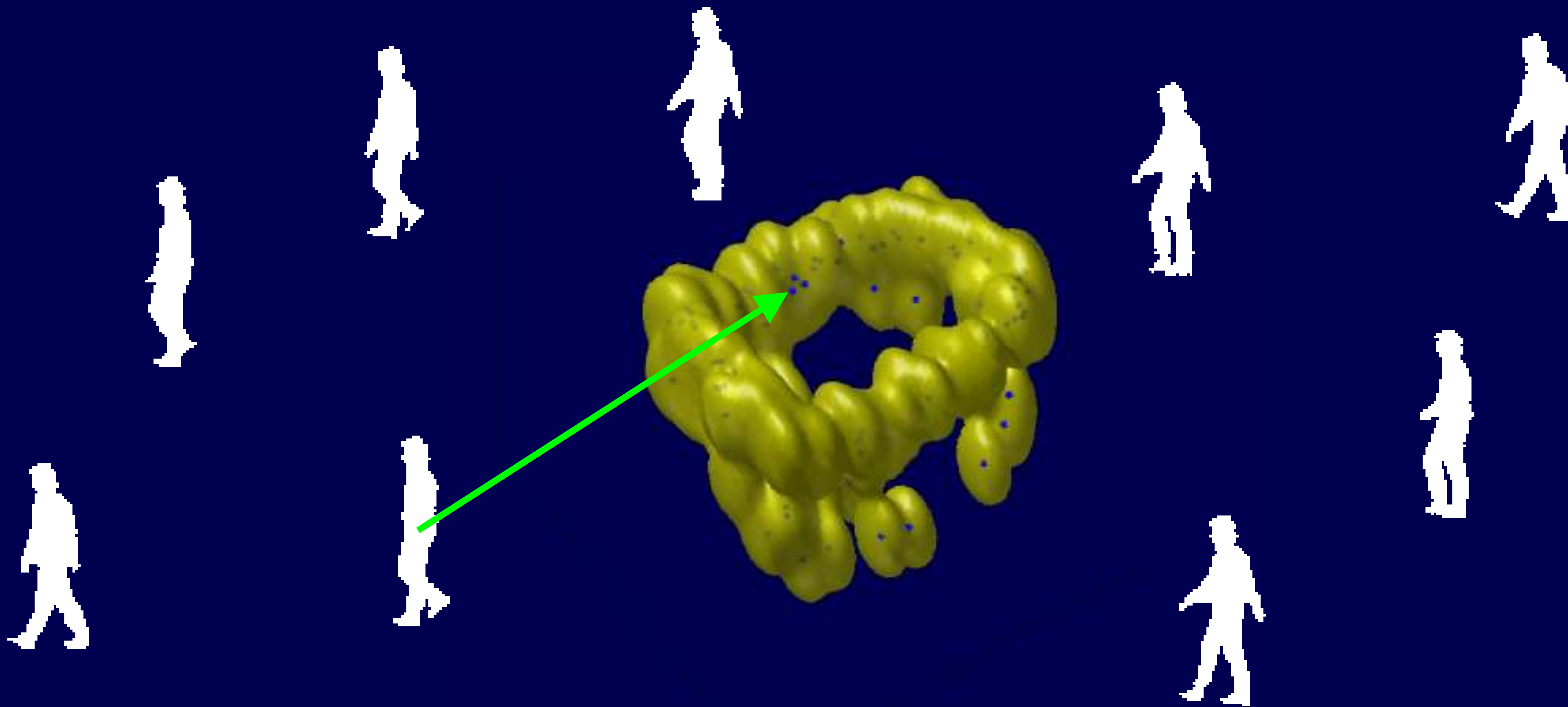
$$\text{s.t.} \quad \sum_i v_i = 1$$



2 label segmentation

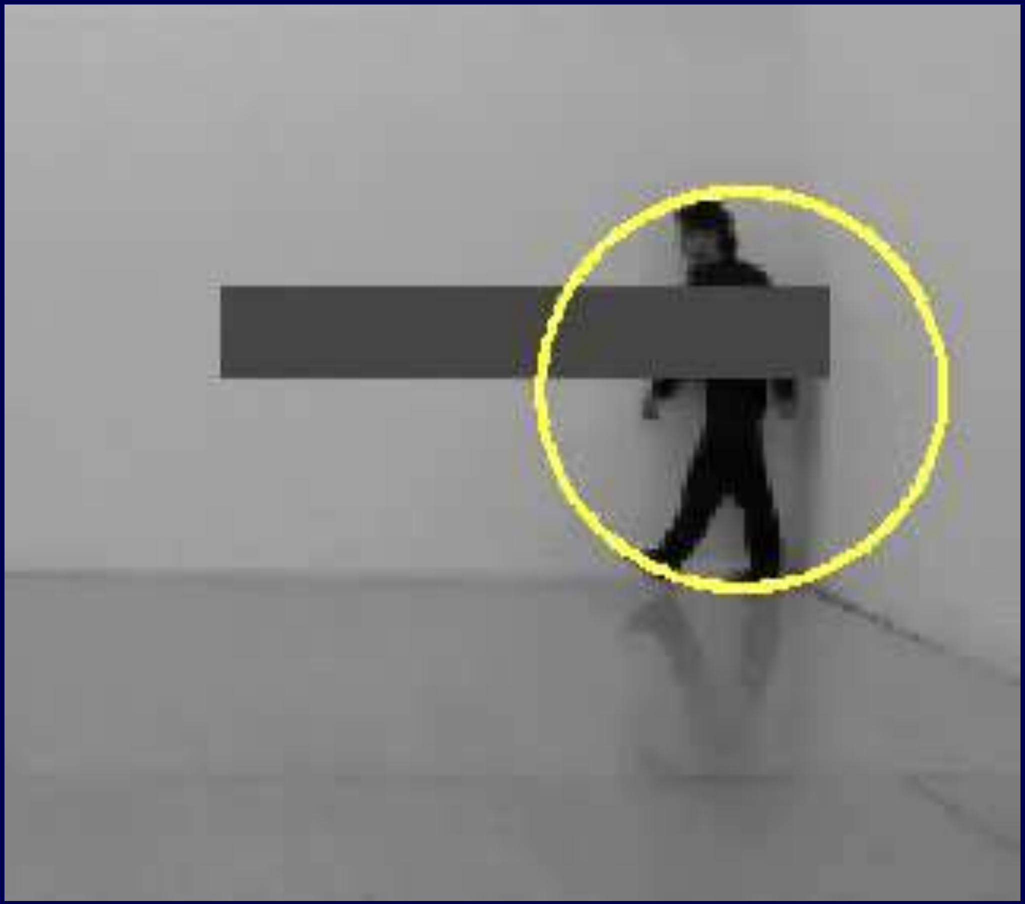
10 label segmentation

Prior Knowledge for Segmentation

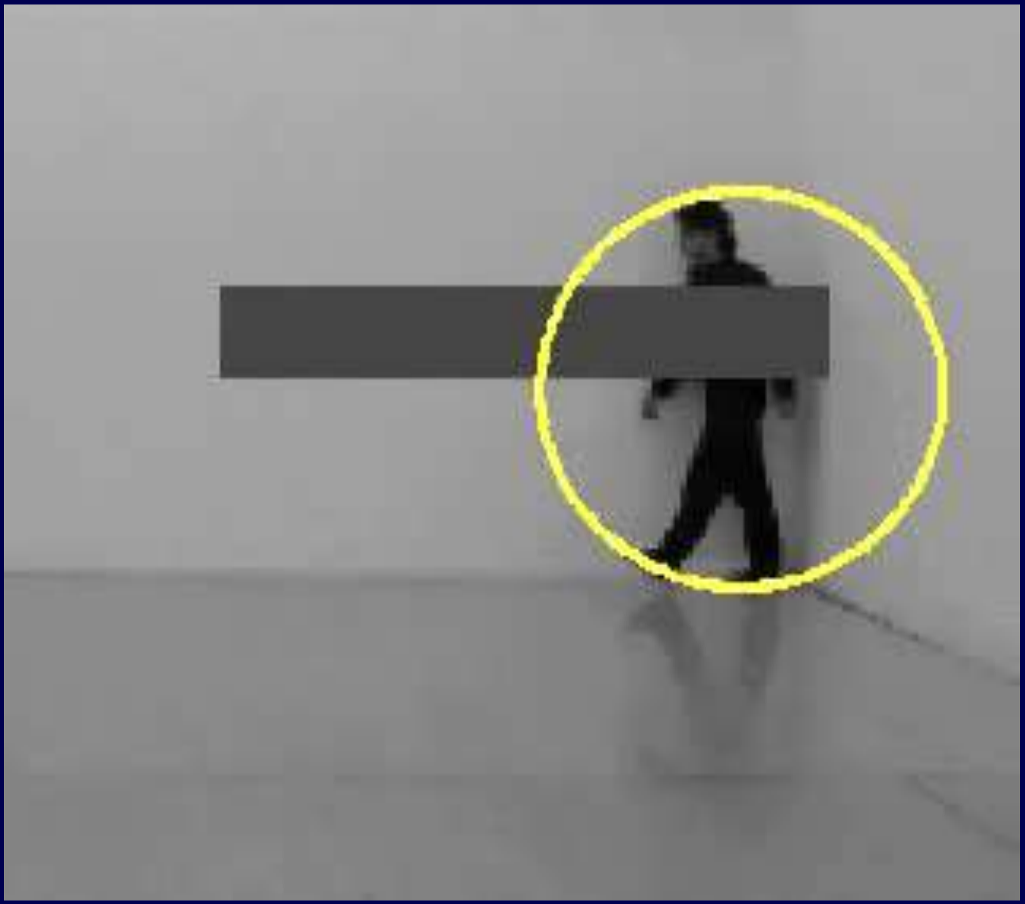


$$\mathcal{P}(C \mid I) = \frac{\mathcal{P}(I \mid C)\,\mathcal{P}(C)}{\mathcal{P}(I)} \xrightarrow{-\log} E = E_{data} + E_{prior}$$

Cremers, Osher, Soatto, IJCV ’06

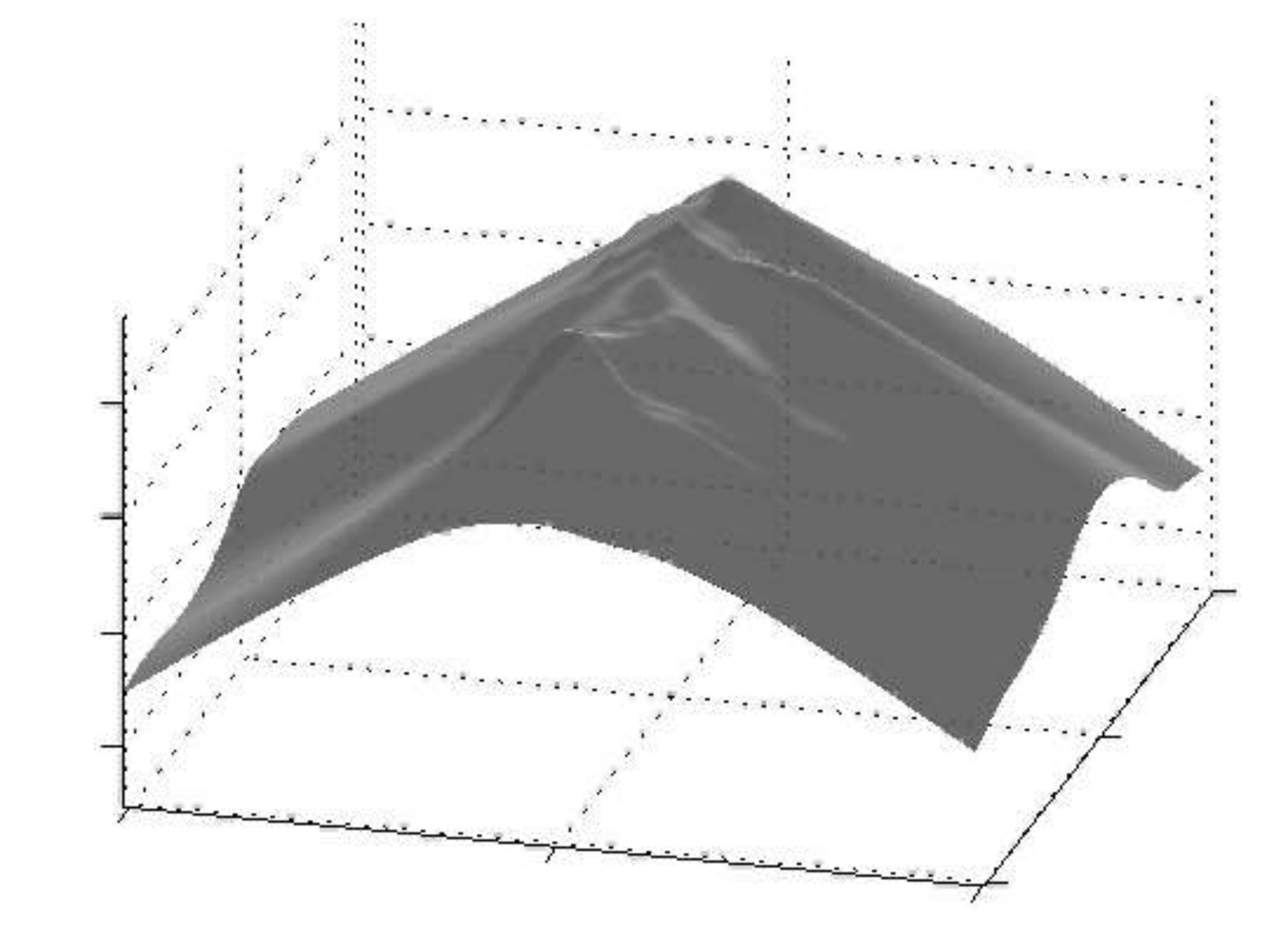


$E_{data} + E_{length} \rightarrow \min$



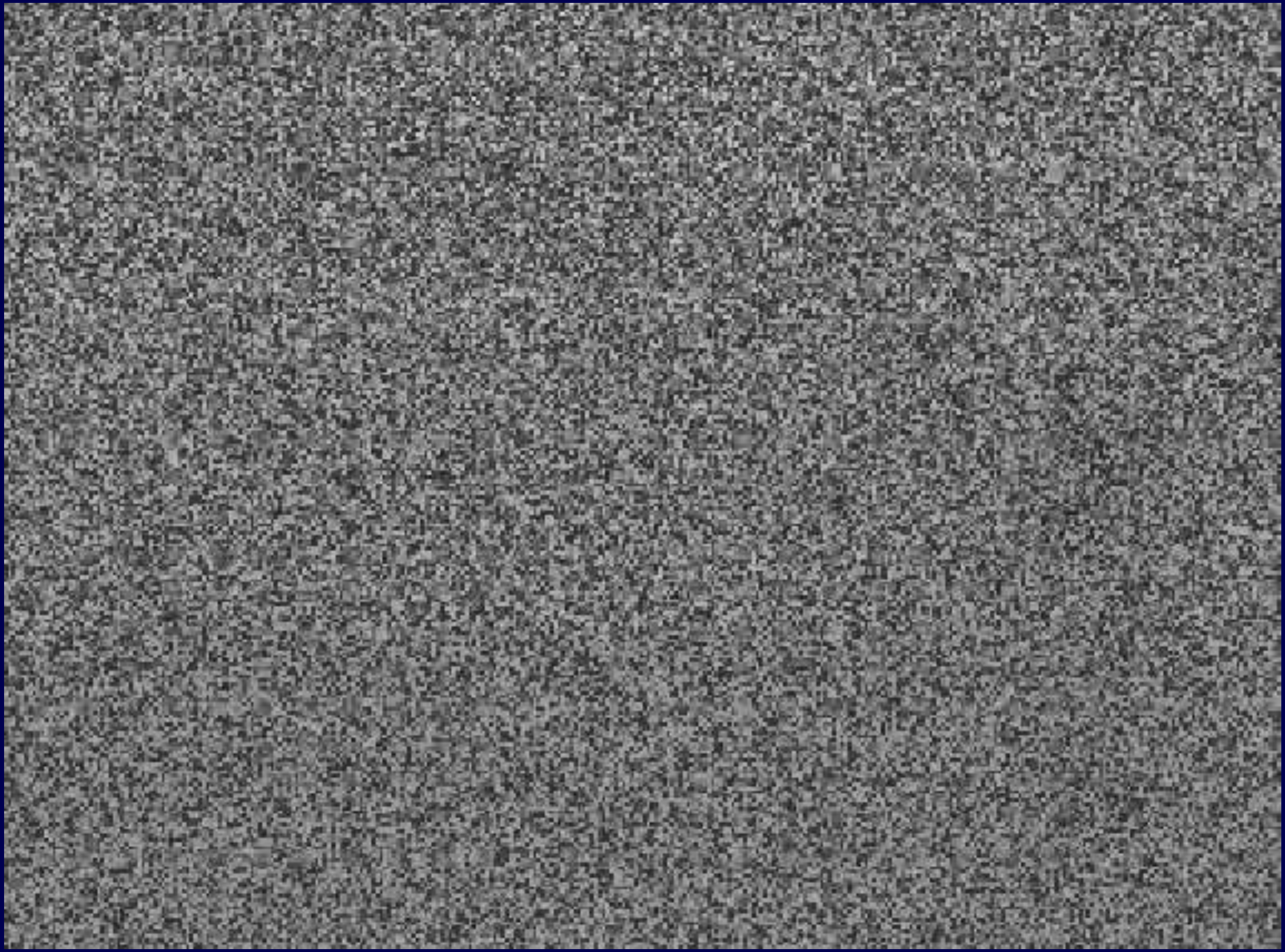
$E_{data} + E_{shape} \rightarrow \min$

Cremers, Osher, Soatto, IJCV '06



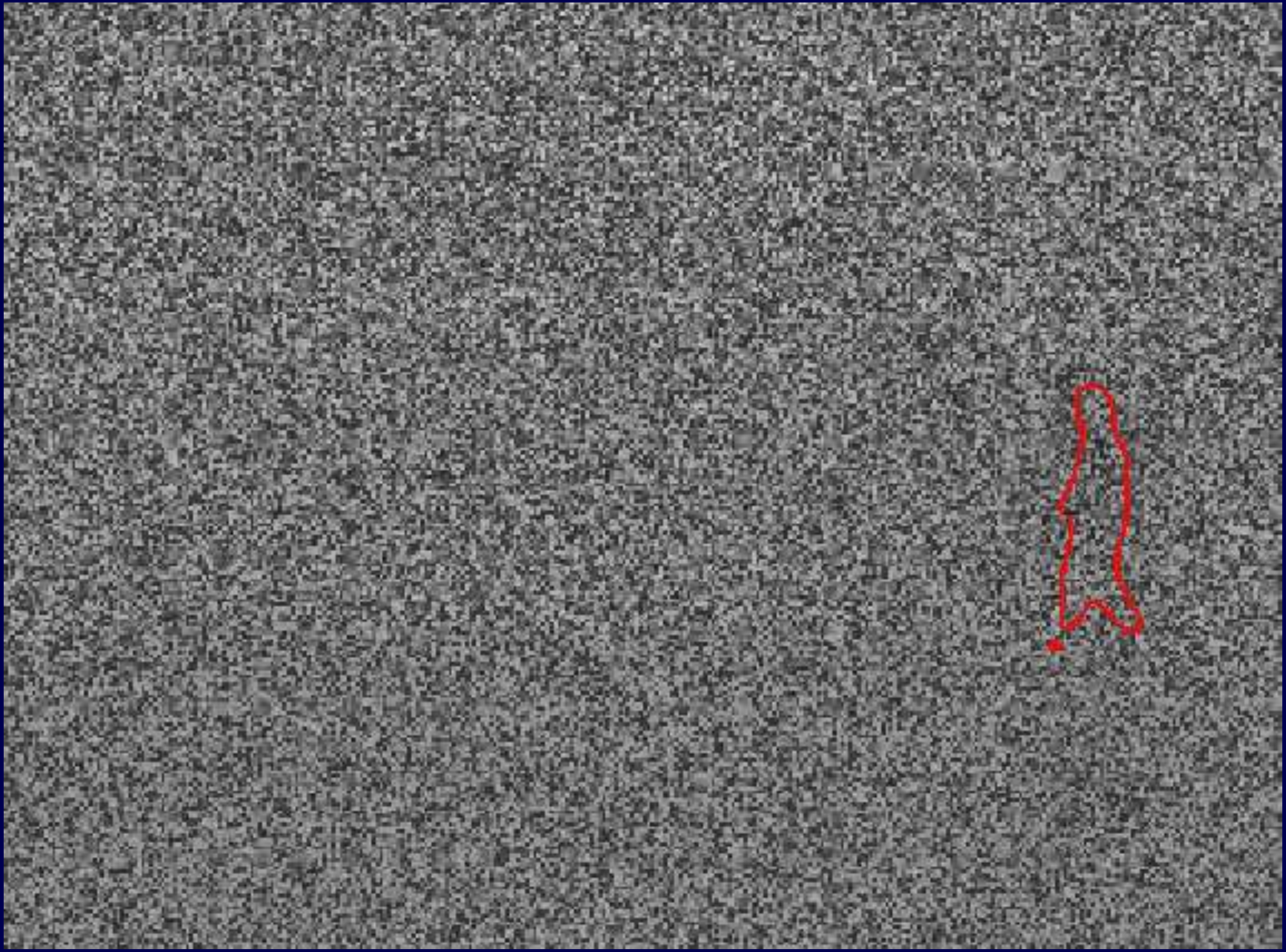
Statistically synthesized embedding functions

Cremers, IEEE PAMI '06



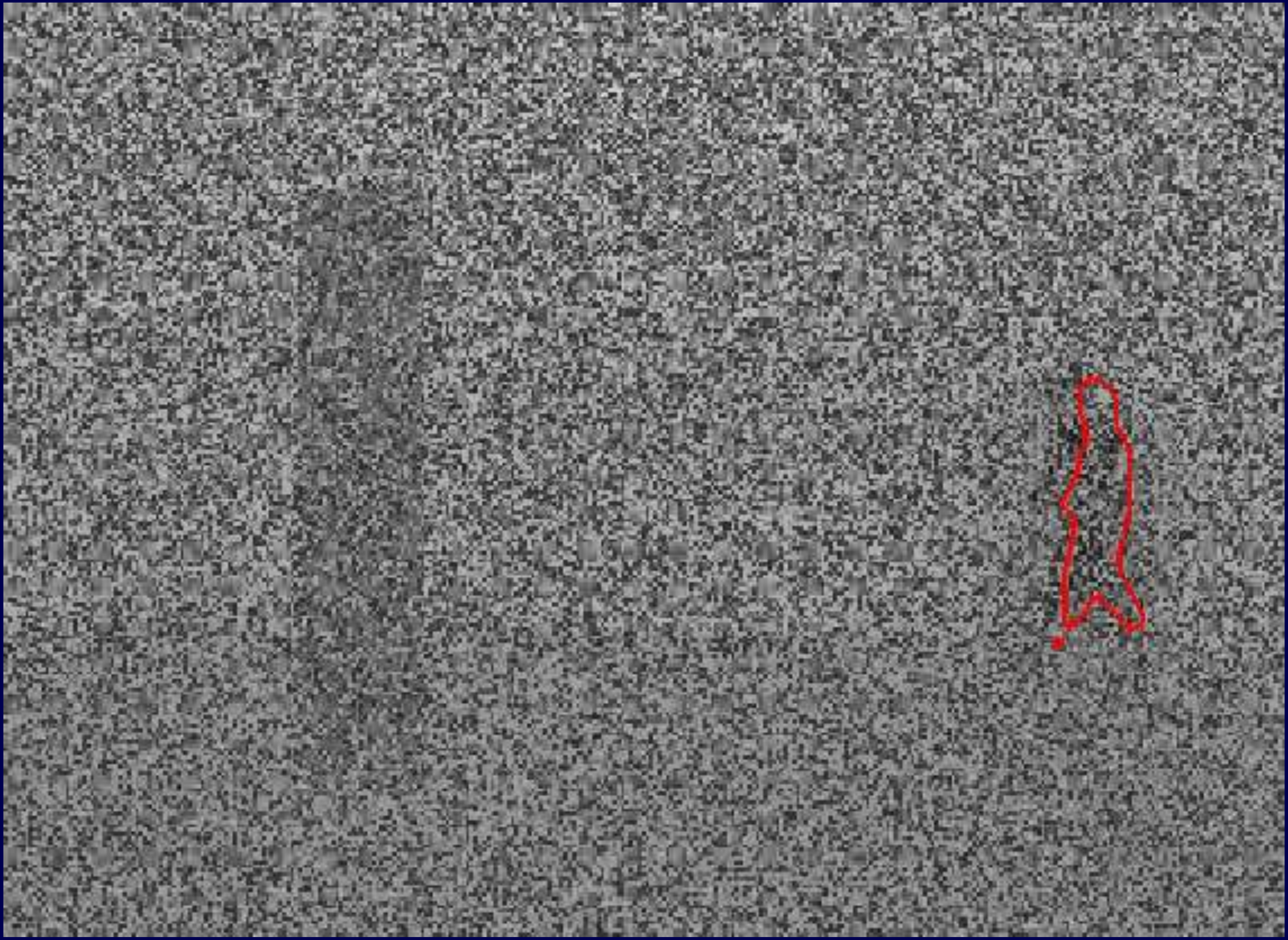
Cremers, IEEE PAMI ’06

Dynamical Shape Priors



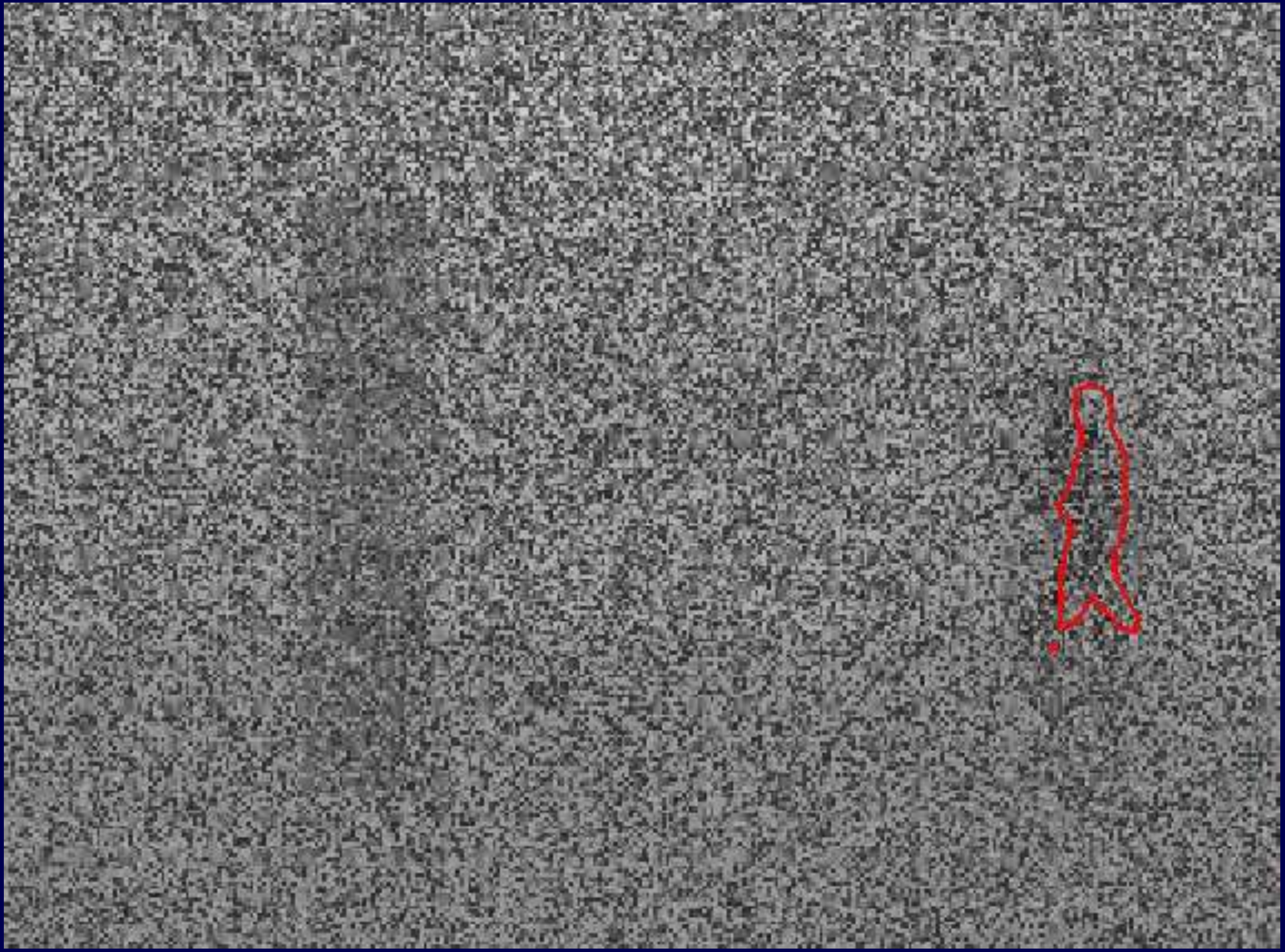
Cremers, IEEE PAMI '06

Dynamical Shape Priors



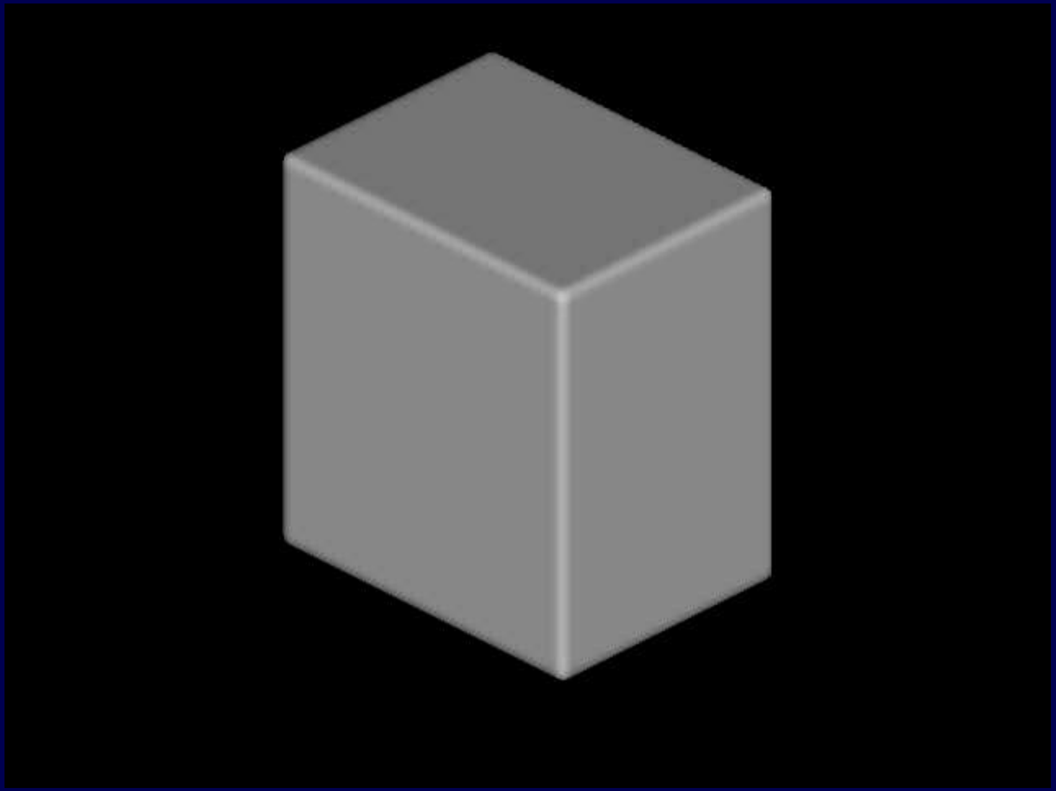
Dynamical shape prior

Cremers, IEEE PAMI ’06

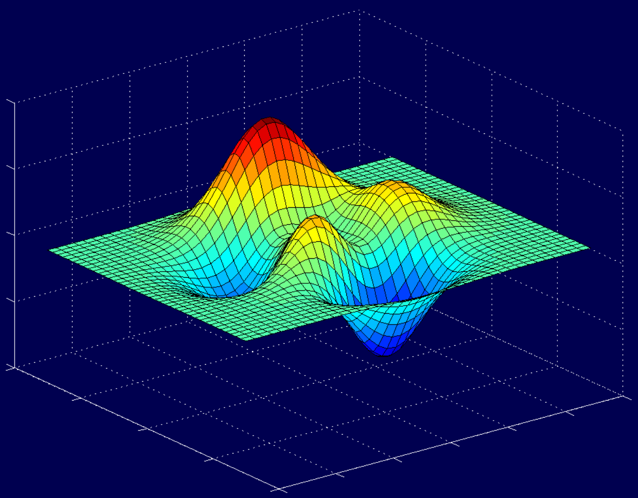


Dynamical prior of shape and translation

Cremers, IEEE PAMI '06



Kolev et al., IJCV '09

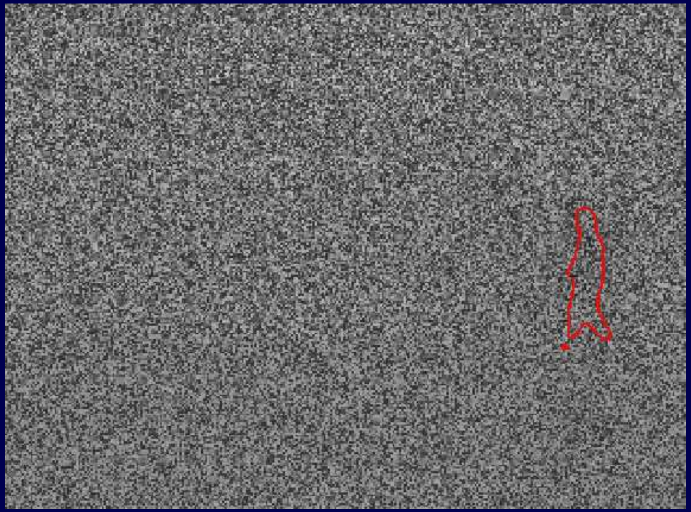


variational methods

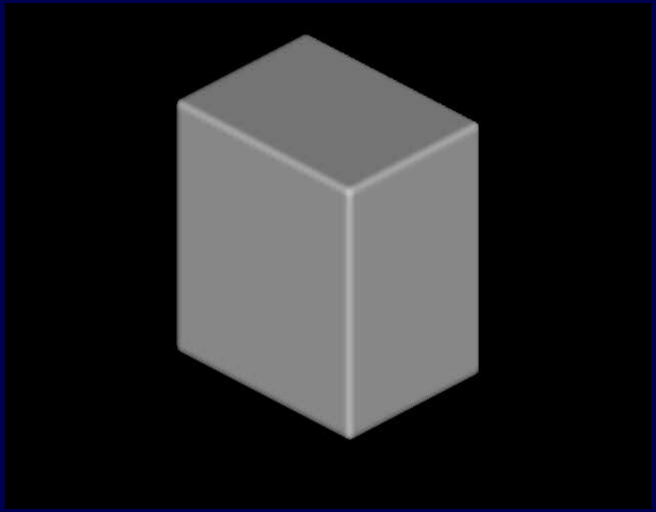
optical flow

scene flow

image segmentation



statistical shape priors



multiview reconstruction