Convex Optimization and Image Segmentation

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Image segmentation:

Geman, Geman '84, Blake, Zisserman '87, Kass et al. '88, Mumford, Shah '89, Caselles et al. '95, Kichenassamy et al. '95, Paragios, Deriche '99, Chan, Vese '01, Tsai et al. '01, …

Multiview stereo reconstruction:

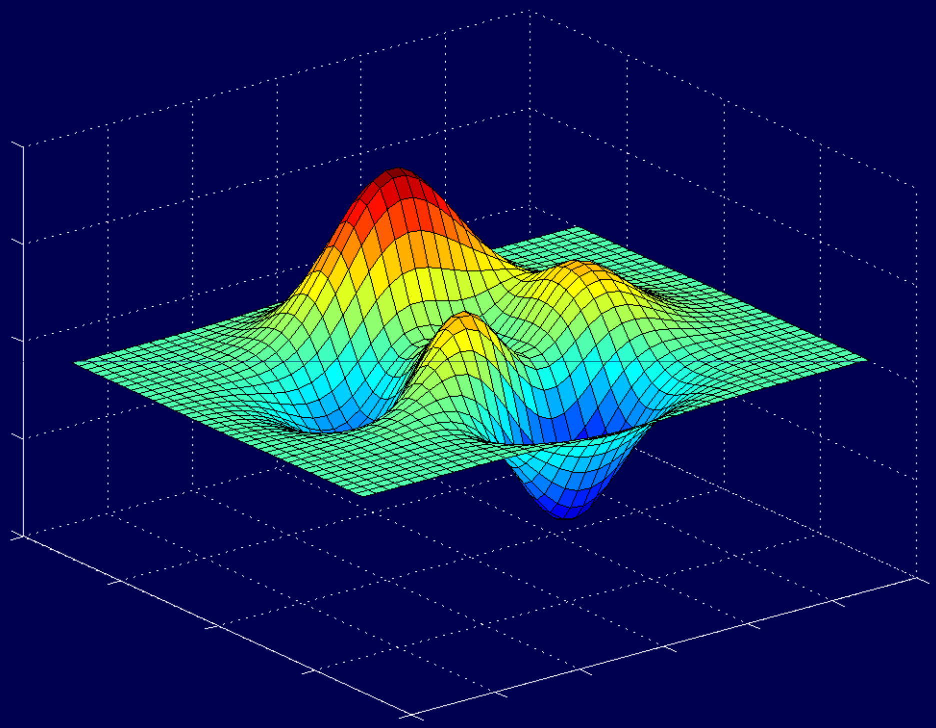
Faugeras, Keriven '98, Duan et al. '04, Yezzi, Soatto '03, Seitz et al. '06, Hernandez et al. '07, Labatut et al. '07, …

Non-convex energies

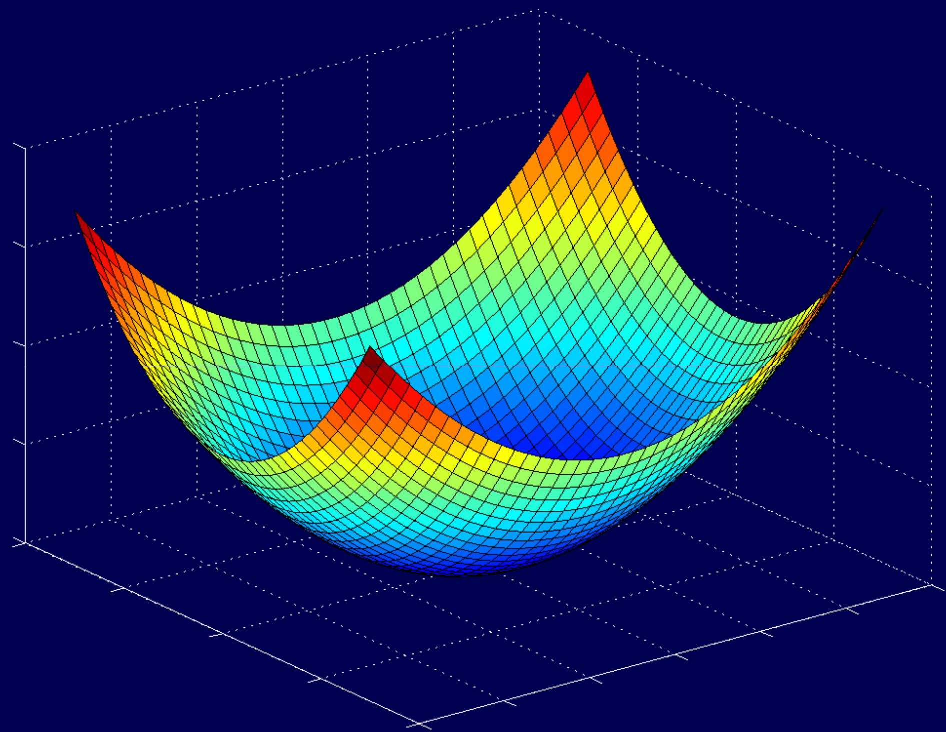
Optical flow estimation:

Horn, Schunck '81, Nagel, Enkelmann '86, Black, Anandan '93, Alvarez et al. '99, Brox et al. '04, Baker et al. '07, Zach et al. '07, Sun et al. '08, Wedel et al. '09, …

Non-convex versus Convex Energies



Non-convex energy



Convex energy

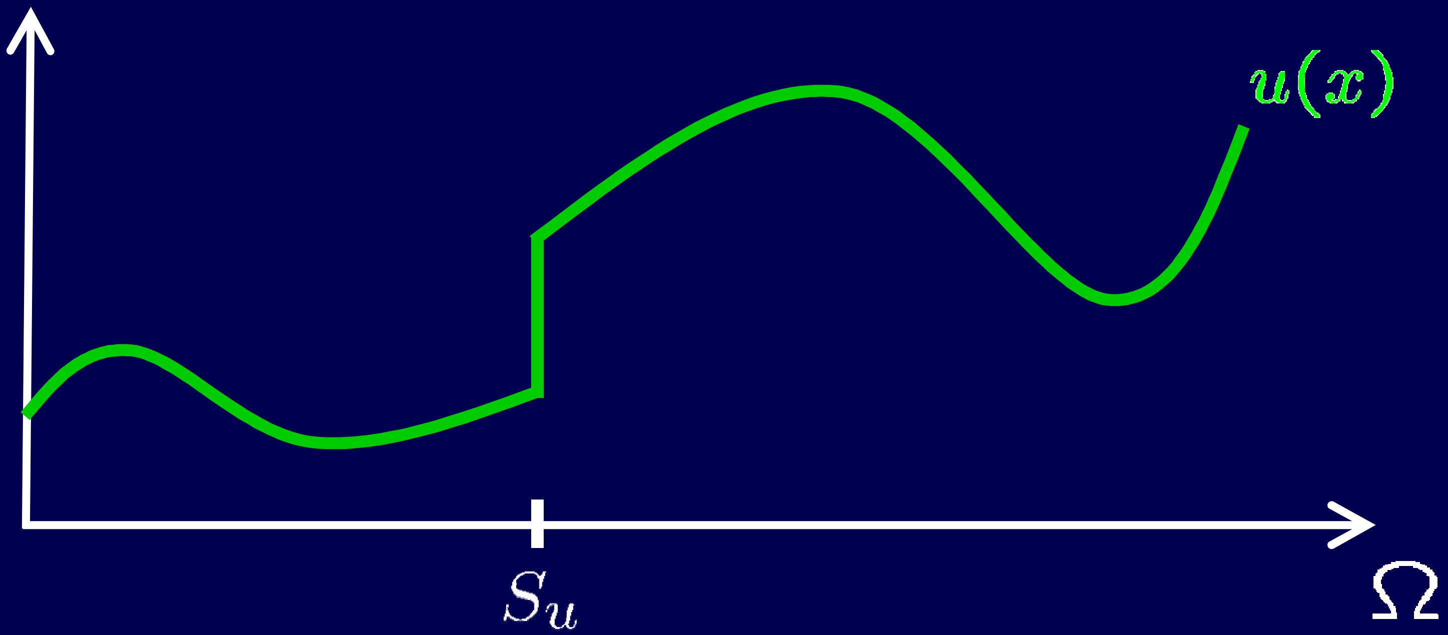
Some related work: *Brakke '95, Alberti et al. '01, Ishikawa '01, Chambolle '01, Attouch et al. '06, Nikolova et al. '06, Bresson et al. '07, Zach et al. '08, Lellmann et al. '08, Zach et al. '09, Brown et al. '10, Bae et al. '10, Yuan et al. '10,…*

The Mumford-Shah Functional

Let $\Omega \subset \mathbb{R}^d$ and $f, u : \Omega \to \mathbb{R}^k$.

$$E(u) = \int_{\Omega} |f - u|^2 dx + \lambda \int_{\Omega \setminus S_u} |\nabla u|^2 dx + \nu \mathcal{H}^1(S_u)$$

Mumford, Shah '89, Blake, Zisserman '87
Ambrosio, Tortorelli '90, Vese, Chan '02

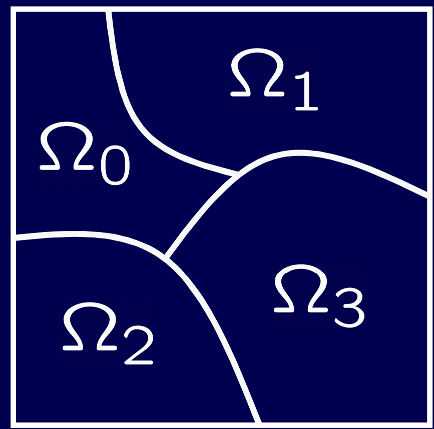


Let $\Omega \subset \mathbb{R}^d$ and $f, u : \Omega \to \mathbb{R}^k$.

$$E(u) = \int_{\Omega} |f - u|^2 dx \;+\; \lambda \int_{\Omega \setminus S_u} |\nabla u|^2 dx \;+\; \nu \, |S_u|$$

Mumford, Shah '89, Blake, Zisserman '87
Ambrosio, Tortorelli '90, Vese, Chan '02

Piecewise constant approximation for $\lambda \to \infty$:



$$E(\{\Omega_i, \mu_i\}_i) = \sum_i \int_{\Omega_i} |f(x) - \mu_i|^2 \, dx \;+\; \frac{\nu}{2} \left| \partial \Omega_i \right|$$

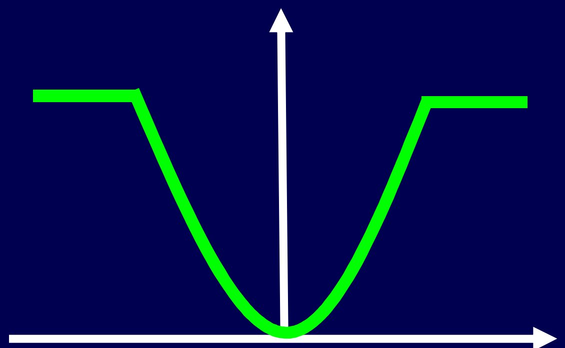
s.t. $\bigcup_i \Omega_i = \Omega$, and $\Omega_i \cap \Omega_j = \emptyset \;\; \forall i \neq j$

Mumford, Shah '89, Chan, Vese '01, Potts '52, Ising '25

Comparison of Regularizers

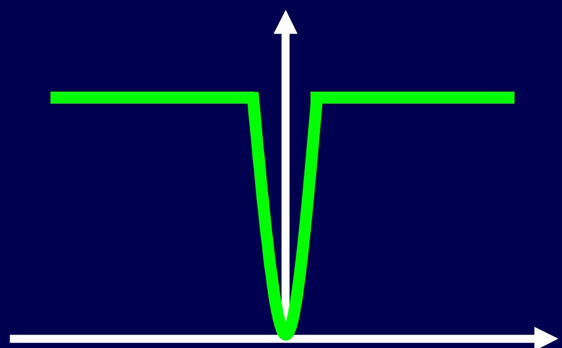
$$E(u) = \int_\Omega |f - u|^2 dx \; + \; \int_\Omega \psi(\nabla u)\, dx$$

$$\psi(s) = \min(|s|^2, \nu)$$



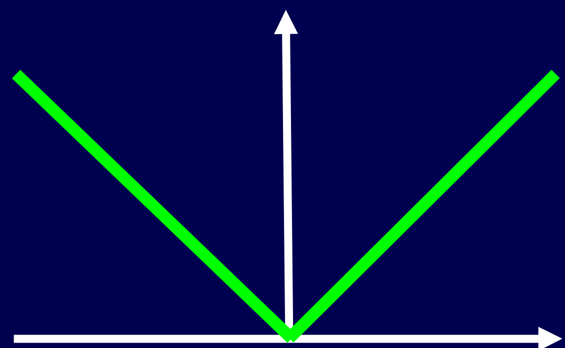
Truncated quadratic

$$\psi(s) = \begin{cases} 0, & \text{if } s = 0 \\ \nu, & \text{else} \end{cases}$$



Potts model

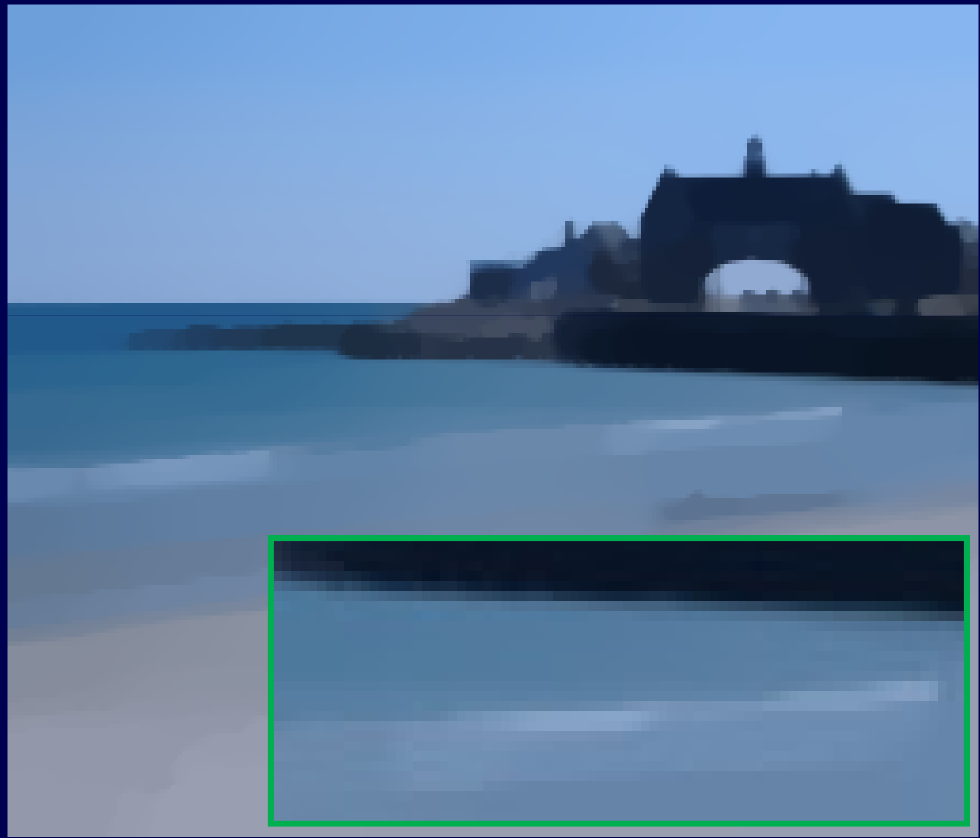
$$\psi(s) = |s|$$



Linear / TV

Total Variation Denoising

Input image

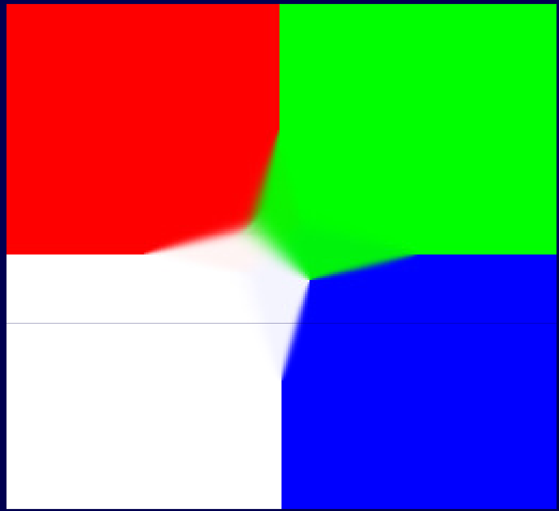


TV-denoised

+ Convex & fast to minimize
- Oversmoothing in flat regions (staircasing)
- Reduces contrast at edges

Overview

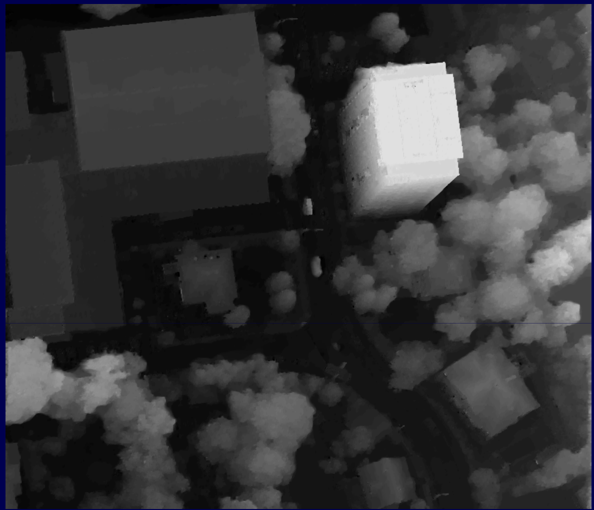
Convex multilabel optimization



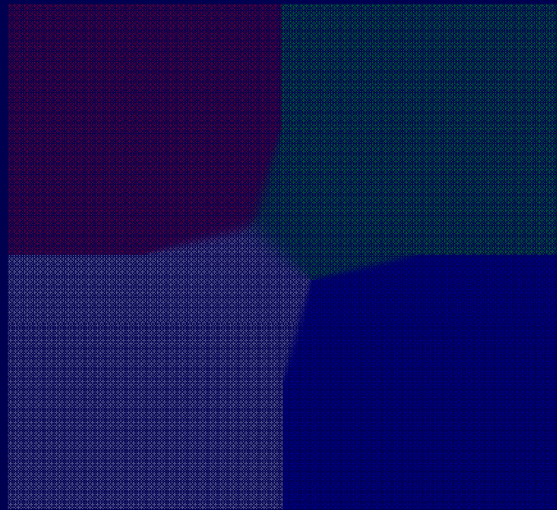
Minimal partitions

Semantic segmentation

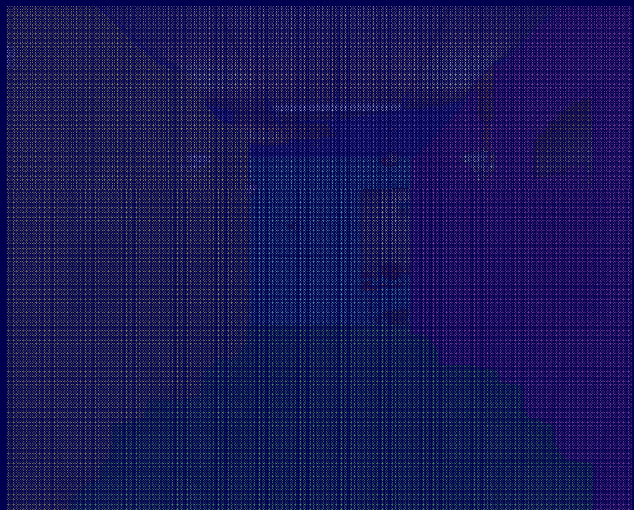
Mumford-Shah



Convex multilabel optimization



Minimal partitions



Semantic segmentation

Mumford-Shah

$$u : \Omega \to \Gamma = [\gamma_{min}, \gamma_{max}]$$

$$E(u) = \underbrace{\int_\Omega \rho(x, u(x))\, dx}_{\text{nonconvex data term}} + \underbrace{\int_\Omega |\nabla u(x)|\, dx}_{\text{label regularity}} \qquad (*)$$

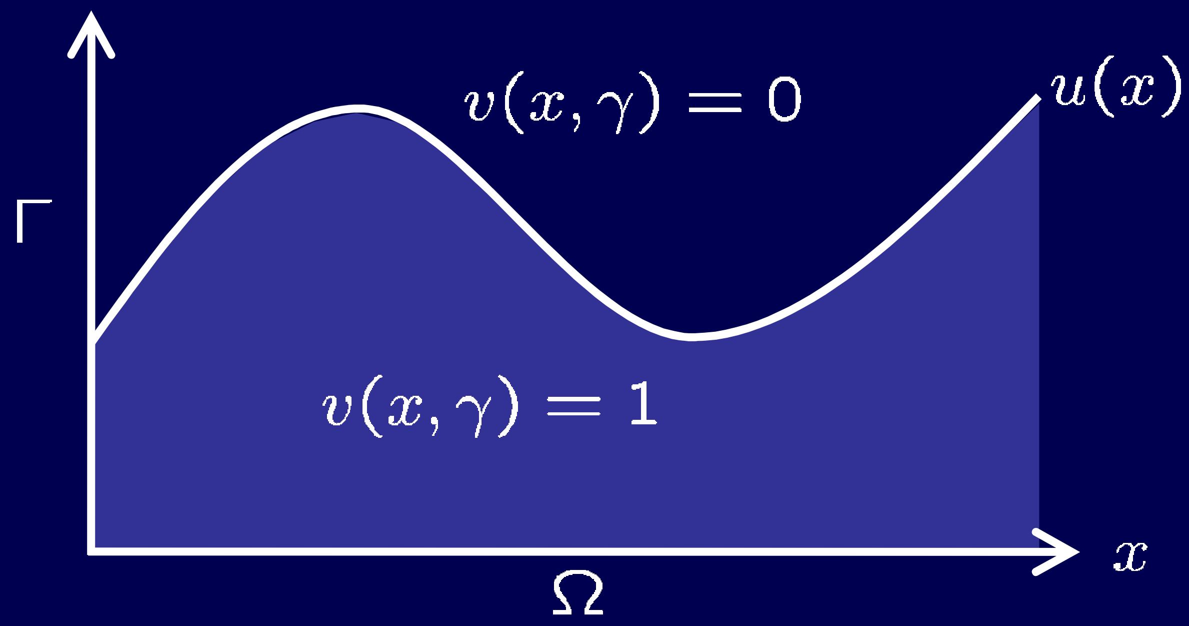
Pock , Schoenemann, Graber, Bischof, Cremers ECCV '08

Cartesian Currents and Relaxation

$$u : \Omega \rightarrow \Gamma = [\gamma_{min}, \gamma_{max}]$$

$$E(u) = \int_\Omega \rho(x, u(x))\, dx + \int_\Omega |\nabla u(x)|\, dx \qquad (*)$$

Let $v : (\Sigma = \Omega \times \Gamma) \rightarrow \{0, 1\}$ $\qquad v(x, \gamma) = \mathbf{1}_{u \geq \gamma}(x)$



Ishikawa , PAMI '03

Pock , Schoenemann, Graber, Bischof, Cremers ECCV '08

$$u : \Omega \to \Gamma = [\gamma_{min}, \gamma_{max}]$$

$$E(u) = \int_\Omega \rho(x, u(x))\, dx + \int_\Omega |\nabla u(x)|\, dx \qquad (*)$$

nonconvex functional

Let $v : (\Sigma = \Omega \times \Gamma) \to \{0, 1\}$ $\qquad v(x,\gamma) = \mathbf{1}_{u \geq \gamma}(x)$

Theorem: Minimizing $(*)$ is equivalent to minimizing

$$F(v) = \int_\Sigma \rho(x,\gamma)|\partial_\gamma v(x,\gamma)| + |\nabla v(x,\gamma)|\, dx\, d\gamma \qquad (**)$$

convex functional

Solve $(**)$ in relaxed space ($v : \Sigma \to [0,1]$) and threshold to obtain a globally optimal solution.

Pock , Schoenemann, Graber, Bischof, Cremers ECCV '08

Evolution to Global Minimum

Global Optima for Convex Regularizers

Let

$$E(u) = \int_\Omega f(x, u, \nabla u)\, dx$$

be continuous in $x \in \mathbb{R}^d$ and u, and convex in ∇u.

Theorem:

$E(u)$ can be minimized globally by solving the saddle point problem

$$\min_v \sup_{\phi \in \mathcal{K}} \int_{\Omega \times \mathbb{R}} \phi \cdot D\, v,$$

where ϕ is constrained to the convex set

$$\mathcal{K} = \Big\{ \phi = (\phi^x, \phi^t) \in C_0\left(\Omega \times \mathbb{R}; \mathbb{R}^d \times \mathbb{R}\right) \; : \; \phi^t(x,t) \geq f^*(x, t, \phi^x(x,t)) \,, \; \forall x, t \in \Omega \times \mathbb{R} \Big\} \,.$$

Pock, Cremers, Bischof, Chambolle, SIAM J. on Imaging Sciences '10

Given the saddle point problem

$$\min_{v \in C} \max_{\phi \in K} \langle Av, \phi \rangle + \langle g, v \rangle - \langle h, \phi \rangle$$

with closed convex sets C and K and linear operator A of norm L.

The iterative algorithm

$$\begin{cases} \phi^{n+1} = \Pi_K\big(\phi^n + \sigma(A\bar{v}^n - h)\big) \\ v^{n+1} = \Pi_C\big(v^n - \tau(A^*\phi^{n+1} + g)\big) \\ \bar{v}^{n+1} = 2\,v^{n+1} - v^n \end{cases}$$

converges with rate $O(1/n)$ to a saddle point for $\sigma\,\tau\,L^2 \leq 1$.

Pock, Cremers, Bischof, Chambolle, ICCV '09, Chambolle, Pock '10

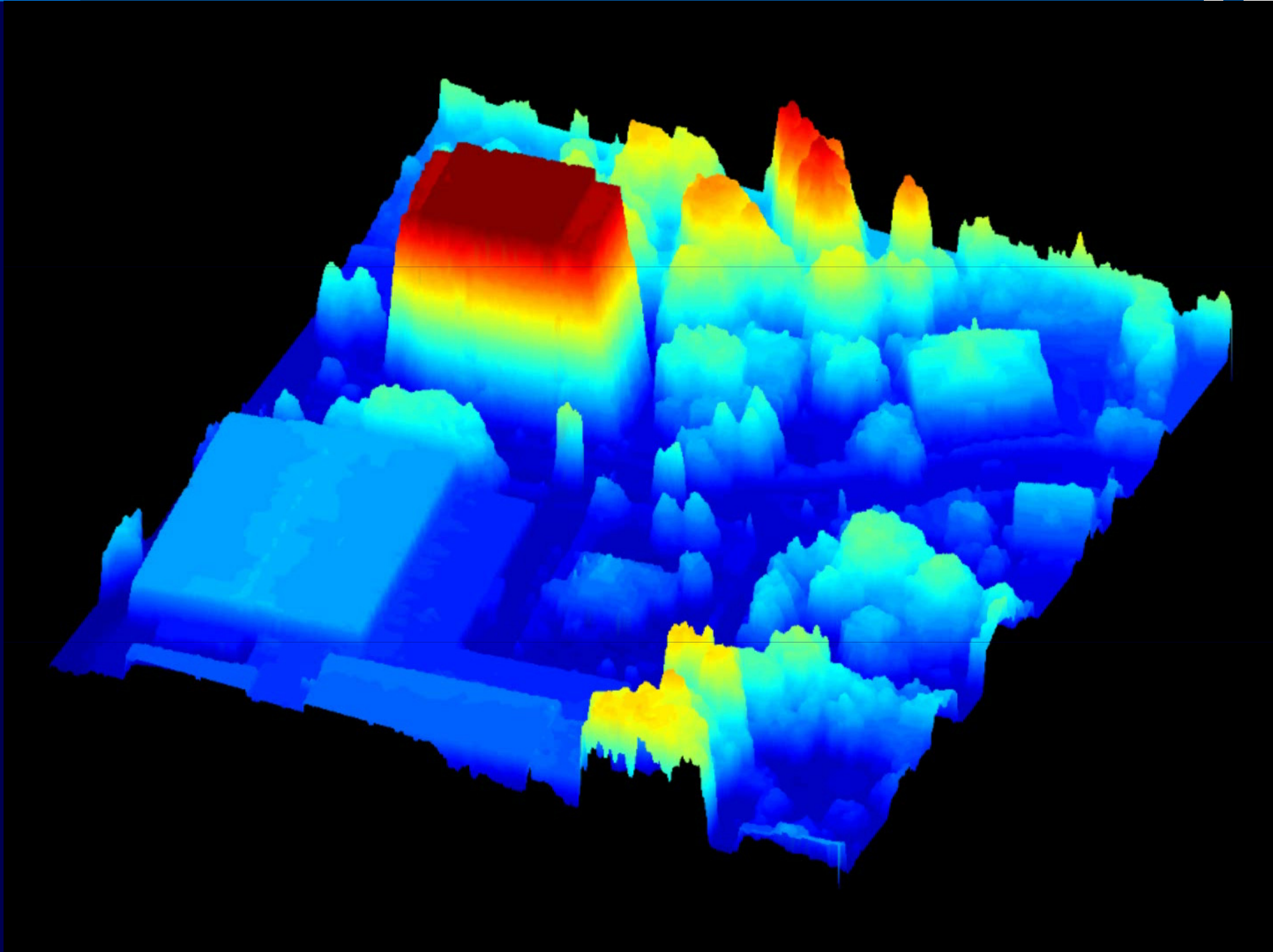
Reconstruction from Aerial Images

One of two input images

Courtesy of Microsoft

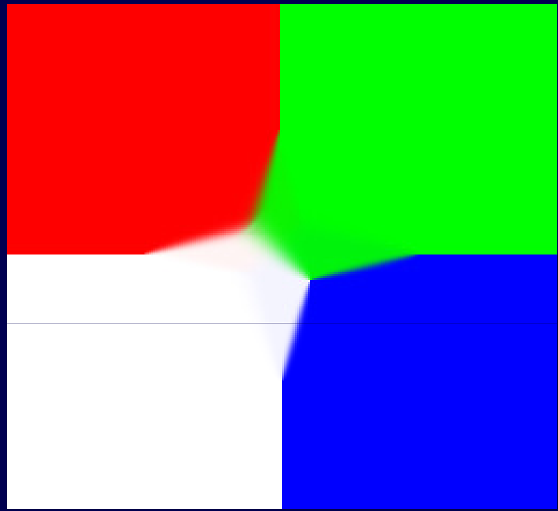
Depth reconstruction

Reconstruction from Aerial Images

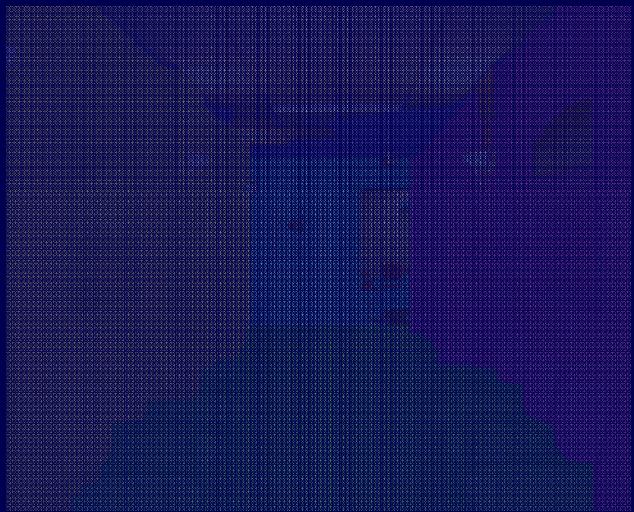


Overview

Convex multilabel optimization



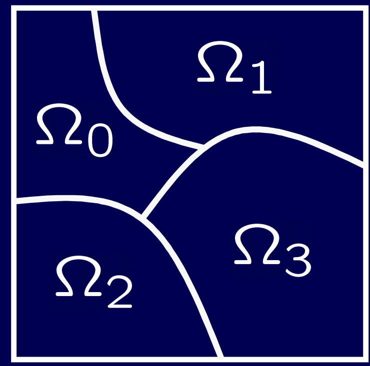
Minimal partitions



Semantic segmentation

Mumford-Shah

$$\min_{\Omega_0,\dots,\Omega_n} \frac{1}{2}\sum_i |\partial\Omega_i| + \sum_i \int_{\Omega_i} f_i(x)\,dx$$



$$\text{s.t.} \quad \bigcup_i \Omega_i = \Omega \subset \mathbb{R}^d, \quad \text{and } \Omega_i \cap \Omega_j = \emptyset \;\; \forall i \neq j$$

Potts '52, Blake, Zisserman '87, Mumford-Shah '89, Vese, Chan '02

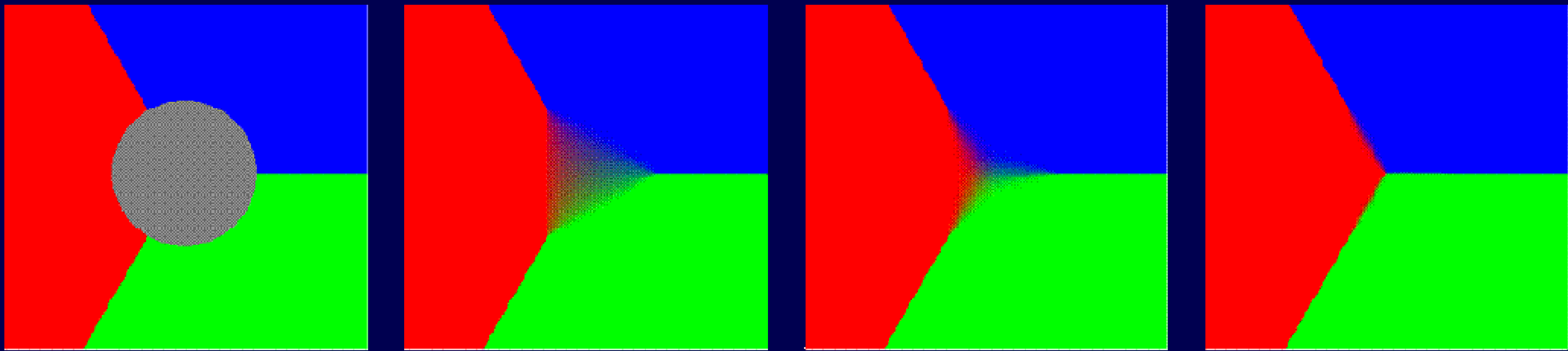
Proposition: With $v_i = 1_{\Omega_i}$, this is equivalent to

$$\min_{v\in\mathcal{B}} \frac{1}{2}\sum_i \int_\Omega |Dv_i| + \int_\Omega v_i\, f_i\, dx = \min_{v\in\mathcal{B}} \sup_{p\in\mathcal{K}} \sum_i \int_\Omega v_i \operatorname{div} p_i\, dx + \int_\Omega v_i\, f_i\, dx$$

$$\text{where} \quad \mathcal{K} = \left\{ p = (p_1,\dots,p_n)^\top \in \mathbb{R}^{n\times d} : \left|p_i - p_j\right| \leq 1, \; \forall i < j \right\}$$

Chambolle, Cremers, Pock '08, SIIMS '12

Test Case: The Triple Junction

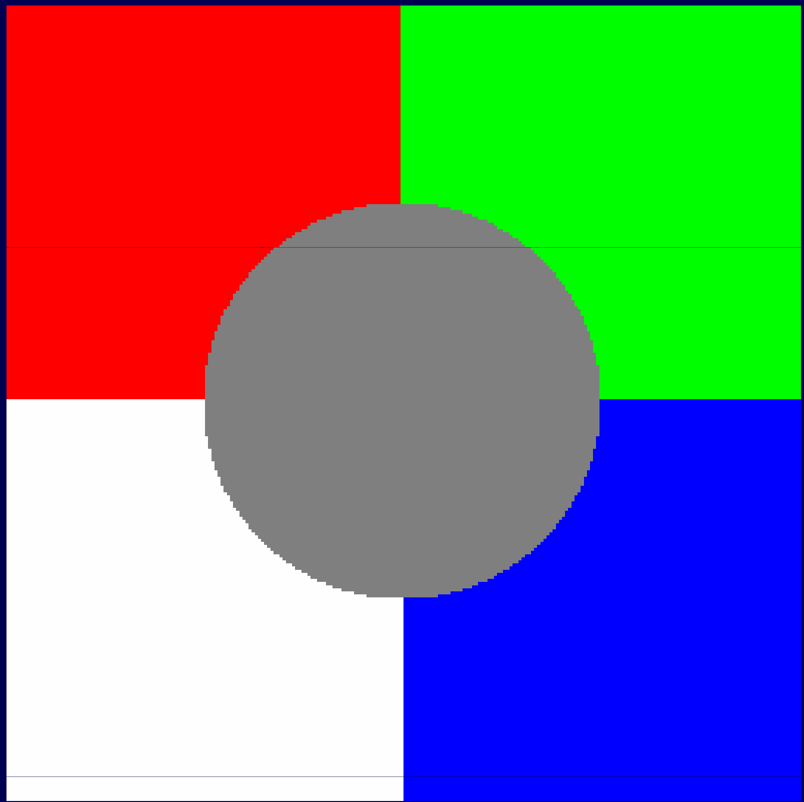


Input image Lellmann et al. '08 Zach et al. '08 our approach

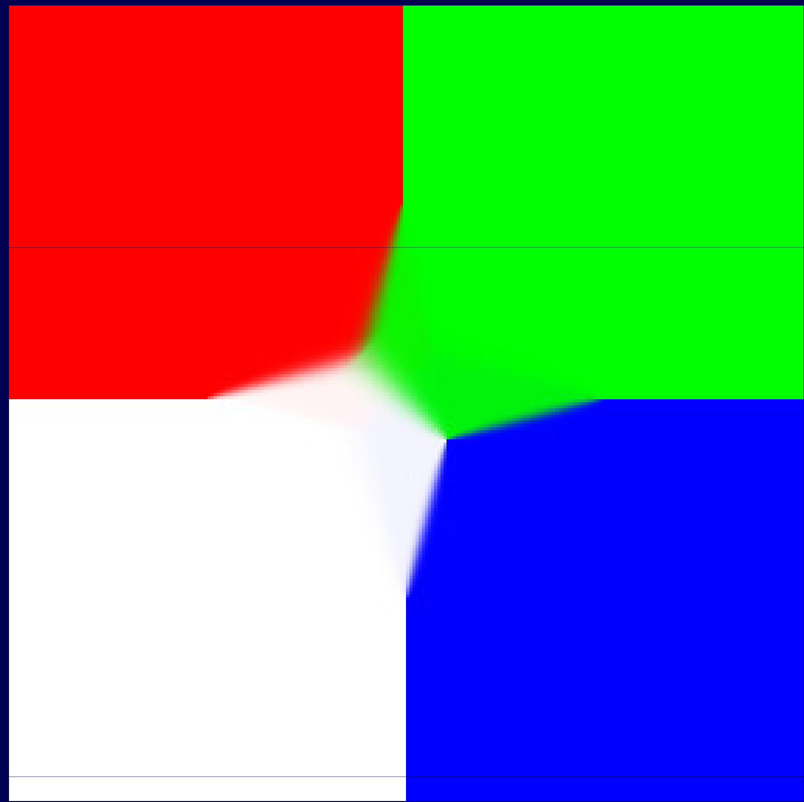
Proposition: The proposed relaxation strictly dominates alternative relaxations.

Chambolle, Cremers, Pock '08, SIIMS '12

Four-Region Case

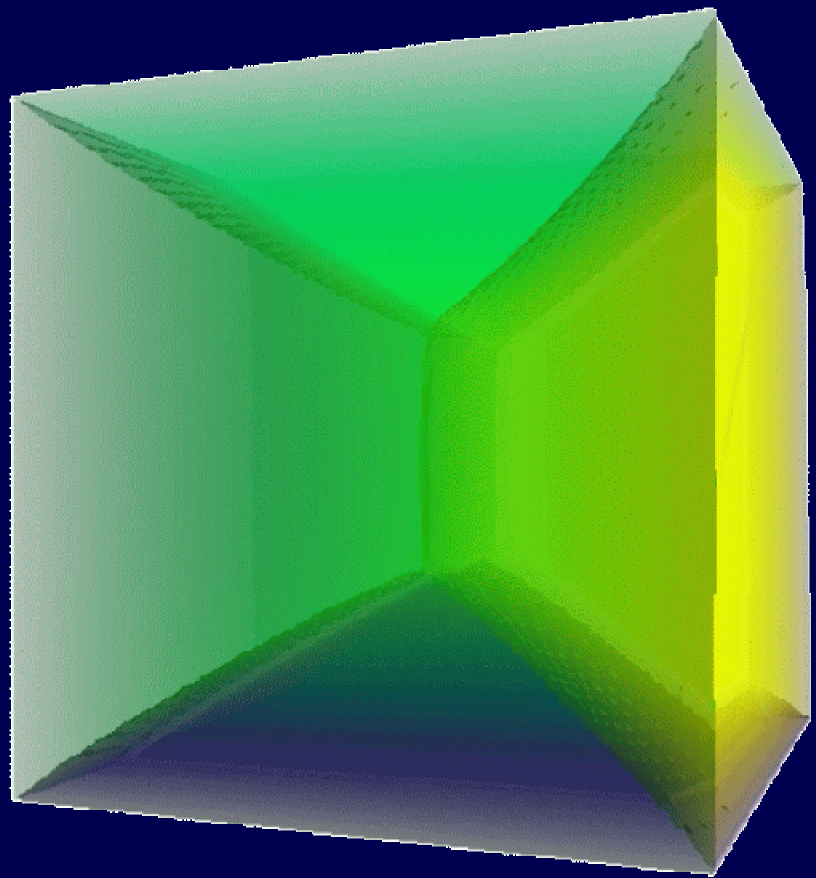


Input image

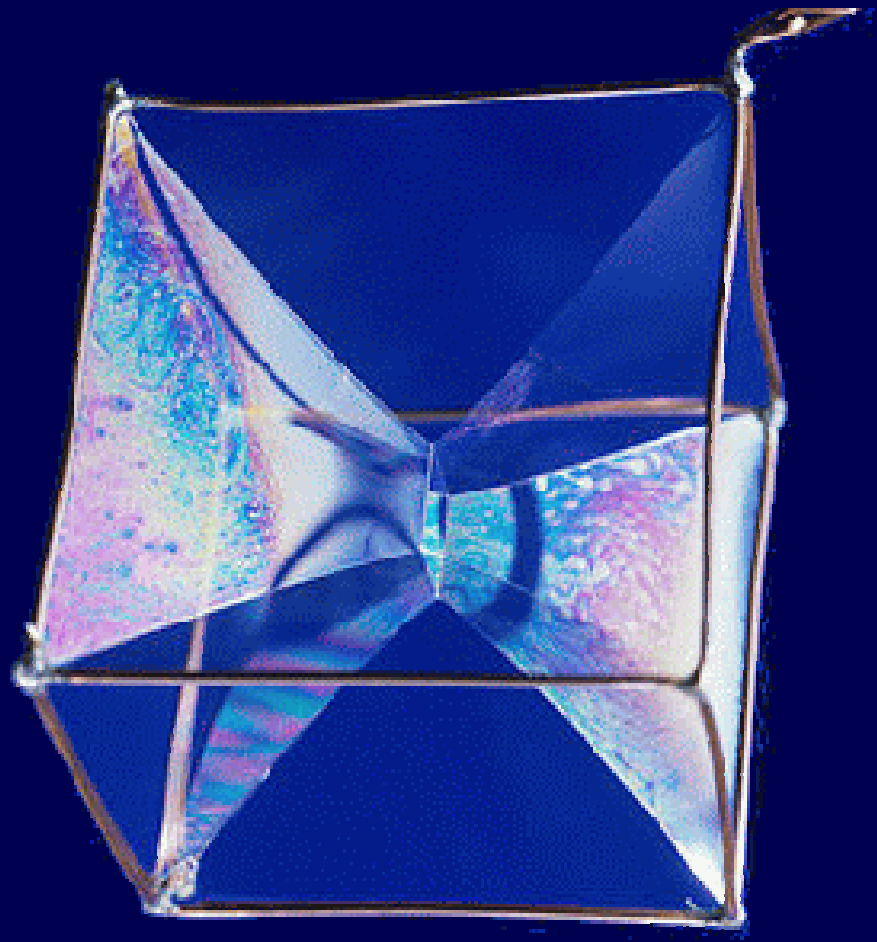


Inpainted

Chambolle, Cremers, Pock '08, SIIMS '12



3D min partition inpainting



Photograph of a soap film

Chambolle, Cremers, Pock '08, SIIMS '12

The Minimal Partition Problem

Input color image

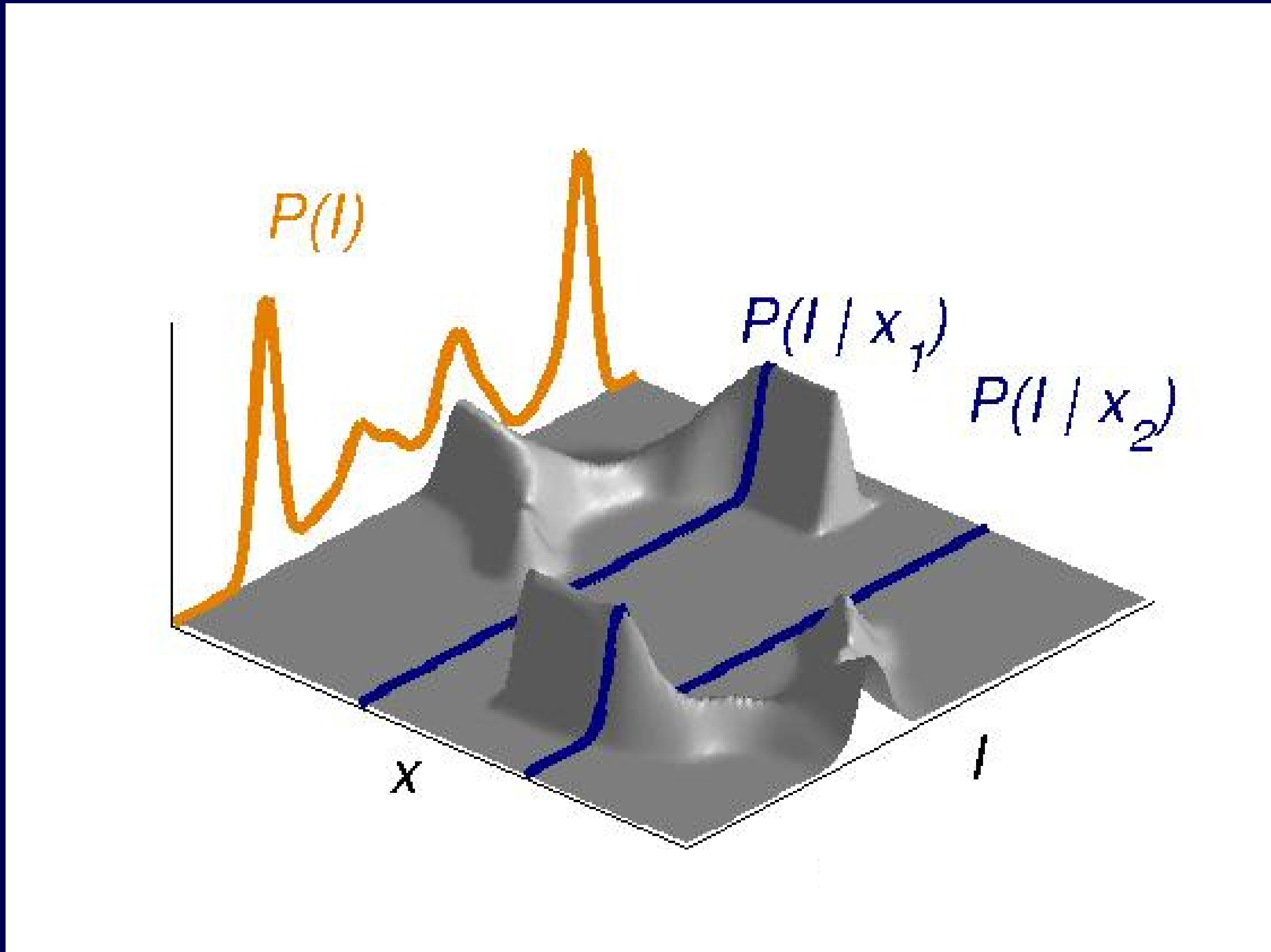
10 label segmentation

Chambolle, Cremers, Pock ’08, SIIMS ‘12

Interactive Segmentation

Nieuwenhuis, Cremers, PAMI '12

Space-dependent Color Likelihoods



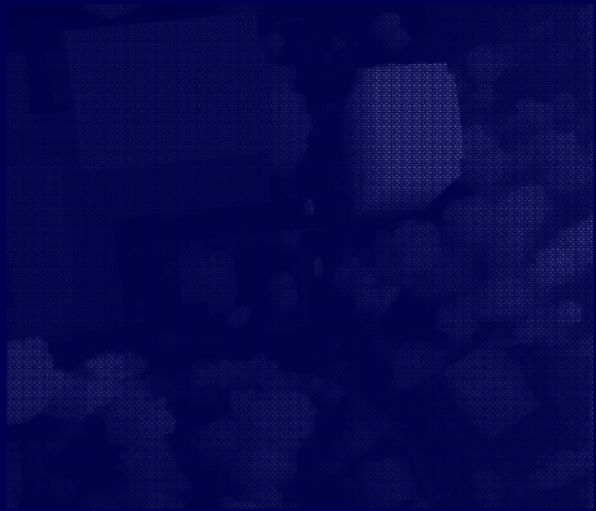
Nieuwenhuis, Cremers, PAMI '12

Interactive Segmentation

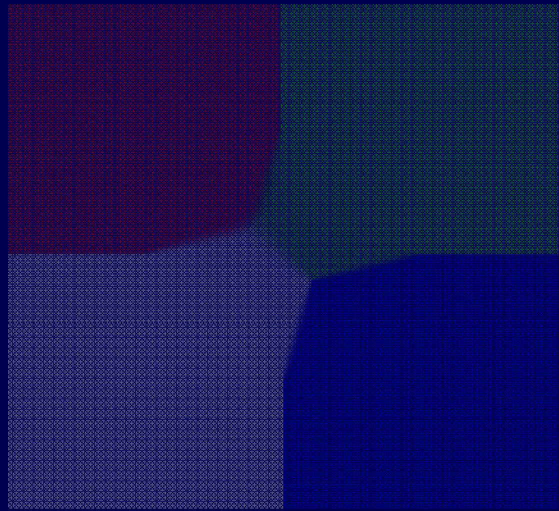
Nieuwenhuis, Cremers, PAMI ‘12

Interactive Segmentation

Nieuwenhuis, Cremers, PAMI '12

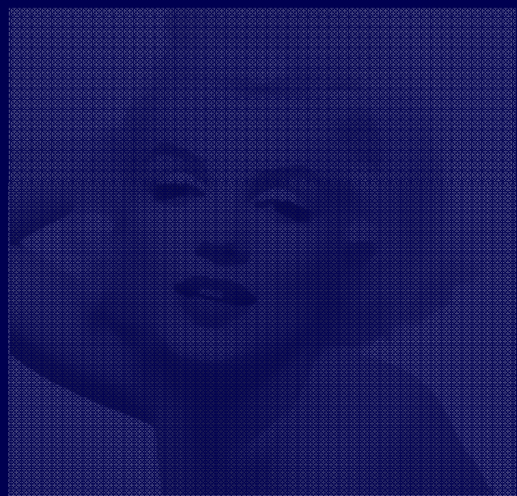


Convex multilabel optimization



Minimal partitions

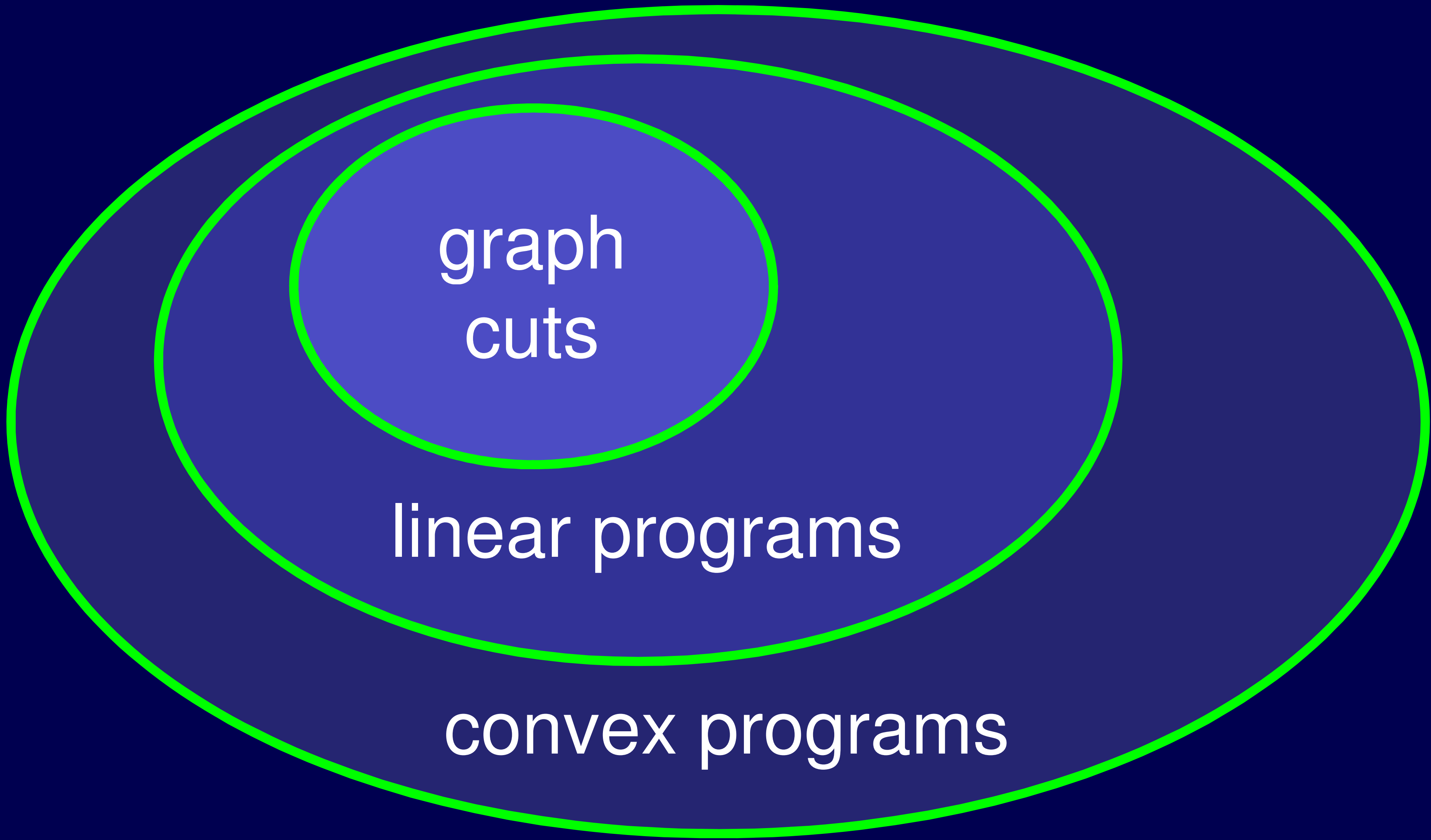
Semantic segmentation



Mumford-Shah

Semantic Image Segmentation

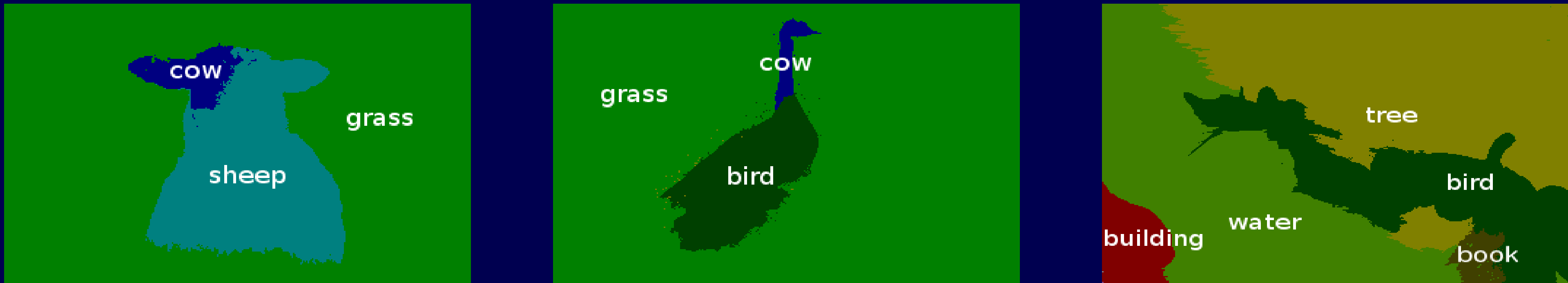
Ladicki et al. ECCV '10, Souiai et al. EMMCVPR '13



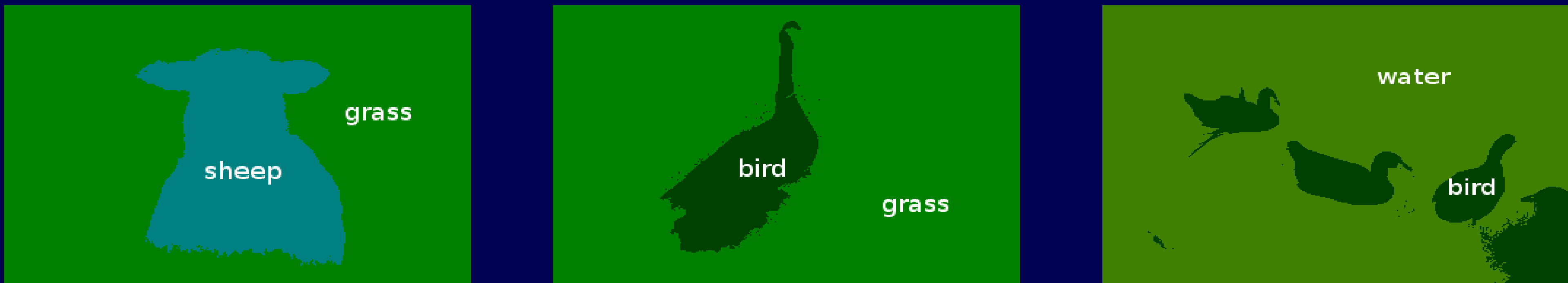
Klodt et al., ECCV '08, Nieuwenhuis et al. PAMI '13

Semantic Image Segmentation

Input images

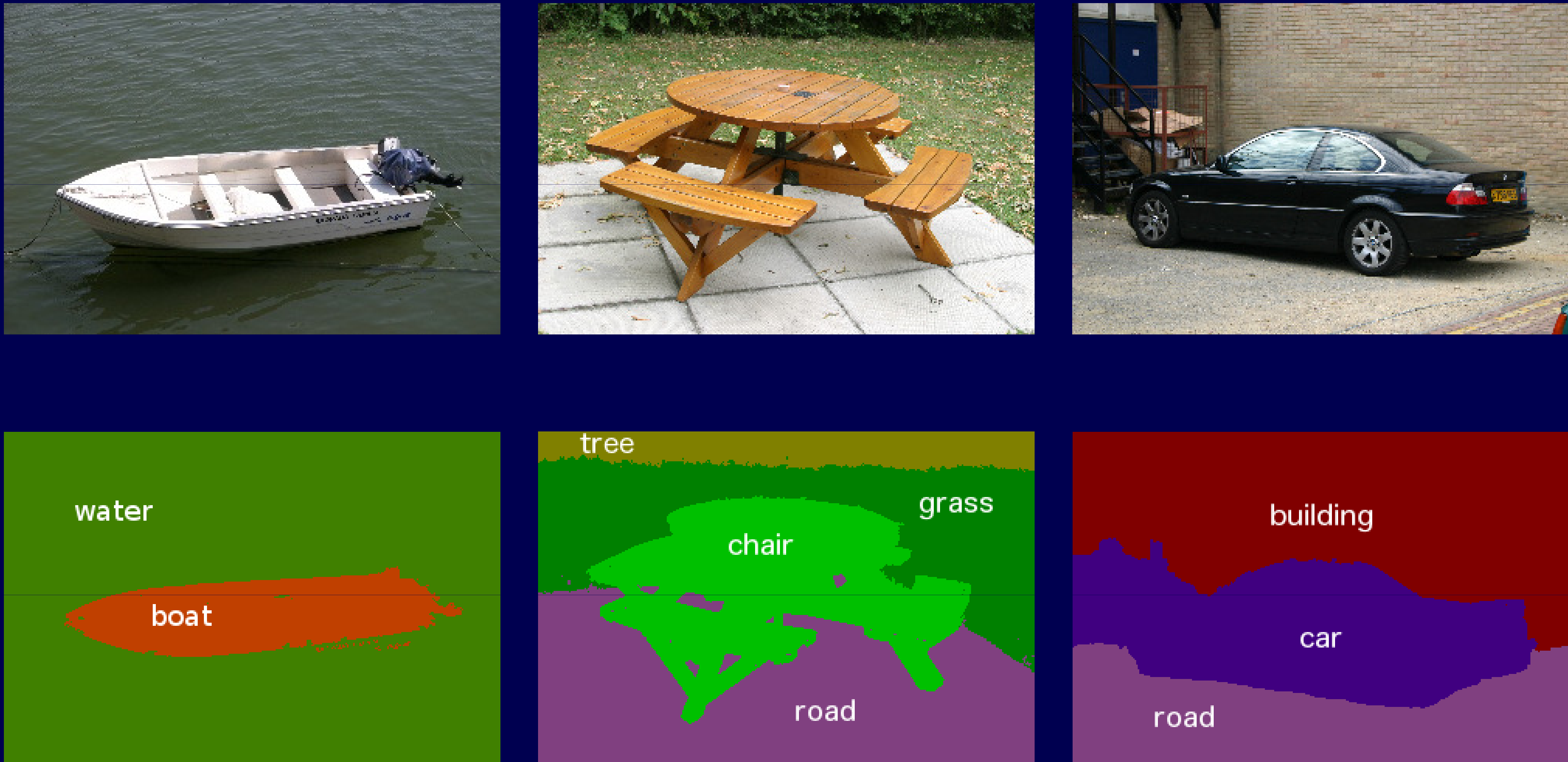


Segmentation with boundary length regularity



Segmentation with label configuration prior

Semantic Image Segmentation



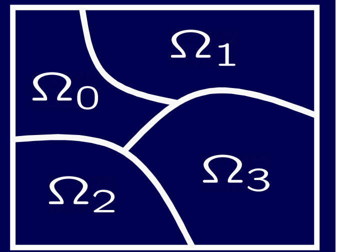
Souiai et al. EMMCVPR '13

General Ordering Constraints

Strekalovskiy, Cremers, ICCV 2011

Related discrete approach: *Liu et al. PAMI '10*

General Ordering Constraints



<u>Reminder:</u> With $v_i = 1_{\Omega_i}$, the minimal partition problem is:

$$\min_{v \in \mathcal{B}} \frac{1}{2} \sum_i \int_\Omega |Dv_i| + \int_\Omega v_i \, f_i \, dx = \min_{v \in \mathcal{B}} \sup_{p \in \mathcal{K}} \sum_i \int_\Omega v_i \, \text{div} \, p_i \, dx + \int_\Omega v_i \, f_i \, dx$$

where $\mathcal{K} = \left\{ p = (p_1, \ldots, p_n)^\top \in \mathbb{R}^{n \times m} : \left| p_j - p_i \right| \leq 1, \forall i < j \right\}$

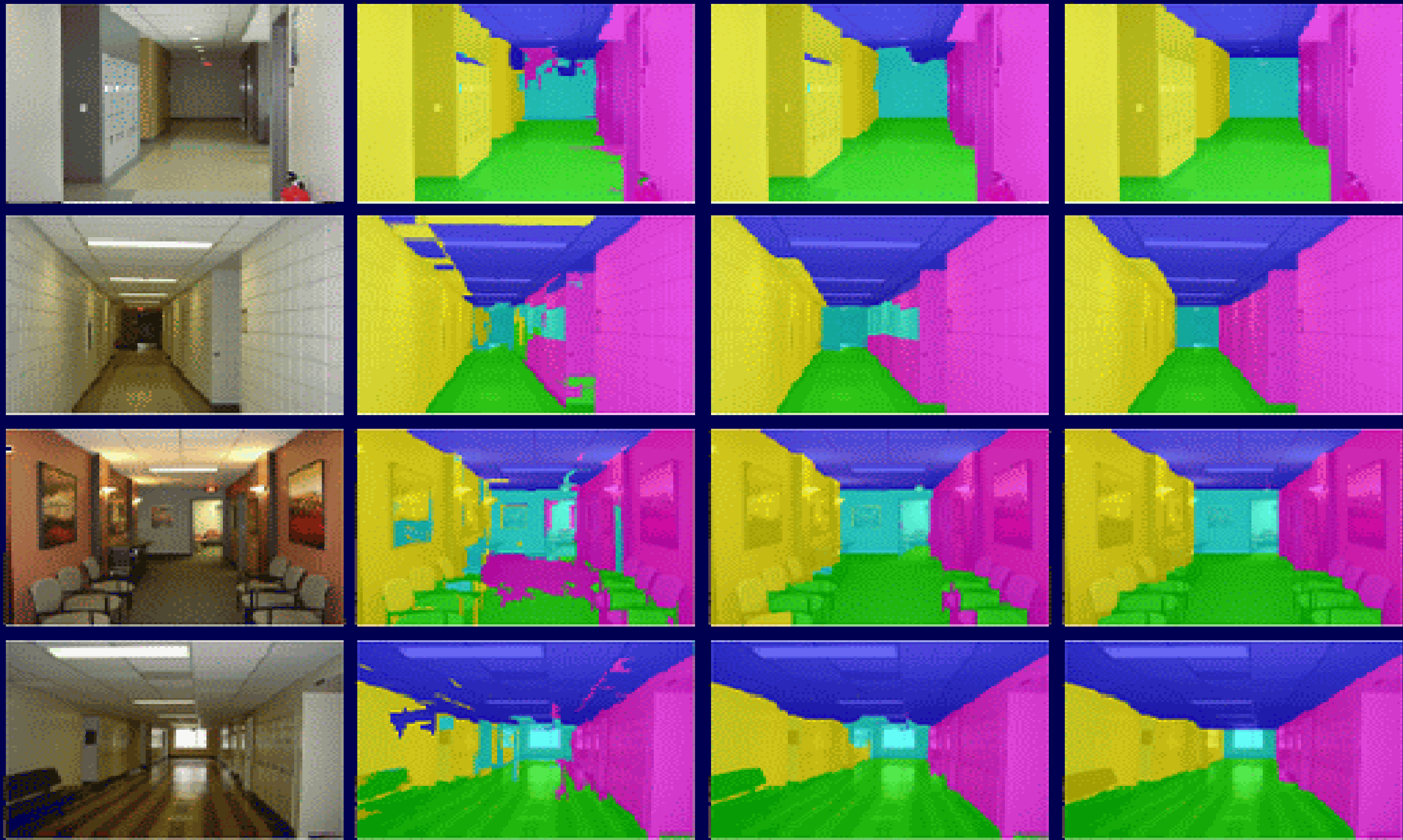
Consider instead the more general convex set:

$$\mathcal{K}_d = \left\{ p \in \mathbb{R}^{n \times m} : \langle p_j - p_i, \nu \rangle \leq d(i, j, \nu), \forall i < j, \, \nu \in \mathbb{S}^{m-1} \right\}$$

Penalize transitions depending on label values i, j and orientation ν.

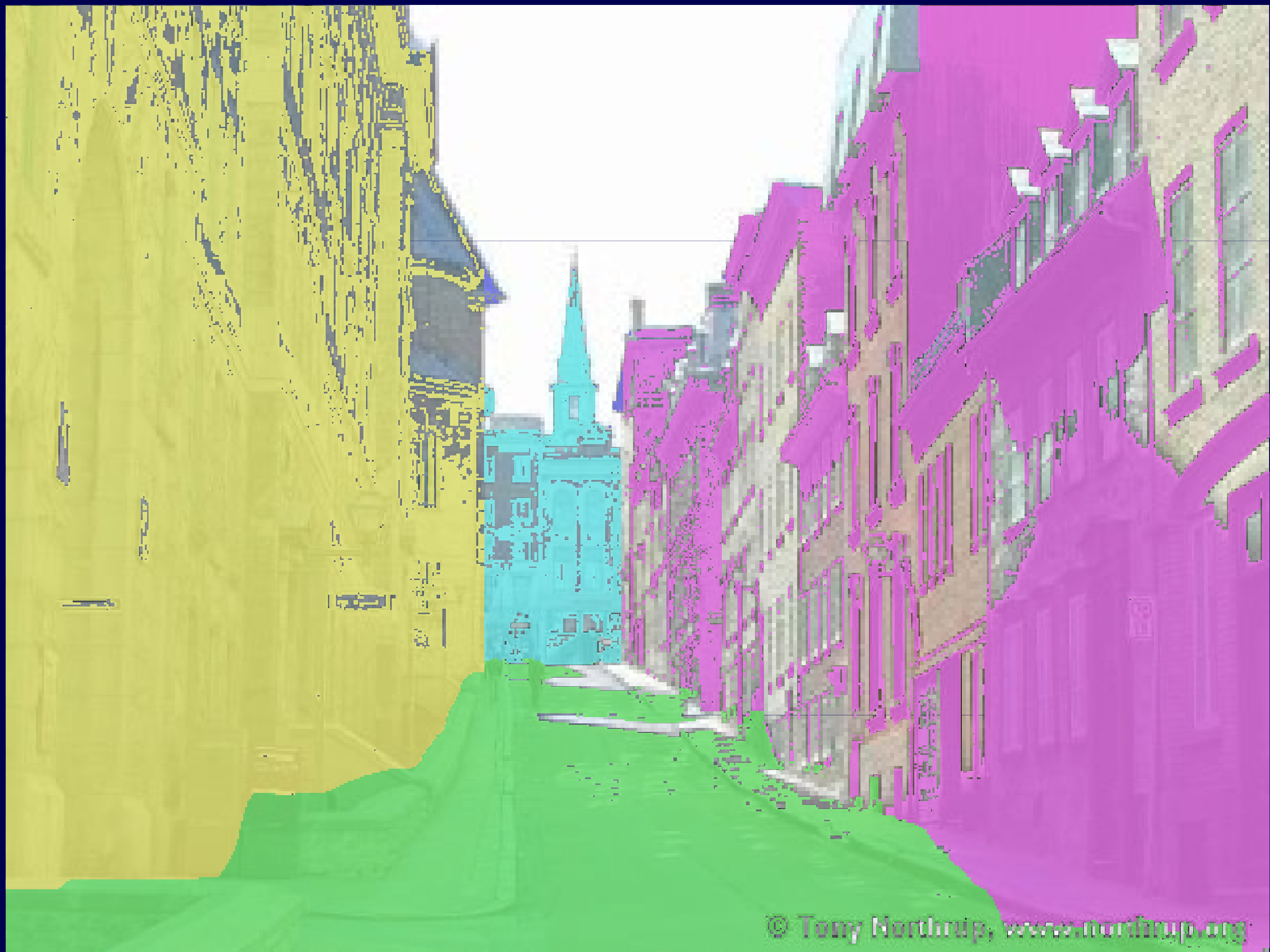
Strekalovskiy, Cremers, ICCV 2011

General Ordering Constraints



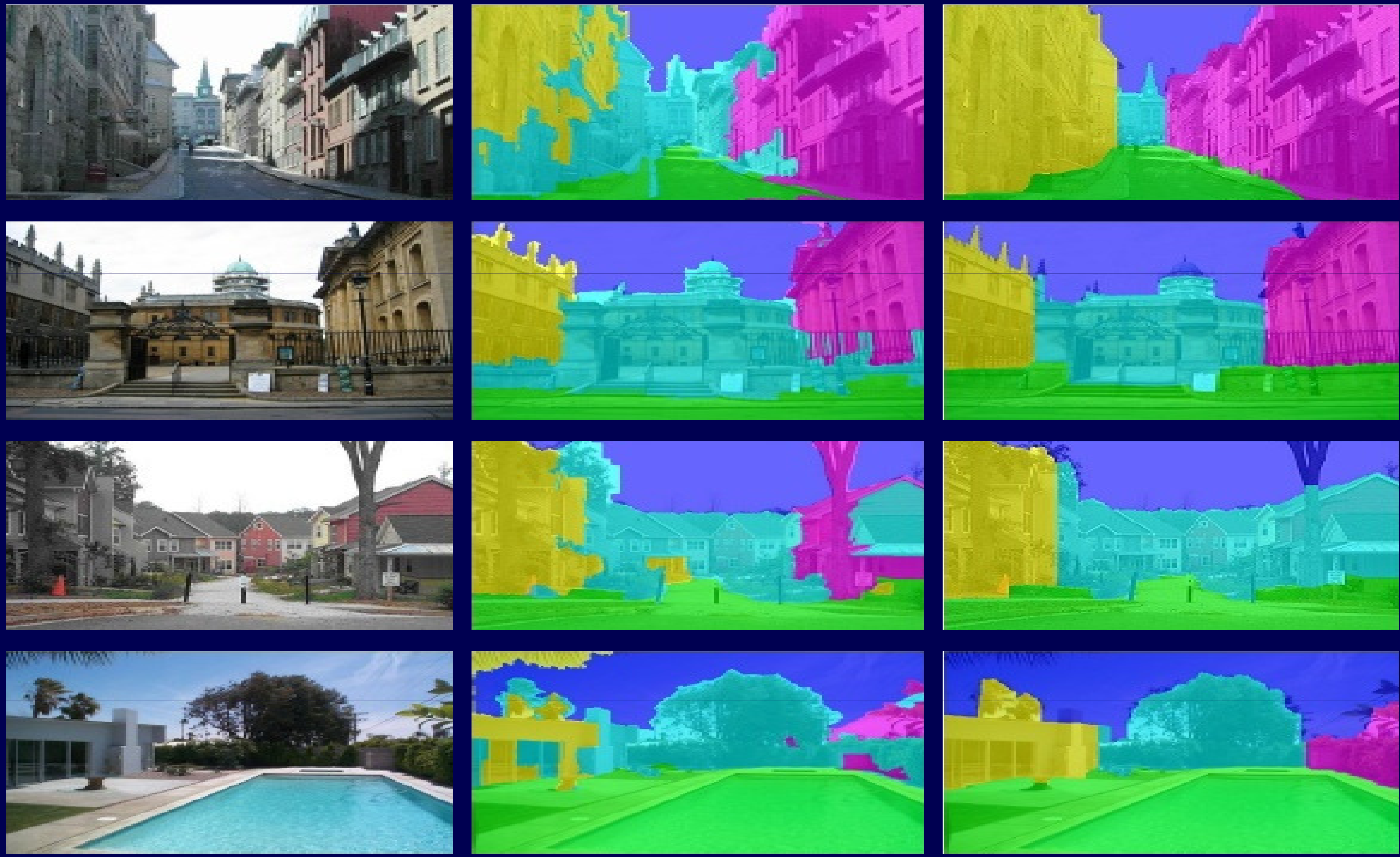
Input Data term Min. partition Ordering

Strekalovskiy, Cremers, ICCV 2011



Strekalovskiy, Cremers, ICCV 2011

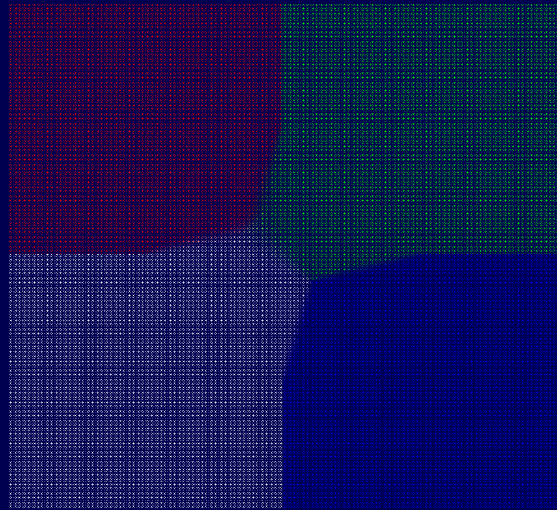
General Ordering Constraints



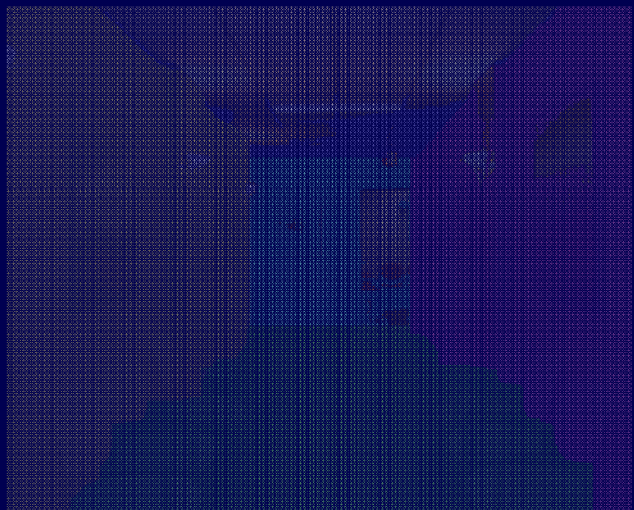
Input Min. partition Ordering

Strekalovskiy, Cremers, ICCV 2011

Convex multilabel optimization



Minimal partitions



Semantic segmentation

Mumford-Shah

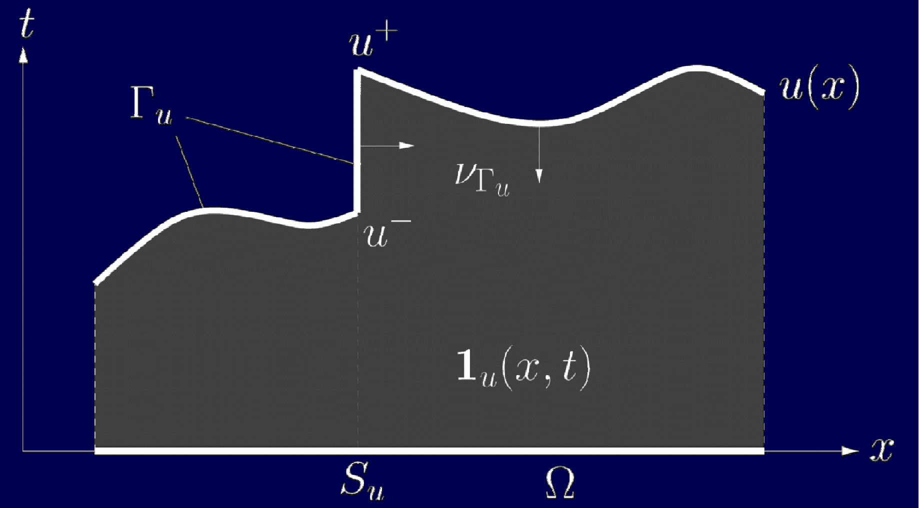
Piecewise Smooth: Scalar Case

$$E(u) = \lambda \int_{\Omega} (f-u)^2 dx + \int_{\Omega \setminus S_u} |\nabla u|^2 dx + \nu \mathcal{H}^1(S_u) \qquad (*)$$

Mumford, Shah '89

For $u \in SBV(\Omega, \mathbb{R})$, $\Omega \subset \mathbb{R}^n$, $(*)$ can be written as

$$E(u) = \sup_{\varphi \in K} \int_{\Omega \times \mathbb{R}} \varphi D\mathbf{1}_u,$$



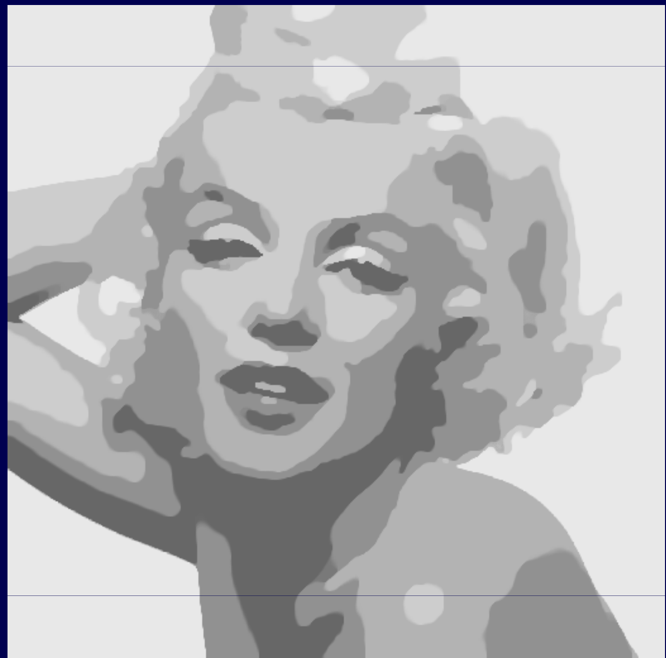
with a convex set

$$K = \Big\{ \varphi \in C_0(\Omega \times \mathbb{R}; \mathbb{R}^n \times \mathbb{R}) :$$
$$\varphi^t(x,t) \geq \frac{\varphi^x(x,t)^2}{4} - \lambda(t - f(x))^2, \ \left| \int_{t_1}^{t_2} \varphi^x(x,s)ds \right| \leq \nu \Big\},$$

Alberti, Bouchitte, Dal Maso '04

Piecewise Smooth: Scalar Case

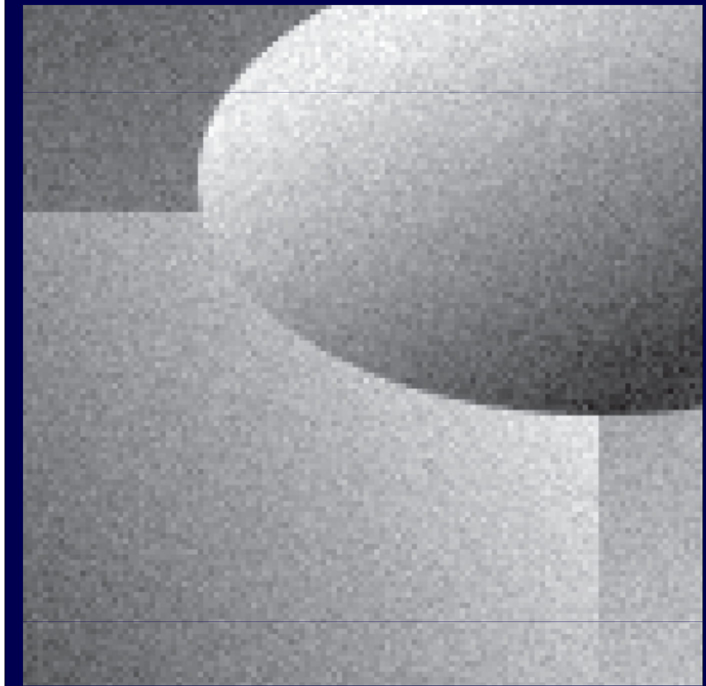
Input image



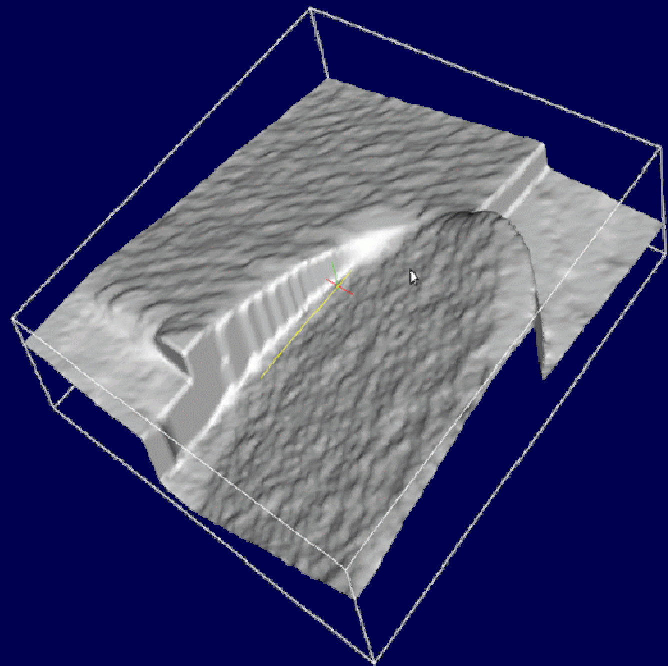
piecewise constant

piecewise smooth

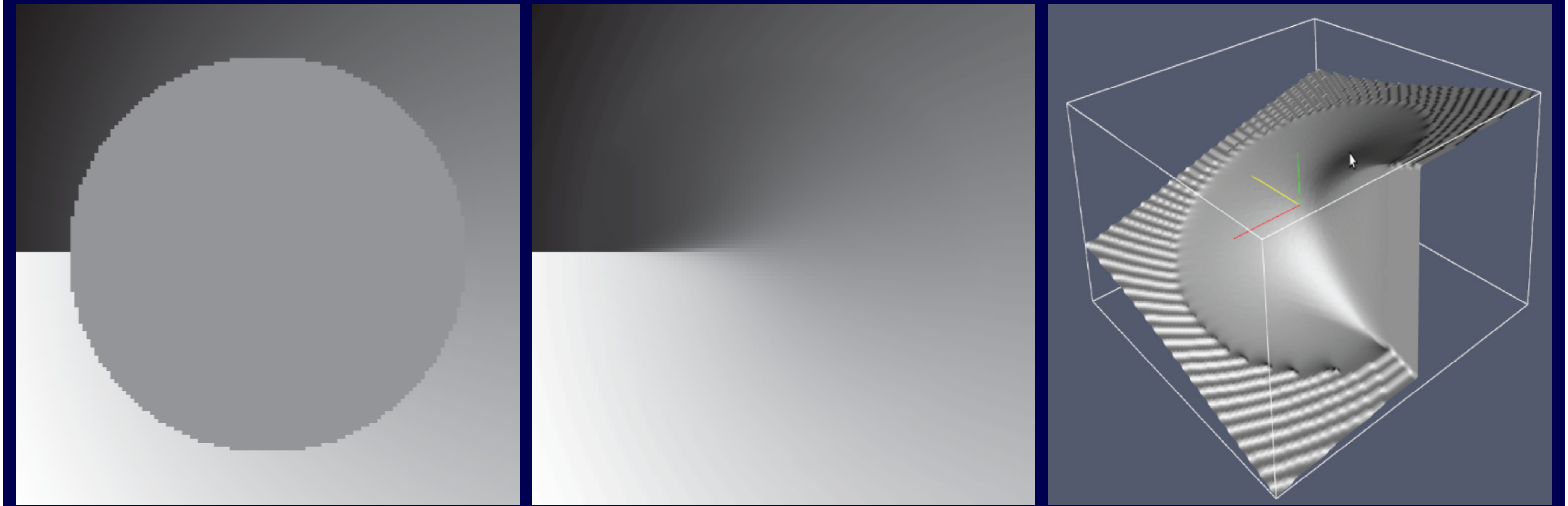
Pock, Cremers, Bischof, Chambolle ICCV '09


noisy input


restoration


surface plot

Pock, Cremers, Bischof, Chambolle ICCV '09



fixed boundary values | inpainted crack tip | surface plot

Pock, Cremers, Bischof, Chambolle ICCV '09

The Vectorial Mumford-Shah Problem

For $u \in SBV(\Omega, \mathbb{R}^k)$, we consider the functional

$$E(u) = \int_\Omega |f-u|^2 dx + \lambda \int_{\Omega \setminus S_u} \sum_{i=1}^k |\nabla u_i|^2 dx + \nu \mathcal{H}^1(S_u).$$

Proposition: For $v = 1_u = (1_{u_1}, \dots, 1_{u_k})$, we have:

$$E(u) = \mathcal{F}(v) := \sup_{\sigma \in \mathcal{K}} \sum_{i=1}^k \int_{\Omega \times \mathbb{R}} \sigma_i(x,t) \cdot Dv_i(x,t)$$

with the convex set:

$$\mathcal{K} = \Big\{ \sigma \,\Big|\, (\sigma_i^x, \sigma_i^t) \in C_c^\infty(\Omega \times \mathbb{R}; \mathbb{R}^n \times \mathbb{R}),$$

$$\sigma_i^t(x,t_i) \geq \frac{1}{4\lambda} |\sigma_i^x(x,t_i)|^2 - (t_i - f_i(x))^2,$$

$$\sum_{j=1}^k \left| \int_{t_j}^{t'_j} \sigma_j^x(x,s)\, ds \right| \leq \nu, \quad \forall\, 1 \leq i \leq k,\ x \in \Omega,\ t_j < t'_j \Big\}.$$

$\mathcal{O}(n_1^2 \cdots n_k^2)$ constraints!

Strekalovskiy, Chambolle, Cremers, CVPR '12

An Efficient Reformulation

<u>Proposition:</u> The constraint set $\mathcal{O}(n_1^2 \cdots n_k^2)$ constraints

$$\mathcal{K} = \left\{ \sigma \,\middle|\, (\sigma_i^x, \sigma_i^t) \in C_c^\infty(\Omega \times \mathbb{R}; \mathbb{R}^n \times \mathbb{R}), \right.$$

$$\sigma_i^t(x, t_i) \geq \frac{1}{4\lambda}|\sigma_i^x(x, t_i)|^2 - (t_i - f_i(x))^2,$$

$$\left. \boxed{\sum_{j=1}^k \left| \int_{t_j}^{t_j'} \sigma_j^x(x, s)\, ds \right| \leq \nu,} \quad \forall\, 1 \leq i \leq k,\, x \in \Omega,\, t_j < t_j' \right\}.$$

is equivalent to the constraint set $\mathcal{O}(n_1^2 + \ldots + n_k^2)$ constraints

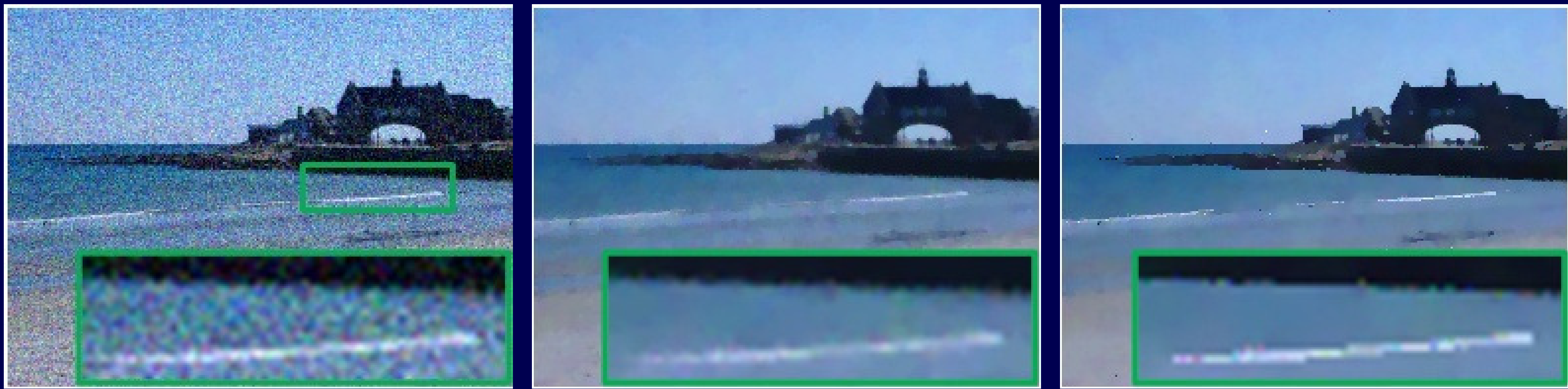
$$\mathcal{K}' := \left\{ (\sigma, m) \,\middle|\, (\sigma_i^x, \sigma_i^t) \in C_c^\infty(\Omega \times \mathbb{R}; \mathbb{R}^n \times \mathbb{R}), \right.$$

$$\sigma_i^t(x, t_i) \geq \frac{1}{4\alpha}|\sigma_i^x(x, t_i)|^2 - (t_i - f_i(x))^2,$$

$$\left. \boxed{\left| \int_{t_i}^{t_i'} \sigma_i^x(x, s)\, ds \right| \leq m_i(x), \quad \sum_{j=1}^k m_j(x) \leq \nu} \quad \forall i,\, x \in \Omega,\, t_i < t_i' \right\}.$$

→ **Same complexity as channel-wise processing.**

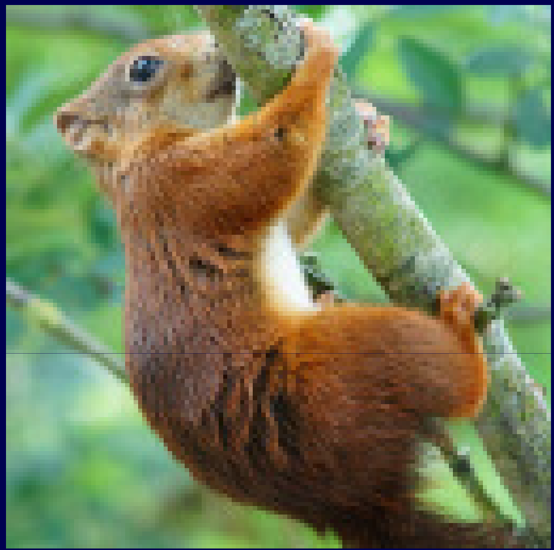
Strekalovskiy, Chambolle, Cremers, CVPR '12



Input image TV denoised Vectorial Mumford-Shah

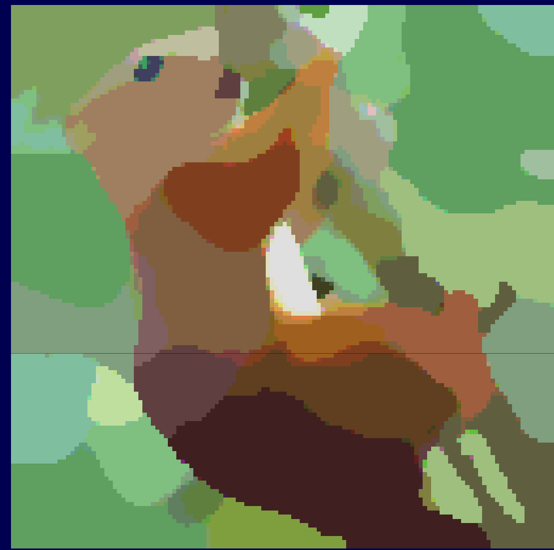
Strekalovskiy, Chambolle, Cremers, CVPR '12

Channelwise versus Vectorial

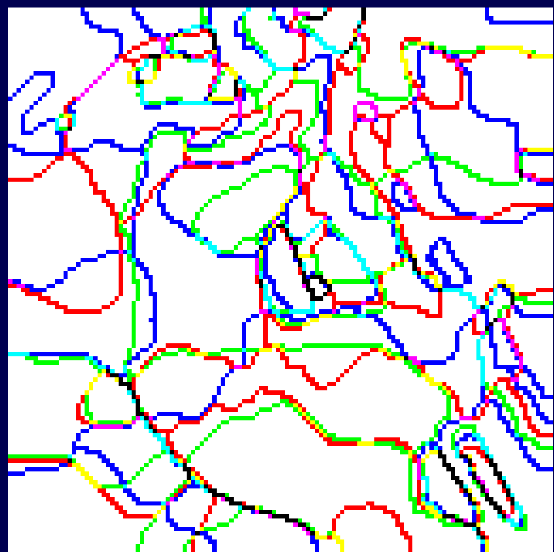


Input image

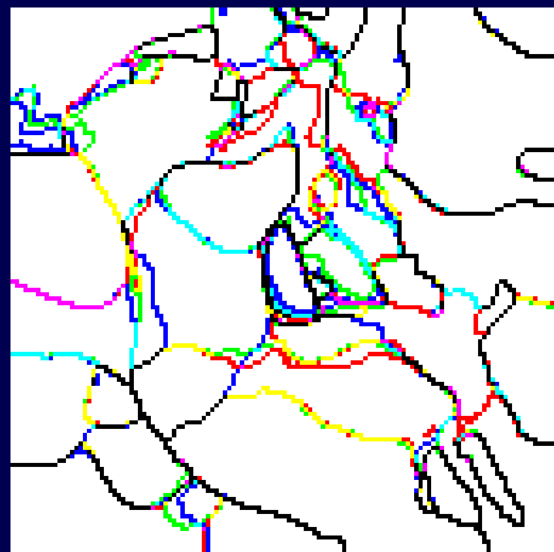
Channelwise MS



Vectorial MS



Jump set S_u



Jump set S_u

Channelwise versus Vectorial

Input image

Channelwise MS

Vectorial MS

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Piecewise Constant Color Segmentation

Input image

$\lambda = \infty, \nu = 0.05$

$\lambda = \infty, \nu = 0.1$

$\lambda = \infty, \nu = 0.2$

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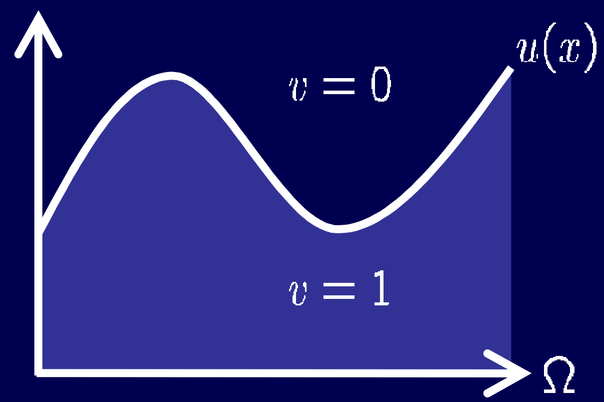
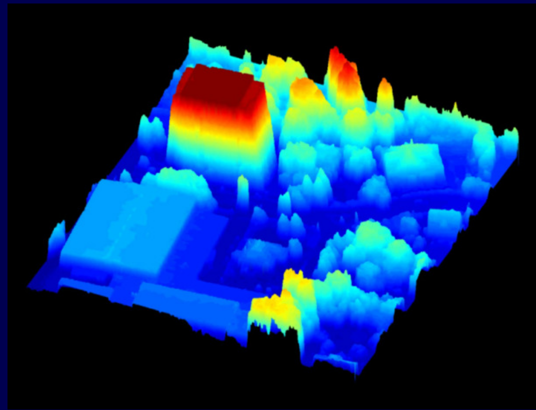
Convex relaxations for real-valued estimation problems can be derived by discretizing the space of permissible values.



In the scalar-valued case we obtain provably optimal solutions for convex regularizers.



For nonconvex regularizers (Mumford-Shah and min. partition) we get near-optimal solutions independent of the initialization.