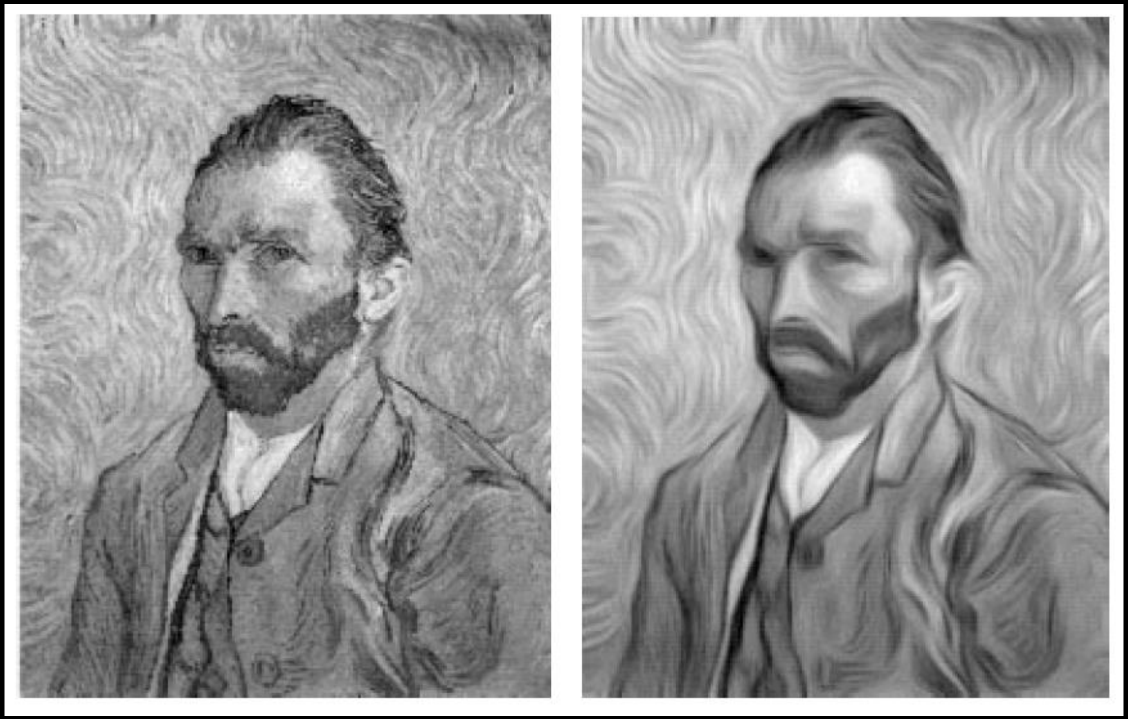
Introduction to Nonlinear Image Processing



IPAM Summer School on Computer Vision

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Ecole Centrale Paris / INRIA Saclay

Mean and median

outlier

Observations in a 3x3 window: $[1, 2, 100, 1, 3, 2, 1, 5, 3]$

Mean:

$$(1 + 2 + 100 + 1 + 3 + 2 + 1 + 5 + 3)/9 = 13.1$$

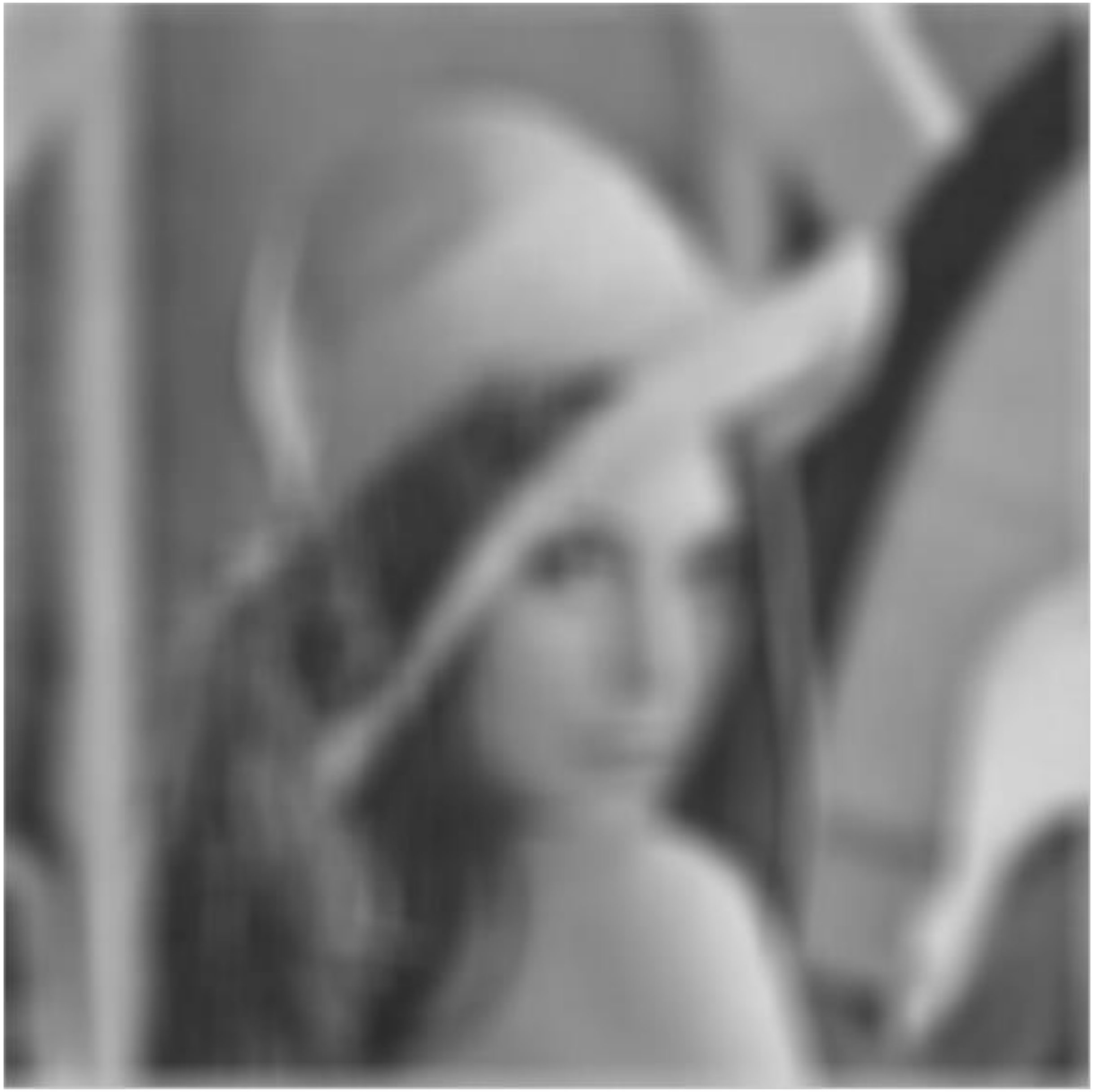
Median:

1) Sort: $[1, 1, 1, 2, 2, 3, 3, 5, 100]$
2) Pick mid-point $\uparrow$

robust to outliers

non-linear

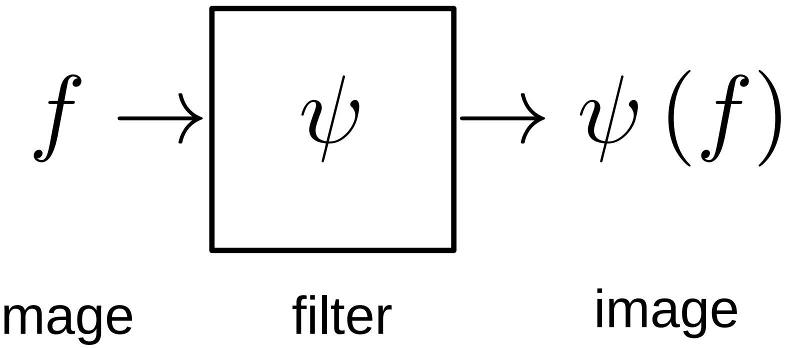
Mean



Median

Gaussian blur

Image Processing



Previous lecture: $\psi(\alpha f + \beta h) = \alpha \psi(f) + \beta \psi(h)$

This lecture: remove this constraint

freedom! at the expense of control

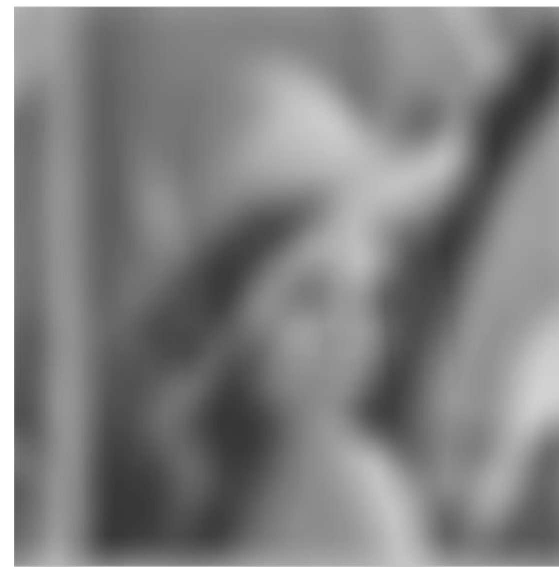
Multi-scale Gaussian smoothing

$$g_\sigma(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right)$$

$\sigma = 5$

$\sigma = 10$

$\sigma = 20$



Gaussian scale space

Rewrite Gaussian: $g_t(x,y) = \frac{1}{4\pi t}\exp\left(-\frac{x^2+y^2}{4t}\right)$ $\quad t = \frac{\sigma^2}{2}$

'time'

Scale space: $u(x,y,t) = g_t(x,y) * u_0(x,y), \quad t > 0$

$$u(x,y,0) = u_0(x,y)$$

A. Witkin, Scale-space filtering, IJCAI, 1983.
J. Koenderink, The structure of images, Biological Cybernetics, 1984
J. Babaud, A. P. Witkin, M. Baudin, and R. O. Duda, 'Uniqueness of the Gaussian kernel for scale-space filtering', PAMI, 1986.
A. Yuille, T.A. Poggio: Scaling theorems for zero crossings. PAMI, 1986.
T. Lindeberg, Scale-Space Theory in Computer Vision, Kluwer, 1994
L. Florack, Image Structure, Kluwer, 1997
B. Romeny, Front-End Vision and Multi-Scale Image Analysis, Kluwer, 2003.
J. Weickert, Linear scale space has first been proposed in Japan. JMIV, 1999.

Heat diffusion and image processing

Gaussian satisfies: $$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2}$$

Scale-space satisfies: $$u(x,y,t) = g_t(x,y) * u_0(x,y)$$

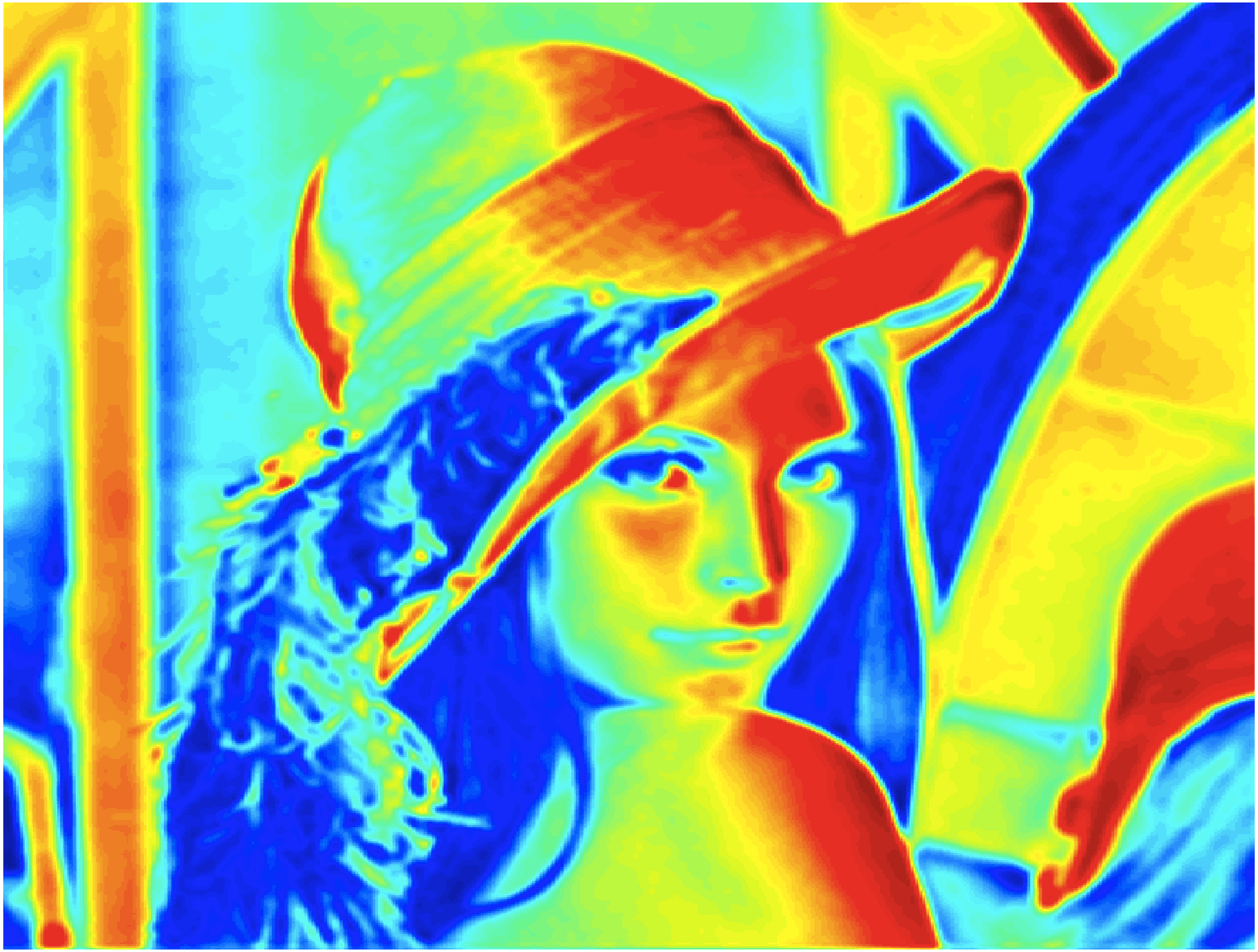
Associative property: $$f * [g * h] = [f * g] * h$$

Scale-space satisfies: $$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$u(x,y,0) = u_0(x,y)$$

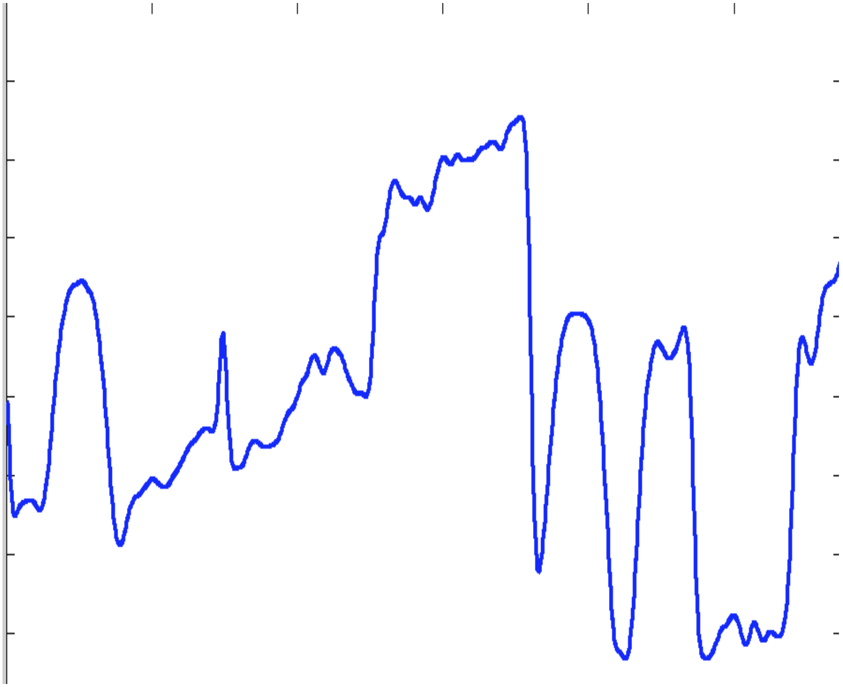
Heat diffusion PDE (Partial Differential Equation)

Heat diffusion and image processing



Heat diffusion in 1D

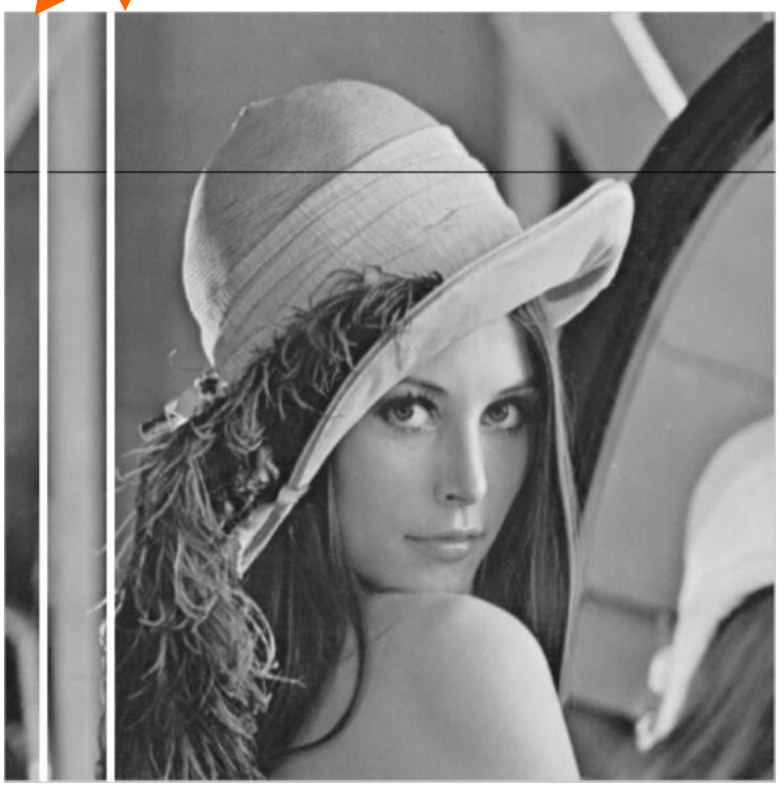
$$u_t = u_{xx}$$

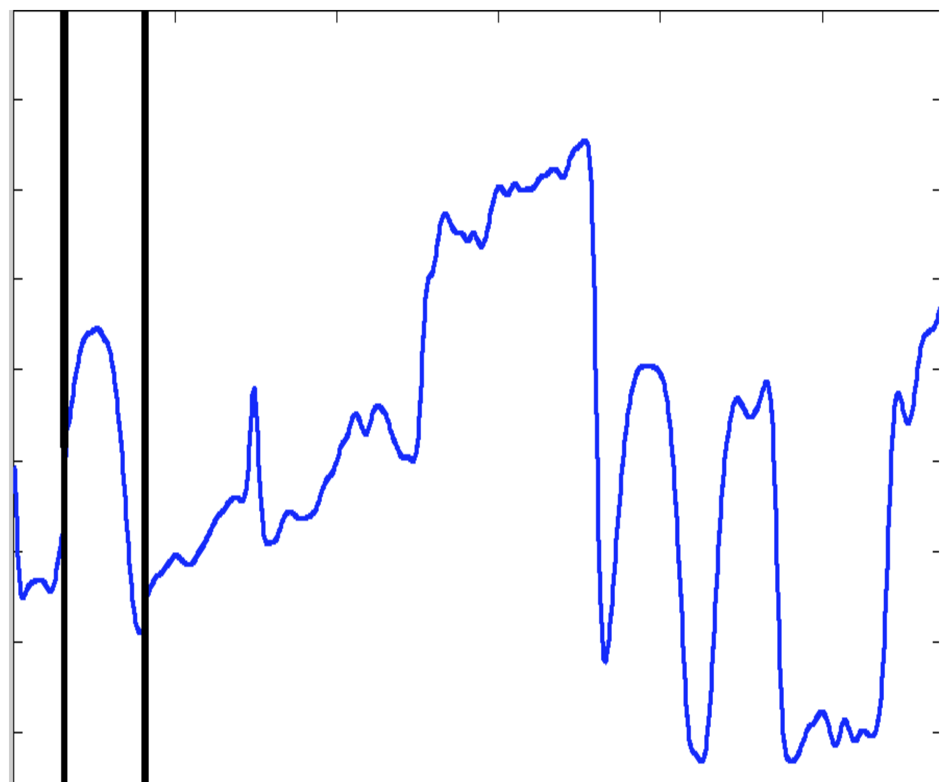


Heat diffusion in 1D - inhomogeneous material

$$u_t = \frac{d}{dx}\left(c\frac{d}{dx}u\right)$$

conductivity





Heat diffusion in 2D

Homogeneous material

$$u_x \;\to\; \nabla u = (u_x, u_y)$$

$$\frac{d}{dx} u_x \to \; \operatorname{div}(\nabla u) = \frac{\partial}{\partial x} u_x + \frac{\partial}{\partial y} u_y$$

$$u_t = u_{xx} + u_{yy}$$

Inhomogeneous material

$$\frac{d}{dx}(c u_x) \to \; \operatorname{div}(c \nabla u)$$

$$u_t = \operatorname{div}(c \nabla u)$$

Perona-Malik Diffusion

Image-dependent conductivity

$$u_t = \operatorname{div}\left(g\left(|\nabla u|\right)\nabla u\right) \qquad u(x,y,0) = u_0(x,y)$$

$$g(s) = \exp\left(-\frac{s^2}{a^2}\right)$$

Diffusion stops at strong image gradients (structure-preserving)

CLMC formulation: $|\nabla u| \to |\nabla G * u|$

P. Perona and J. Malik, Scale-space and edge detection using anisotropic diffusion, PAMI 1990
F. Catte, P.L. Lions, J.M. Morel, T. Coll, Image selective smoothing and edge detection by nonlinear diffusion, SIAM J. Numer. Analysis, 1992

Nonlinear vs. linear diffusion

(a) Linear diffusion at t=2

(b) Nonlinear diffusion at t=4.4

Extension to vectorial images

- Extension of nonlinear diffusion to vectorial images:

$$\boldsymbol{u} = (u_1, u_2, \ldots, u_N)$$

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(g(\|\nabla u\|)\nabla u\right)$$

generalization

$$\frac{\partial u_i}{\partial t} = \operatorname{div}\left(g(\|\nabla \boldsymbol{u}\|)\nabla u_i\right), \; i = 1, .., N$$

where: $$\|\nabla \boldsymbol{u}\| = \sqrt{\sum_{i=1}^{N} \|\nabla u_i\|^2}$$

Nonlinear diffusion for color image denoising

(a) Color Image with Noise

(b) *Perona-Malik* diffusion

Variational interpretation of heat diffusion

- Cost functional:

$$\begin{aligned} E[u] &= \iint_\Omega \|\nabla u\|^2 dxdy \\ &= \iint_\Omega \left(u_x^2 + u_y^2\right) dxdy \end{aligned}$$

- Euler-Lagrange:

$$\begin{aligned} \frac{\delta E}{\delta u} &= \frac{\partial E}{\partial u} - \frac{\partial}{\partial x}\left(\frac{\partial E}{\partial u_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial E}{\partial u_y}\right) \\ &= -2\frac{\partial u_x}{\partial x} - 2\frac{\partial u_y}{\partial y} \\ &= -2(u_{xx} + u_{yy}) \end{aligned}$$

- Heat diffusion: modifies temperature to decrease E quickly

Variational techniques

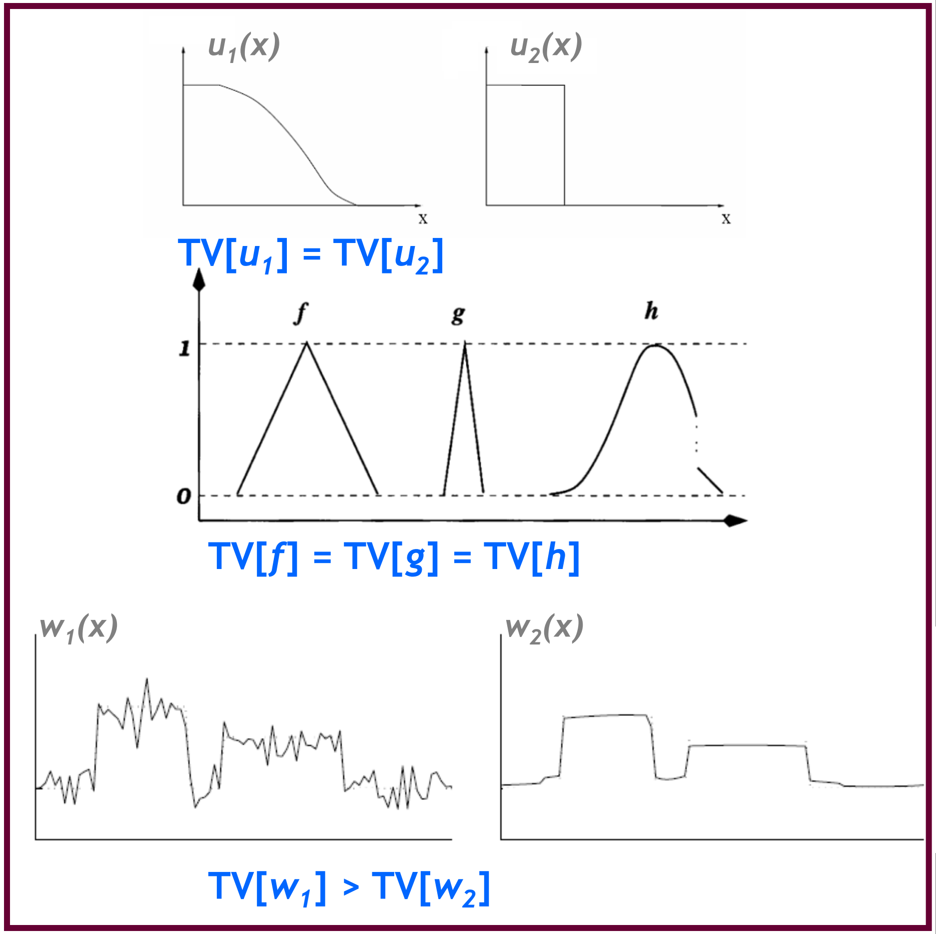
- Denoising as functional minimization
  - Functional: encodes *undesirable* properties

- *Total Variation*:

$$\mathrm{TV}[u] = \iint_\Omega \sqrt{u_x^2 + u_y^2}\,dx\,dy$$

Minimization flow

$$\frac{\partial u}{\partial t} = \operatorname{div}\left(\frac{\nabla u}{\|\nabla u\|}\right)$$



Rudin, L. I.; Osher, S.; Fatemi, E. "Nonlinear total variation based noise removal algorithms". Physica D 60, 1992

Total Variation diffusion

(a) Noisy image

(b) Total Variation diffusion

Perona-Malik versus Total Variation

(a) Perona-Malik diffusion

$$g(s) = \exp\left(-\frac{s^2}{K^2}\right)$$

(b) Total-Variation diffusion

$$g(s) = \frac{1}{s}$$

What is the 'right' cost functional?

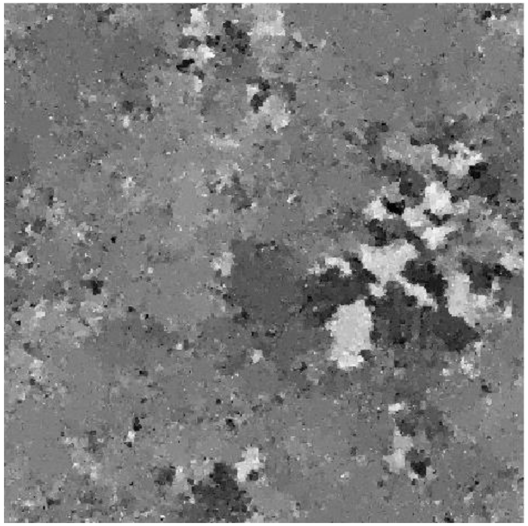
Can we learn the cost function?

S.C. Zhu, Y. N. Wu and D. Mumford, 'FRAME', IJCV 1997

- *Filters:* Use Gabors, Difference-of-Gaussians, Gaussian filters,
- *Random Fields:* Construct distribution that reproduces their histograms
- *And Maximum Entropy:* while being as random as possible

Training:

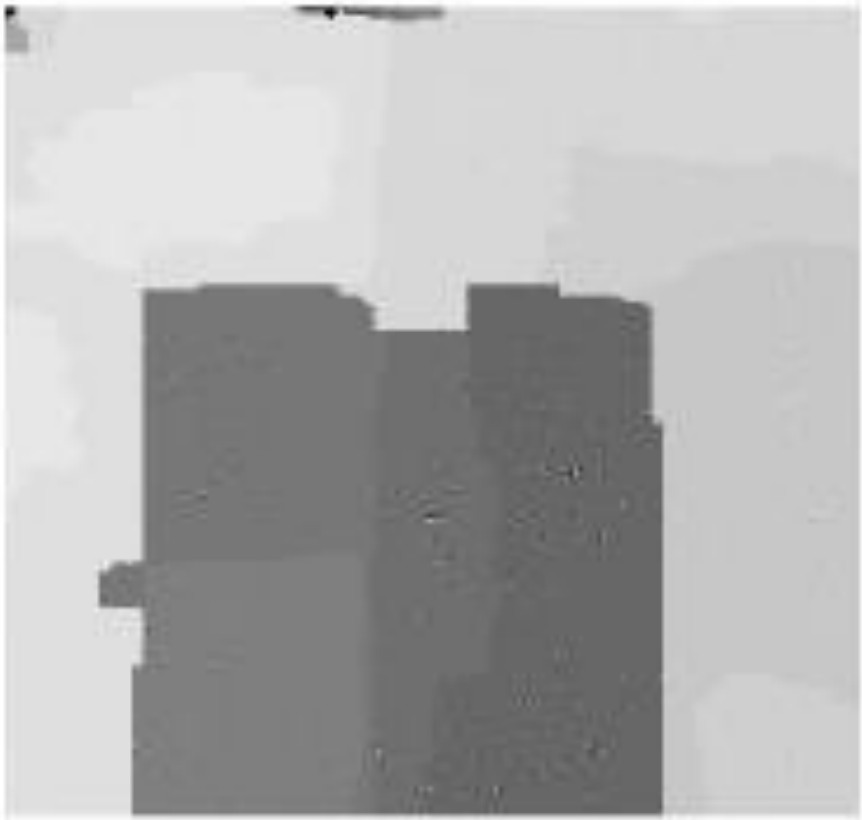
Sample:



From FRAME to GRADE

GRADE: Gibbs Reaction & Diffusion Equation

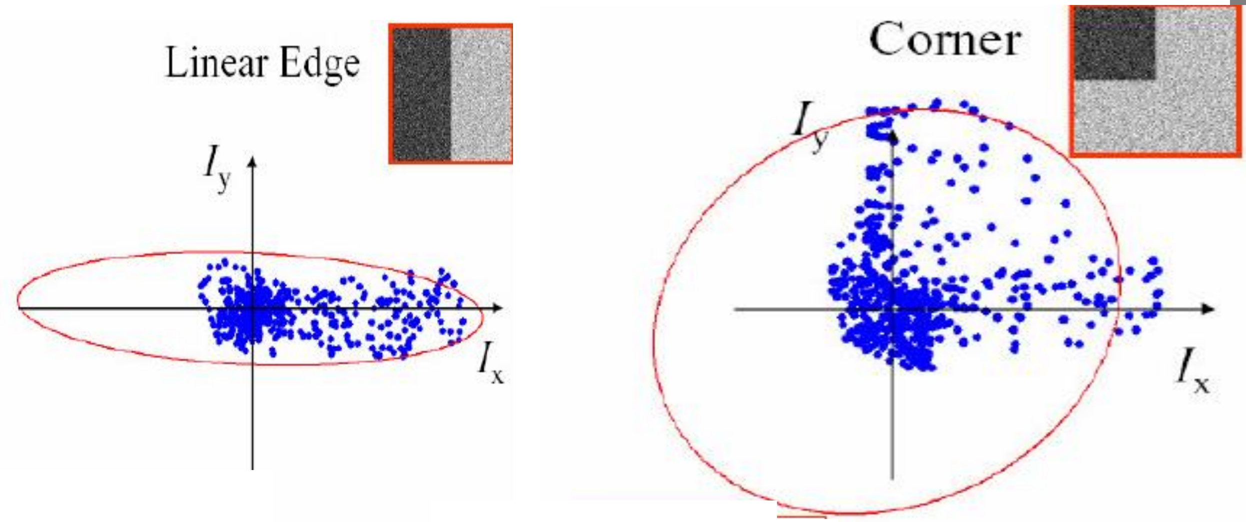
GRADE: maximize image probability using Euler-Largange PDEs



S.C. Zhu, Y. N. Wu, D. Mumford, ‘Filters, Random Fields and Max. Ent.’, IJCV 1997.
S.C. Zhu and D. Mumford, ‘Gibbs Reaction and Diffusion Equation’, PAMI 1998.
S. Roth and M. Black, ‘Fields of Experts’, IJCV 2009

Second Moment Matrix

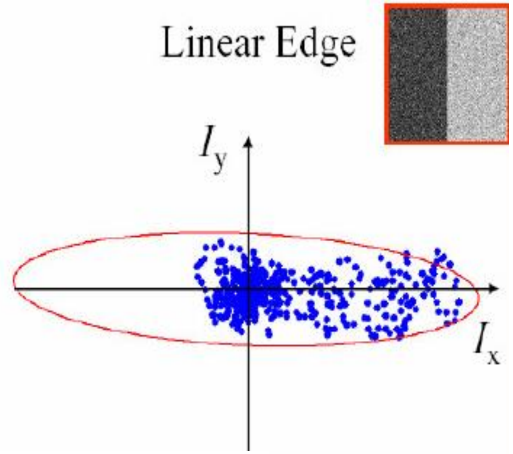
Distribution of gradients:

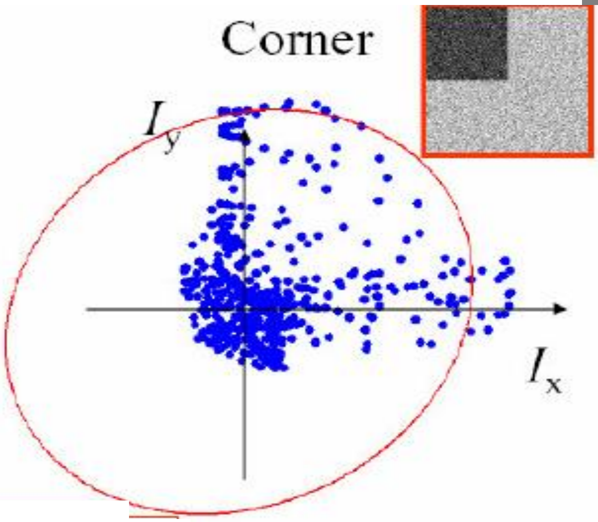


$$J = \begin{bmatrix} \sum_{x',y'} I_x^2 & \sum_{x',y'} I_x I_y \\ \sum_{x',y'} I_x I_y & \sum_{x',y'} I_y^2 \end{bmatrix}$$

Second Moment Matrix

Distribution of gradients:

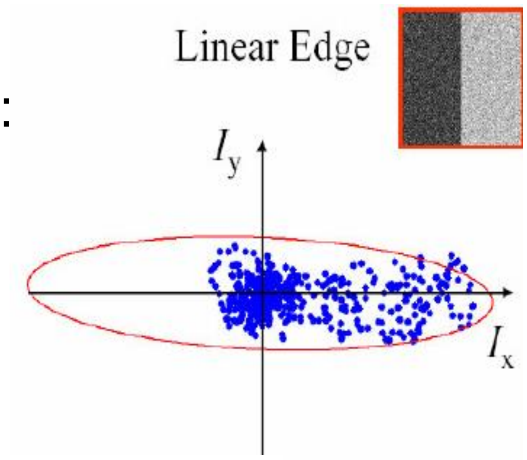


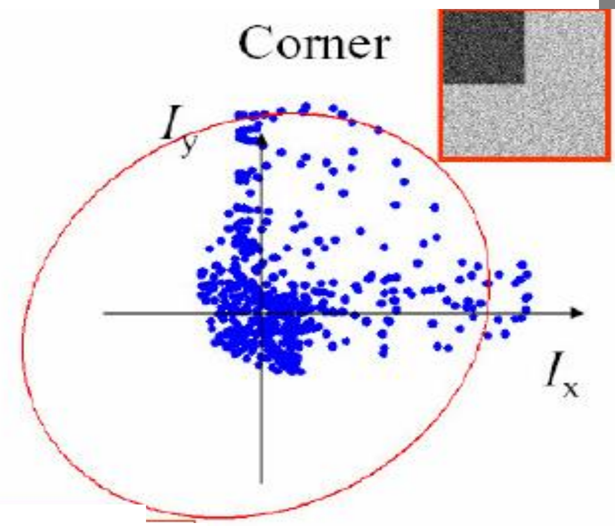


$$J = \sum_{x',y'} (I_x, I_y)^T (I_x, I_y)$$

Second Moment Matrix

Distribution of gradients:

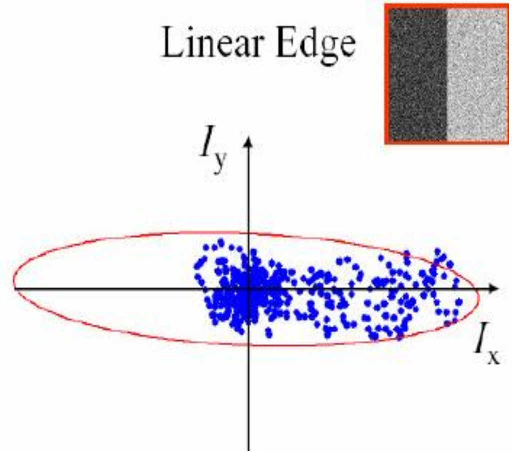


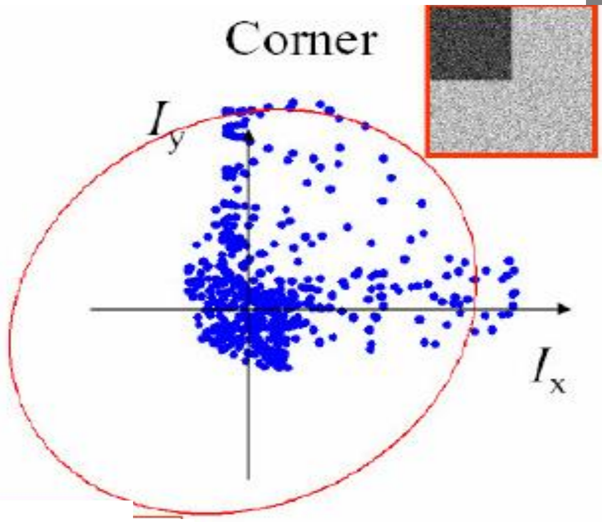


$$J = \sum_{x',y'} \left(\nabla G_\sigma * u\right)^T \left(\nabla G_\sigma * u\right)$$

Second Moment Matrix

Distribution of gradients:



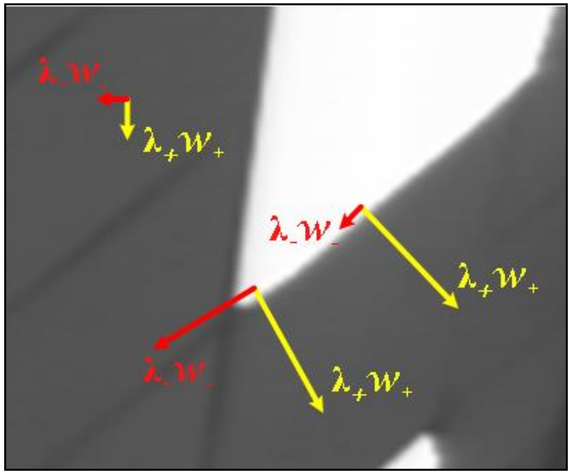


$$J = G_\rho * \left[(\nabla G_\sigma * u)^T (\nabla G_\sigma * u) \right]$$

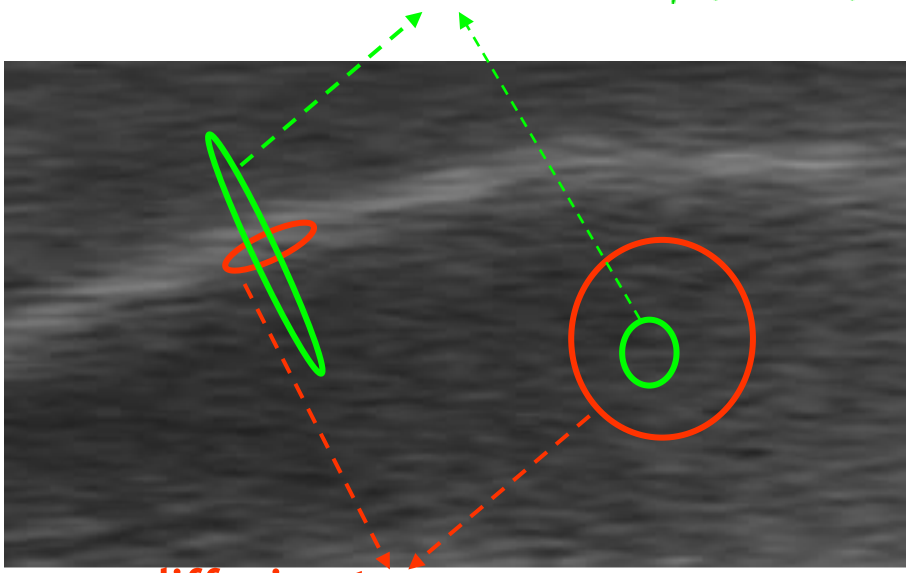
Second Moment Matrix

$$J = G_\rho * \left[(\nabla G_\sigma * u)^T (\nabla G_\sigma * u) \right]$$

- Eigenvectors w_+, w_- : directions of maximal and minimal variation of u
- Eigenvalues: amounts of minimal and maximal variation u



Anisotropic diffusion



Nonlinear diffusion

$$u_t = \operatorname{div}\left(g\left(|\nabla u|\right)\nabla u\right)$$

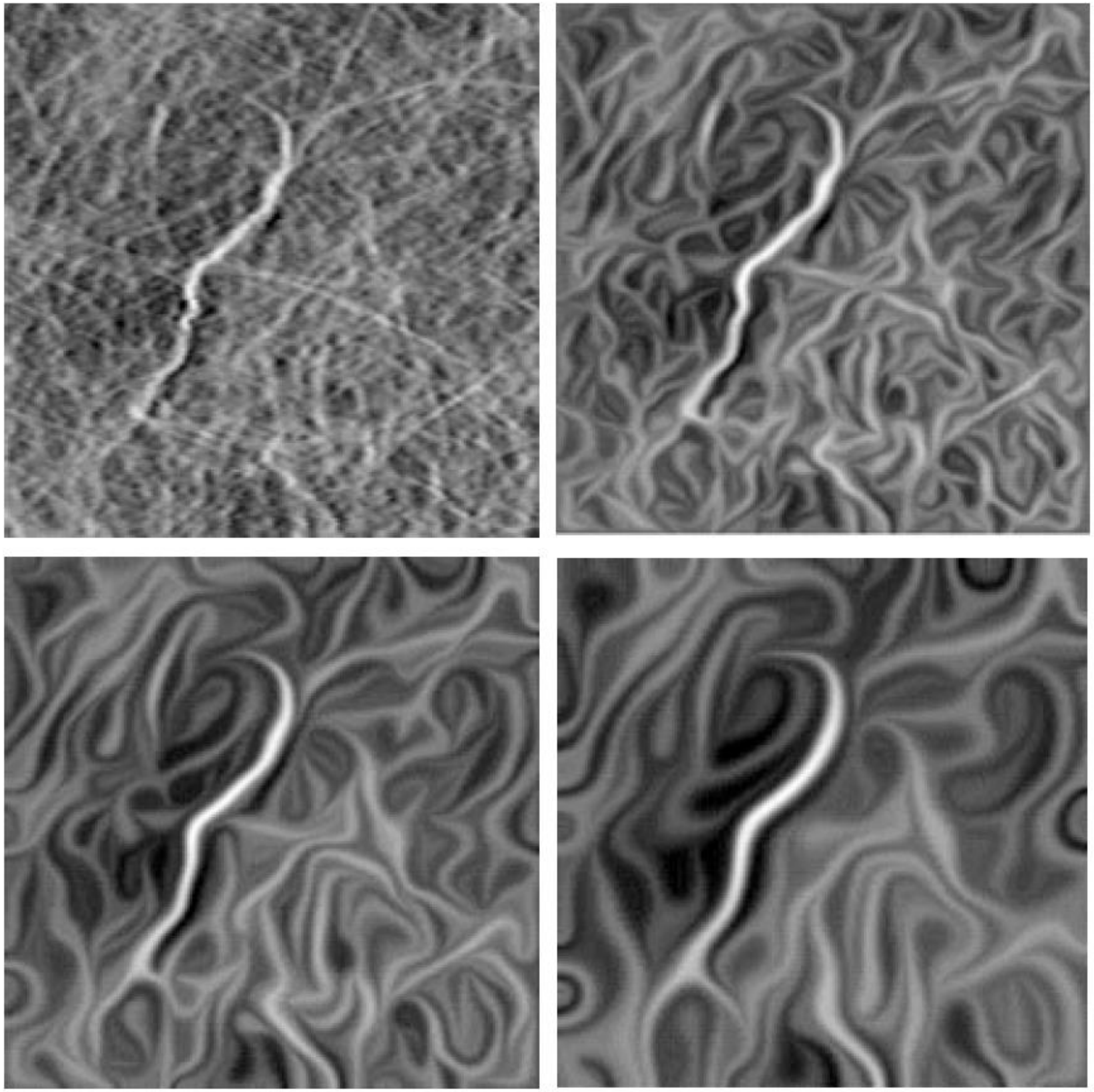
Nonlinear Anisotropic diffusion

$$u_t = \operatorname{div}\left(T(J_\rho\left(\nabla u_\sigma\right)\nabla u\right)$$

Slide credit: A. Roussos

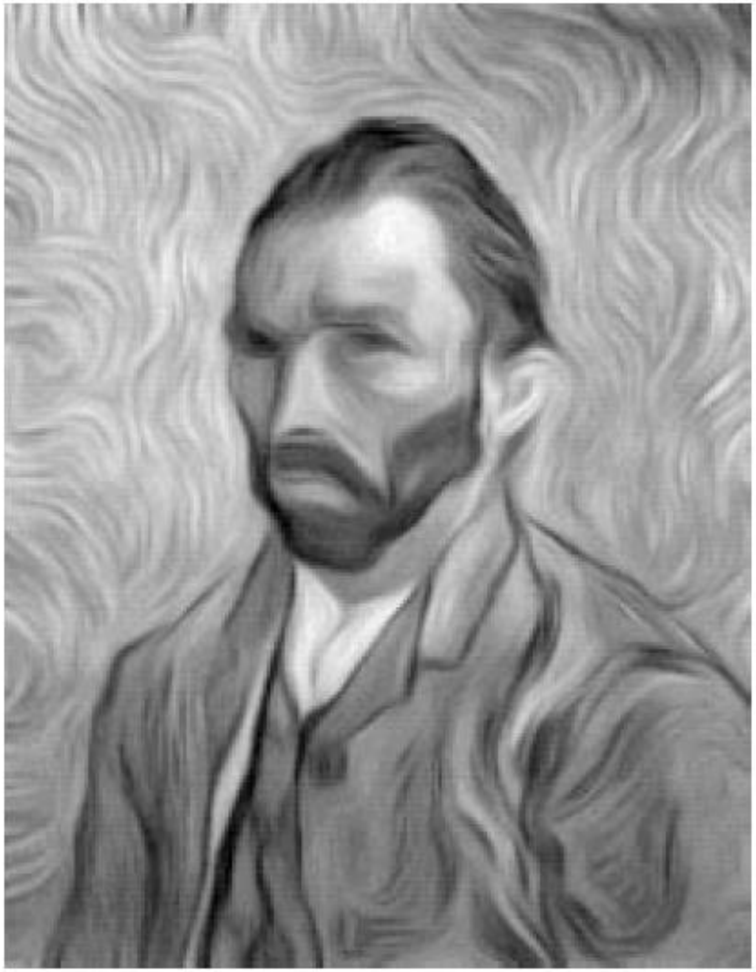
J. Weickert, Coherence-Enhancing Diffusion Filtering, Image and Vision Computing, 1999.
R. Kimmel, R. Malladi, and N. Sochen. Images as Embedded Maps and Minimal Surfaces, IJCV, 2001.
D. Tschumperlé, and R. Deriche (2005), Vector-valued image regularization with PDE's. PAMI, 2005.
A. Roussos and P. Maragos, Reversible Interpolation of vectorial Images, IJCV 2009
M. Lindenbaum, M. Fischer and A. Bruckstein, "On Gabor's contribution to image enhancement", 1994.
D. Gabor, 'Information theory in electron microscopy', 1965

Anisotropic Diffusion example



Slide credits: A. Roussos

Anisotropic Diffusion example



Slide credits: A. Roussos

Nonlinear anisotropic diffusion

$$\frac{\partial u(\boldsymbol{x},t)}{\partial t} = \operatorname{div}\Big(T(J_\rho(\nabla u_\sigma))\, \nabla u \Big)$$

Extension to vectorial images

$$\boldsymbol{u} = (u_1, u_2, \ldots, u_N)$$

$$\frac{\partial u_i\,(\boldsymbol{x},t)}{\partial t} = \operatorname{div}\Big(T(J_\rho(\nabla \boldsymbol{u}_\sigma))\, \nabla u_i \Big), \quad i = 1,..,M$$

Structure tensor for vectorial image:

$$J_\rho(\nabla \boldsymbol{u}_\sigma) = K_\rho * \sum_{i=1}^{N} \nabla u_{i,\sigma}\,(\nabla u_{i,\sigma})^{\mathrm{T}}, \text{ με } \boldsymbol{u}_\sigma = K_\sigma * \boldsymbol{u}$$

Slide credits: A. Roussos

Nonlinear anisotropic diffusion

Noisy input

Nonlinear Anisotropic Diffusion

Slide credits: A. Roussos

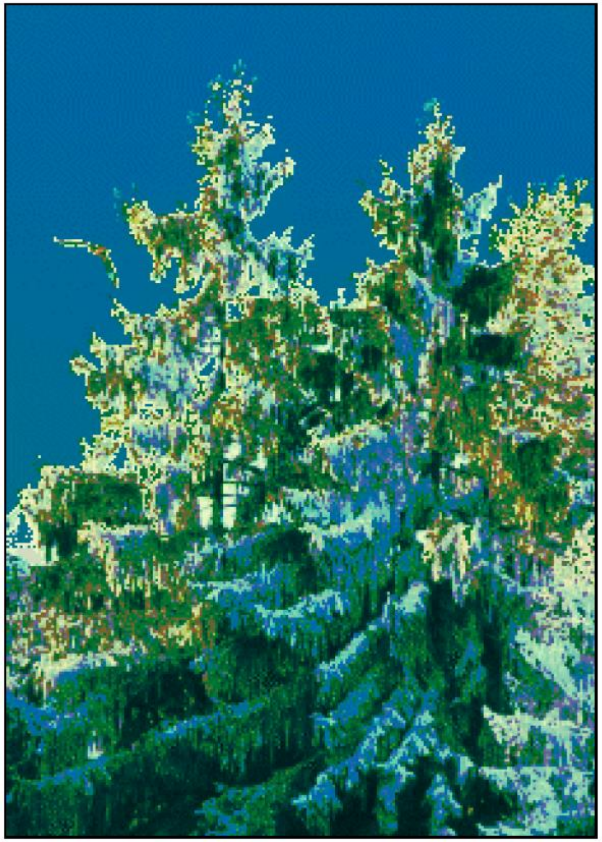
Nonlinear vs. Nonlinear and anisotropic diffusion

Nonlinear Diffusion

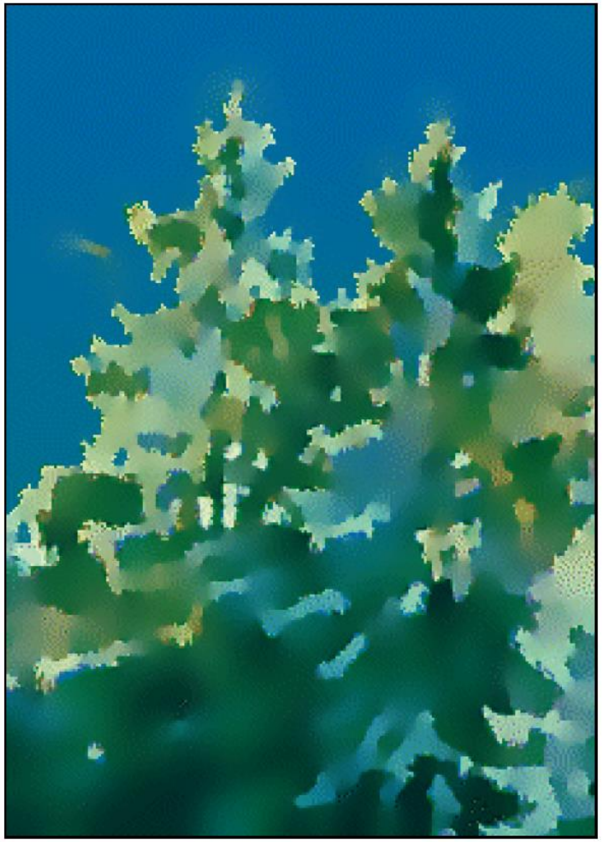
Nonlinear Anisotropic Diffusion

Slide credits: A. Roussos

Nonlinear vs. Nonlinear and anisotropic diffusion



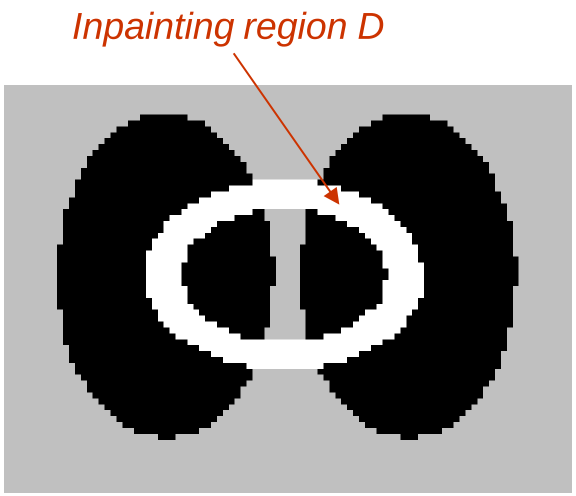
Nonlinear Diffusion



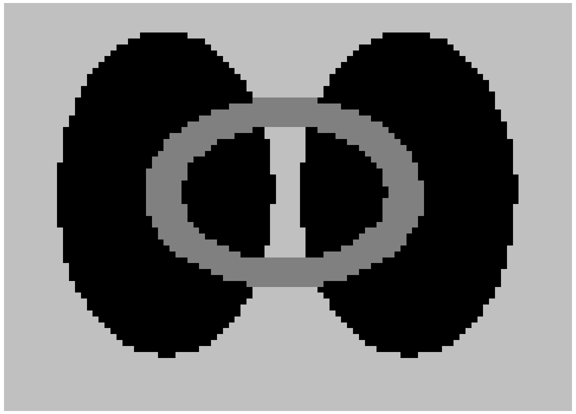
Nonlinear Anisotropic Diffusion

Slide credits: A. Roussos

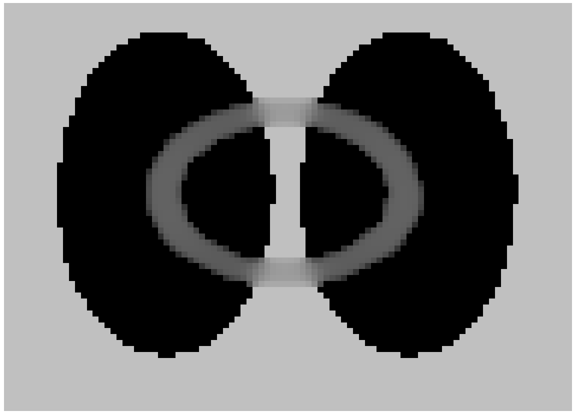
Inpainting problem



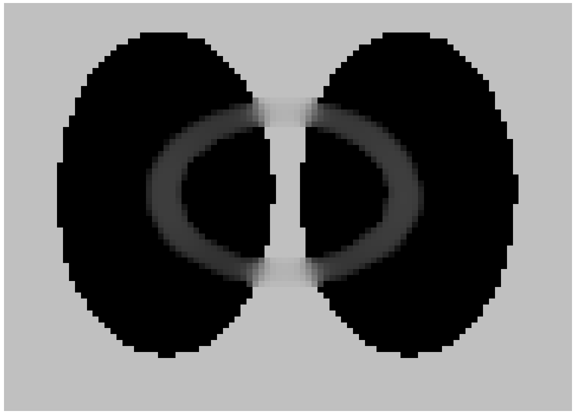
(a) Image with missing region



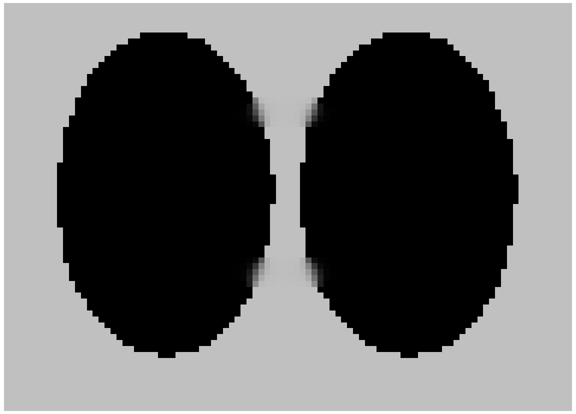
(b) Initial prediction (constant)



(c1) 300 iterations



(c2) 600 iterations



(c3) 1500 iterations (*converged*)

M. Bertalmío, G. Sapiro, V. Caselles and C. Ballester., "Image Inpainting", SIGGRAPH 20
T. F. Chan and J. Shen, "Mathematical Models for Local Nontexture Inpainting", SIAM, 20

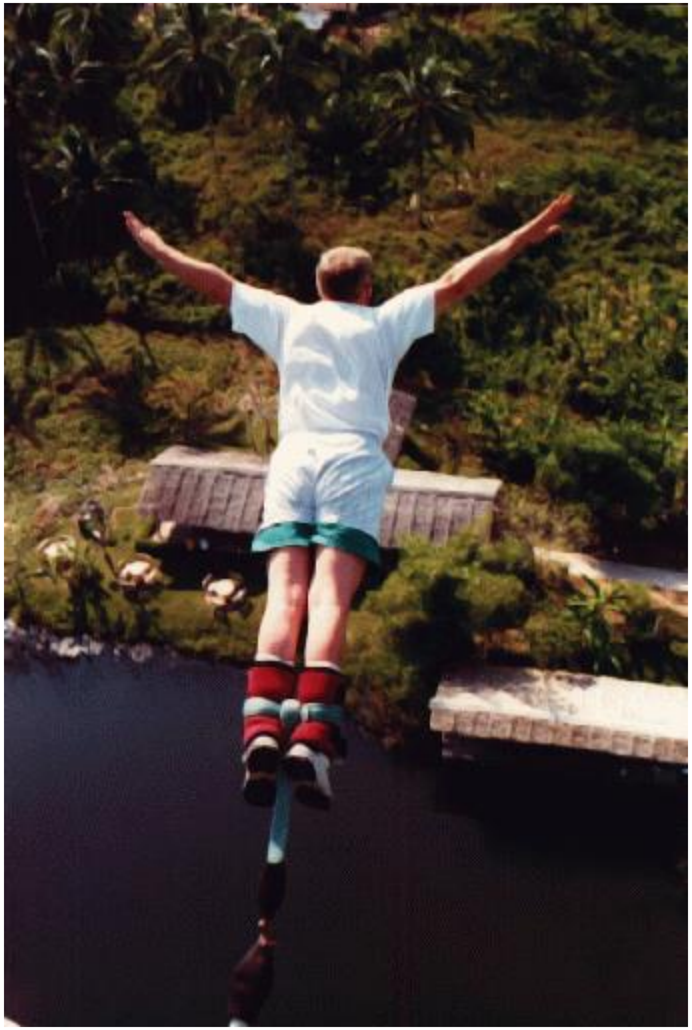
Application: text removal

(a) Input

(b) *Total Variation Inainting*

Slide credits: A. Roussos

Application: fake bravado



Slide credits: A. Roussos

Interpolation problem

(a) Low resolution input image (b) Zero-Padding initialization (c) 4x4 PDE-based magnificatio

A. Roussos and P. Maragos, Reversible Interpolation of vectorial Images, IJCV 2009

PDE-based interpolation

(a) Input Signal

(b) Bicubic interpolation (4 x 4)

(d) PDE-based interpolation (4 x 4)

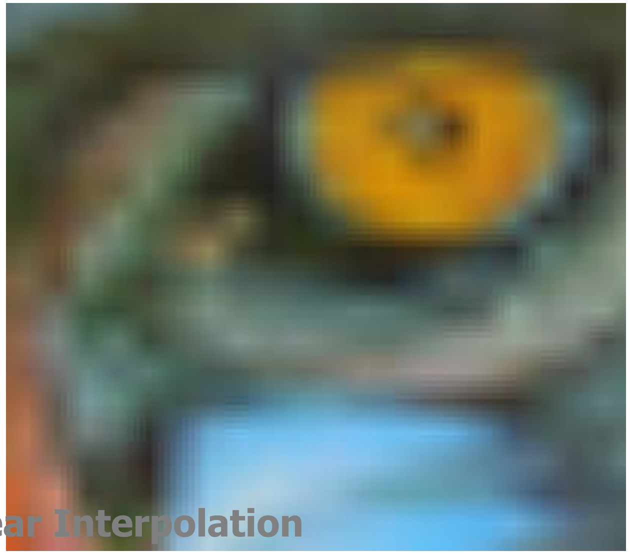
A. Roussos and P. Maragos, Reversible Interpolation of vectorial Images, IJCV 2009

PDE-based interpolation

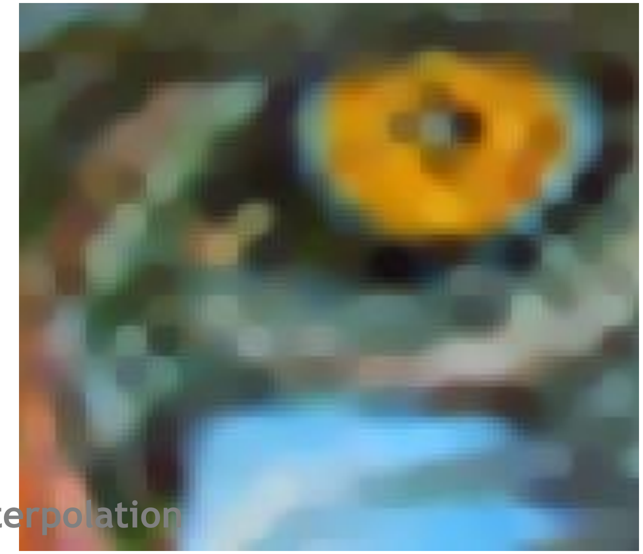
A. Roussos and P. Maragos, Reversible Interpolation of vectorial Images, IJCV 200

PDE-based interpolation

(a) Input



(b) Bilinear Interpolation



(c) TV based Interpolation

(d) Structure-tensor based interpolation

A. Roussos and P. Maragos, Reversible Interpolation of vectorial Images, IJCV

Further study

Fast numerical solutions

G. Papandreou and P. Maragos, Multigrid Geometric Active Contour Models,TIP, 2007.
J. Weickert and B. H. Romeny, 'Efficient Schemes for Nonlinear Diffusion Filtering', TIP '98
A. Chambolle, 'An Algorithm for Total Variation Minimization and Applications', JMIV 2004
T. Goldstein, S. Osher The Split Bregman method for L1-regularized problems, SIAM 2009

Not covered in this talk

Bilteral Filter, Non-Local Means

Buades, B. Coll, and J. Morel, “A non-local algorithm for image denoising,” CVPR, 2005
C. Tomasi and R. Manduchi, “Bilateral filtering for gray and color images,” ICCV 1998
P Milanfar, " A Tour of Modern Image Filtering ", IEEE Signal Processing Magazine, 2013

Online software

http://www.ipol.im/