SHAPE MATCHING WITH IMPLICIT AND EXPLICIT CORRESPONDENCE
Alignment of Deformable Objects

Associated by Non-Rigid Deformations
Representations and Mappings

- **Representations**
  - Point-sets
  - Curves
  - Surfaces
  - Implicit representations (fields)

- **Mappings**
  - Rigid
  - Affine
  - Non-rigid deformations (splines)
  - Diffeomorphisms
  - Geodesics
Non-rigid Point Matching Problem

Difficulties:
* Noise.
* Outliers.
* High dimensional parameter space.
* Hard optimization problem.

Correspondence

Transformation
Correspondence

• Given two point sets: \( X \{x_i, i = 1, 2, \ldots, N \}, \ V \{v_a, a = 1, 2, \ldots, K \} \)

• Match matrix \( M \)
  
  - Inner \( M (K \times N) \):
    \[
    \begin{align*}
    m_{ai} = 1 & \iff \text{Point } x_i \text{ corresponds to point } v_a. \\
    m_{ai} = 0 & \iff \text{Otherwise.}
    \end{align*}
    \]

  - Outlier row and column \( M \):
    \[
    \begin{align*}
    m_{K+1,i} = 1 & \iff \text{Point } x_i \text{ is an outlier.} \\
    m_{a,N+1} = 1 & \iff \text{Point } v_a \text{ is an outlier.}
    \end{align*}
    \]

  - All rows and columns (except outlier) sum up to 1, \( \Rightarrow \) 1-to-1 correspondence.
**Match Matrix \( (M) \)**

- **Correspondence:**
  - \(v_1 \leftrightarrow x_1\)
  - \(v_2 \leftrightarrow x_2\)
  - \(v_3 \leftrightarrow x_3\)
  - Outlier \(\leftrightarrow x_4\)

\[
\begin{array}{cccc|c}
 & x_1 & x_2 & x_3 & x_4 \\
m_{ai} & 1 & 0 & 0 & 0 & 0 \\
 1 & 0 & 1 & 0 & 0 & 0 \\
 2 & 0 & 0 & 1 & 0 & 0 \\
 3 & 0 & 0 & 0 & 1 & 0 \\
\end{array}
\]

Outlier

\[
\begin{array}{cccc|c}
 & x_1 & x_2 & x_3 & x_4 \\
Outlier & 0 & 0 & 0 & 1 & 0 \\
\end{array}
\]
Transformation

- Non-rigid spatial transformation $f$.
- Contains affine transformation as a special case.
- Spatial transformation comprises multiple functions – one per dimension.
- Spatial transformation is regularized using linear operator $L$.
  $$\|Lf\|^2$$ Standard regularization
- Original feature points $v_a$ get mapped to $f(v_a)$
Joint Optimization Formulation

- **Linear Assignment (Outliers) for Correspondence**
- **Least-squares objective for Transformation**

\[
\min_{M, f} E(M, f) = \sum_{a=1}^{K} \sum_{i=1}^{N} m_{ai} \| x_i - f(v_a) \|^2 + \lambda \| Lf \|^2 - \zeta \sum_{a=1}^{K} \sum_{i=1}^{N} m_{ai}
\]

- **Under constraints:**

\[
\begin{align*}
\sum_{a=1}^{K+1} m_{ai} &= 1, \quad \text{for } i = 1, \ldots, N. \\
\sum_{i=1}^{N+1} m_{ai} &= 1, \quad \text{for } a = 1, \ldots, K.
\end{align*}
\]
Discrete solutions to correspondence

- Objective function on correspondence is linear assignment (almost) + outliers.

\[ \min_x c^T x \]
subject to
\[ Ax = b \]

- Totally unimodular property:
  - Unimodular matrix is a square matrix with determinant +1 or -1.
  - Totally unimodular constraint matrix: every square non-singular submatrix is unimodular.

- Constraint matrix for correspondence + outliers is TU.
- The vector \( b \) comprised of integers.
- Optimal values occur for integer values of \( M \).
- Linear programming will yield integer solutions — in this case discrete \( \{0,1\} \) solutions.
- Example: Transportation problem (used in earth mover’s distance).
Modeling the Transformation

The Thin-Plate Spline (TPS)

\[ E_{\text{spline}}(f) = \sum_{i=1}^{N}(u_i - f(v_i))^2 + \lambda \iiint_{\Omega} \left[ \left( \frac{\partial^2 f}{\partial x^2} \right)^2 + 2 \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 + \left( \frac{\partial^2 f}{\partial y^2} \right)^2 \right] dx \, dy \]

Kernelize

\[ E_{\text{TPS}}(A, c) = \sum_{i=1}^{N} \| u_i - Av_i - \sum_{j=1}^{N} K(v_i, v_j)c_j \|^2 + \lambda \text{trace}(c^T KC) \]

- Properties:
  - TPS contains an affine transform
  - One function for each coordinate
  - Least-squares data term can be generalized to include correspondence
The RPM objective function

- Putting it all together:

\[ E_{\text{RPM}}(M,A,c) = \sum_{a=1}^{K} \sum_{i=1}^{N} m_{ai} \| x_a - Av_i - \sum_{j=1}^{N} K(v_i,v_j)c_j \|^2 + \lambda \text{trace}(c^T Kc) - \zeta \sum_{a=1}^{K} \sum_{i=1}^{N} m_{ai} \]

- Linear assignment + outliers for correspondence.
- Least-squares problem for transformation.
- Regularization and outlier parameters.
- TPS-RPM: Alternate between assignments and TPS.
- Iterative closest point (ICP): AVOID.
Implicit Correspondence

Density estimation and $l$-divergence minimization
Shape atlas estimation
Gaussian Mixture Models

Mixture density used: 

\[
p(x|\theta) = \frac{1}{K} \sum_{a=1}^{K} p(x|\mu_a, \sigma)
\]

where

\[
p(x|\mu_a, \sigma) = \frac{1}{(2\pi\sigma^2)^{\frac{D}{2}}} \exp\left\{-\frac{1}{2\sigma^2} \|x - \mu_a\|^2\right\}
\]

multivariate Gaussian, isotropic covariance

negative log-likelihood:

\[
-\log p(X^{(1)}|\mu, \sigma) = -\frac{1}{K} \sum_{i=1}^{N_1} \log \sum_{a=1}^{K} p(X^{(1)}_i|\mu_a, \sigma)
\]
Implicit Correspondence: Density Distances

- Estimate Jensen-Shannon (JS) divergence using law of large numbers approach
- Non-rigid deformation estimation using thin-plate splines
Motivating the Jensen-Shannon divergence

\[
p(X \mid \theta^{(1)}) = \frac{\prod_{i=1}^{N_1} \sum_{a=1}^{K_1} p(X_i | \theta^{(1)}_a) \prod_{j=1}^{N_2} \sum_{b=1}^{K_2} p(Y_j | \theta^{(2)}_b)}{\prod_{i=1}^{N_1} \sum_{a=1}^{K_1} p(X_i | \theta^{(1)}_a) + \prod_{j=1}^{N_2} \sum_{b=1}^{K_2} p(Y_j | \theta^{(2)}_b)}
\]
Motivating the JS divergence (contd.)

Logarithm of likelihood ratio leads to JS divergence (using law of large numbers)
Given multiple sample shape (sample point sets), compute the average shape for which the joint distance between the samples and the average is the shortest.

Difficult if the correspondences between the sample points are unknown.
Atlas estimation

Point Set1

Point Set2

Point Set3

Point Set4

Point Set5

Point Set6

Point Set7

Point-sets Before Registration

Deformed Point-sets
Brain Mapping motivation

Direct Comparison of Subjects

Alignment of Subjects

Comparison of Subjects After Alignment

Distribution Before Alignment

Distribution After Alignment

Brain Functional Image
Nine 3D hippocampal point-sets
Atlas Estimation in 3D

Point-Set 1

Point-Set 2

Point-Set 3

Point-Set 4

Original Pooled Point-Sets

Deformed pointsets
Overlay of point-sets

Estimated atlas

Point-sets warped into atlas space

Affine warping
Diffeomorphisms
Minimize distance between densities

- Closed form L2 distance between Gaussian mixtures
- No correspondences at either the cluster or point level
\[ D[p(x | \theta^{(1)}), p(x | \theta^{(2)})] = \int_{\mathbb{R}^p} \left[ p(x | \theta^{(1)}) - p(x | \theta^{(2)}) \right]^2 dx \]

Distance in closed form for Gaussian mixtures

\[ E_{\ell_2}(A, c) = D[p(x | \theta^{(1)}), p(x | \tilde{\theta}^{(2)})] + \lambda \text{trace}(c^T K c) \]

\( \tilde{\theta}^{(2)} \) is the warped version of \( \theta^{(2)} \)

Nonlinear optimization on \((A, c)\)
Shape L’Âne Rouge

A red donkey solves Klotski
Square-root densities

\[
\sqrt{p(x)} = \sum_k \alpha_{j_0,k} \phi_{j_0,k}(x) + \sum_{j \geq j_0,k} \beta_{j,k} \psi_{j,k}(x)
\]

Shape is a point on hypersphere due to Fisher-Rao geometry
Shape L’Âne Rouge: Sliding Wavelets
Geometry of Shape Matching

Point set representation | Wavelet density estimation

Fast Shape Similarity Using Hellinger Divergence

\[
D(p_1 \parallel p_2) = \int \left( \sqrt{p(x | \Theta_1)} - \sqrt{p(x | \Theta_2)} \right)^2 dx
\]

\[
= 2 - 2 \left( \Theta_1^T \Theta_2 \right)
\]

Or Geodesic Distance

\[
D(p_1, p_2) = \cos^{-1} \left( \Theta_1^T \Theta_2 \right)
\]
Localized Alignment Via Sliding

- Local shape differences will cause coefficients to shift.
- Permutations $\Rightarrow$ Translations
  - Slide coefficients back into alignment.
Penalize Excessive Sliding

- Location operator, $r(j,k)$, gives centroid of each $(j,k)$ basis.
- Sliding cost equal to square of Euclidean distance.
Sliding Objective

- Objective minimizes over penalized permutation assignments

\[
E(\pi) = -\sum_{j_0, k} \alpha_{j_0, k}^{(1)} \alpha_{j_0, \pi(k)}^{(2)} - \sum_{j > j_0, k} \beta_{j, k}^{(1)} \beta_{j, \pi(k)}^{(2)} - \lambda \sum_{j_0, k} \|r(j_0, k) - r(\pi(j_0, k))\|^2 + \sum_{j, k} \|r(j, k) - r(\pi(j, k))\|^2
\]

- Solve via linear assignment using cost matrix

\[
C = \Theta_1 \Theta_2^T + \lambda D
\]

- Where \(\Theta_i\) is vectorized list of \(i^{th}\) shape’s coefficients and

\(D\) is the matrix of distances between basis locations.
Effects of $\lambda$
Discussion

- Implicit and explicit approaches to shape correspondence
- RPM: point-to-point matching
- JS divergence: density matching
- Diffeomorphic matching
- L2 distance: density matching in closed form
- Fast square-root wavelet density matching (Klotski)
- All approaches use separate regularization
The shape matching ecosystem

- Henry Baird thesis (‘84): Linear assignment
- Besl and McKay (‘92): Iterative Closest Point (ICP)
- Yuille and Grzywacz (‘88): Motion coherence theory
- Chui and Rangarajan (‘00): TPS-RPM
- Myronenko et al. (‘09): Coherent Point Drift (CPD)
- Klotski and Earth Mover’s Distance (EMD): (‘08 & earlier)
- Lee and Won (‘11): TPRL (topology preserving relax. lab.)
- L2 distance minimization: (Jian et al. ‘05, Yuille et al. ’13)
- Information-theoretic shape matching: Principe (‘12)
- Shapes and diffeomorphisms: Laurent Younes book