SHAPE MATCHING WITH IMPLICIT AND EXPLICIT CORRESPONDENCE

CV School UCLA

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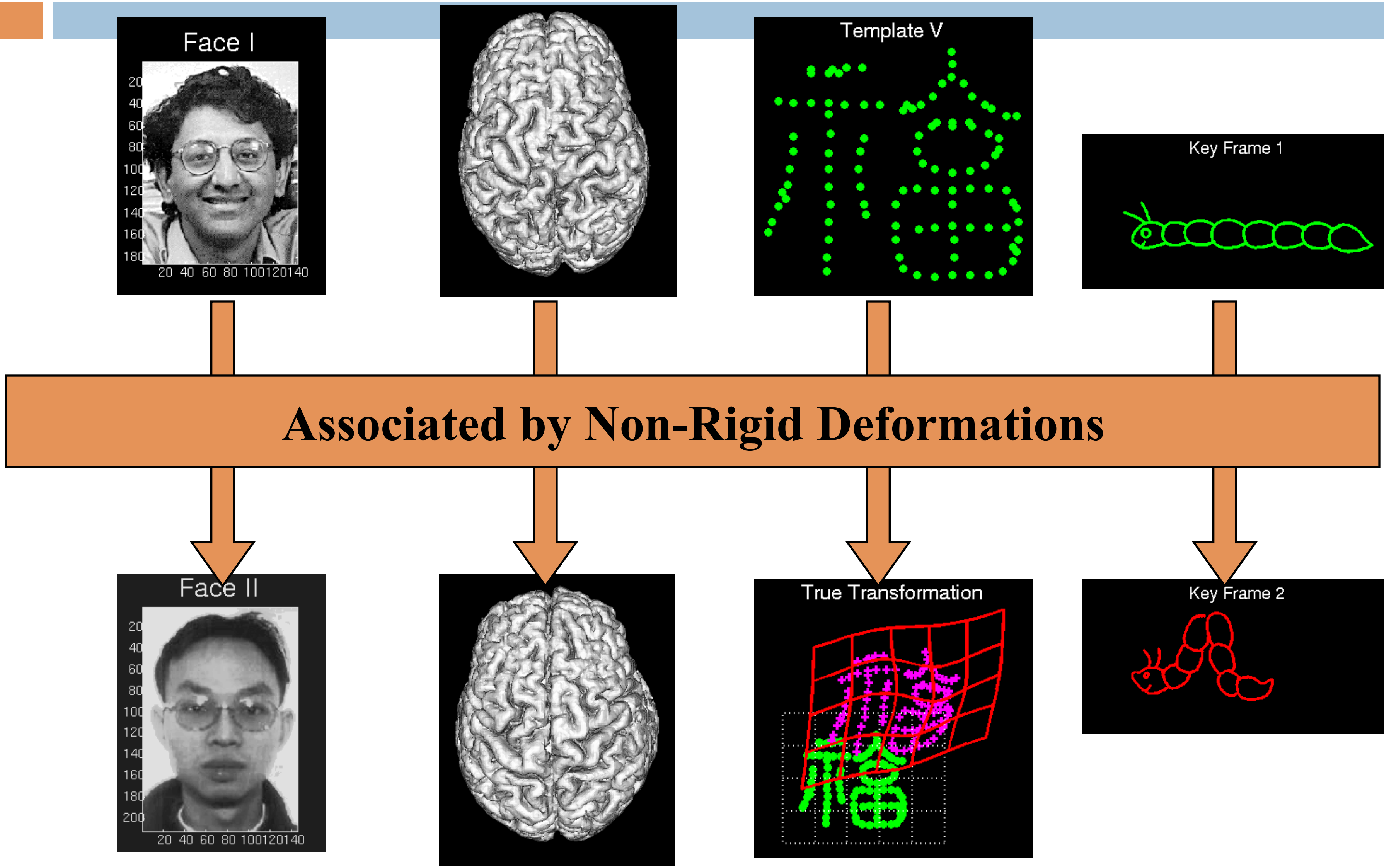
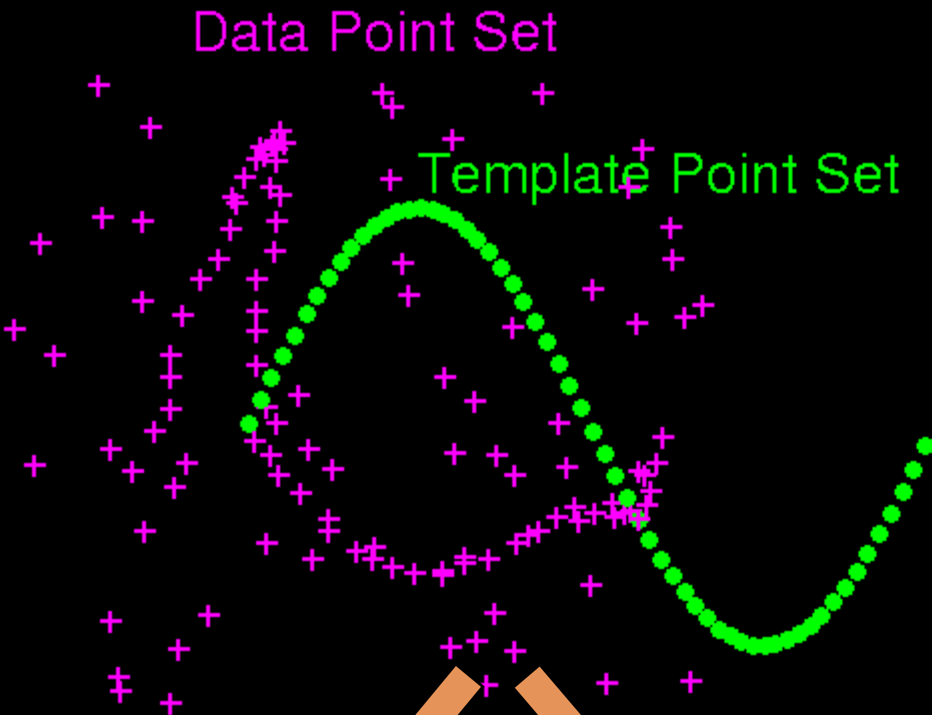
Alignment of Deformable Objects



Representations and Mappings

- Representations
  - Point-sets
  - Curves
  - Surfaces
  - Implicit representations (fields)
- Mappings
  - Rigid
  - Affine
  - Non-rigid deformations (splines)
  - Diffeomorphisms
  - Geodesics

Non-rigid Point Matching Problem



Difficulties:

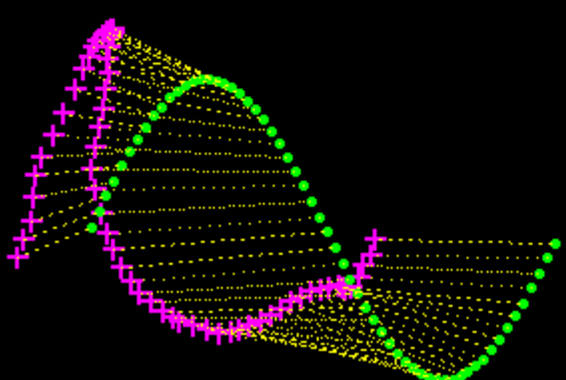
* High dimensional parameter space.

* Hard optimization problem.

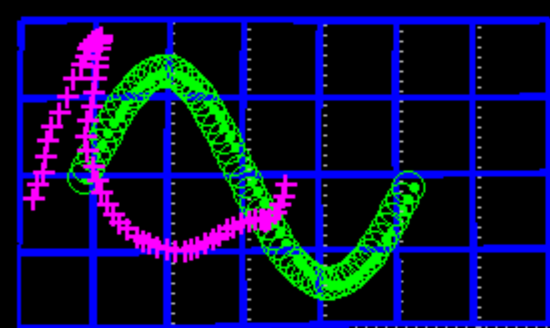
* Noise.

* Outliers.

Correspondence



Transformation



Correspondence

- **Given two point sets:** $X\{x_i, i=1,2,\ldots,N\},\ V\{v_a, a=1,2,\ldots,K\}$

- **Match matrix** M
  - **Inner** $M\,(K\times N)$:

$$\begin{cases} m_{ai}=1 \Leftrightarrow & \text{Point } x_i \text{ corresponds to point } v_\mathrm{a}. \\ m_{ai}=0 \Leftrightarrow & \text{Otherwise.} \end{cases}$$

  - **Outlier row and column** M :

$$\begin{cases} m_{K+1,i}=1 \Leftrightarrow & \text{Point } x_i \text{ is an outlier.} \\ m_{a,N+1}=1 \Leftrightarrow & \text{Point } v_a \text{ is an outlier.} \end{cases}$$

  - **All rows and columns (except outlier) sum up to 1,**
    → 1-to-1 correspondence.

Match Matrix (*M*)

- **Correspondence:**

$v_1 \longleftrightarrow x_1$

$v_2 \longleftrightarrow x_2$

$v_3 \longleftrightarrow x_3$

Outlier $\longleftrightarrow x_4$

$$
\begin{array}{c|cccc|c}
m_{ai} & x_1 & x_2 & x_3 & x_4 & \\
\hline
v_1 & 1 & 0 & 0 & 0 & 0 \\
v_2 & 0 & 1 & 0 & 0 & 0 \\
v_3 & 0 & 0 & 1 & 0 & 0 \\
\hline
\text{Outlier} & 0 & 0 & 0 & \boxed{1} &
\end{array}
$$

Transformation

- Non-rigid spatial transformation *f*.
- Contains affine transformation as a special case.
- Spatial transformation comprises multiple functions – one per dimension.
- Spatial transformation is regularized using linear operator *L*.

  $\|Lf\|^2$ Standard regularization
- Original feature points v_a get mapped to $f(v_a)$

Joint Optimization Formulation

- Linear Assignment (Outliers) for Correspondence
- Least-squares objective for Transformation

Outliers

$$\min_{M,f} E(M,f) = \sum_{a=1}^{K}\sum_{i=1}^{N} m_{ai} \left\| x_i - f(v_a) \right\|^2 + \lambda \| Lf \|^2 - \zeta \sum_{a=1}^{K}\sum_{i=1}^{N} m_{ai}$$

Regularization

- **Under constraints:**

$$\begin{cases} \sum_{a=1}^{K+1} m_{ai} = 1, \quad \text{for } i = 1,\ldots,N. \\ \sum_{i=1}^{N+1} m_{ai} = 1, \quad \text{for } a = 1,\ldots,K. \end{cases}$$

Discrete solutions to correspondence

- Objective function on correspondence is linear assignment (almost) + outliers.

$$\min_x c^T x$$

subject to

$$Ax = b$$

When do we get integer solutions when using an LP solver?

- Totally unimodular (TU) property:
  - Unimodular matrix is a square matrix with determinant +1 or -1.
  - Totally unimodular constraint matrix: every square non-singular submatrix is unimodular.
- Constraint matrix for correspondence + outliers is TU.
- The vector *b* comprised of integers.
- Optimal values occur for integer values of *M*.
- Linear programming will yield integer solutions – in this case discrete {0,1} solutions.
- Example: Transportation problem (used in earth mover's distance).

Modeling the Transformation

The Thin-Plate Spline (TPS)

$$E_{\text{spline}}(f) = \sum_{i=1}^{N} (u_i - f(v_i))^2 + \lambda \iint_{\Omega} \left[\left(\frac{\partial^2 f}{\partial x^2} \right)^2 + 2 \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 + \left(\frac{\partial^2 f}{\partial y^2} \right)^2 \right] dx\, dy$$

Kernelize

$$E_{\text{TPS}}(A, c) = \sum_{i=1}^{N} \| u_i - A v_i - \sum_{j=1}^{N} K(v_i, v_j) c_j \|^2 + \lambda \operatorname{trace}(c^T K c)$$

- Properties:
  - TPS contains an affine transform
  - One function for each coordinate
  - Least-squares data term can be generalized to include correspondence

The RPM objective function

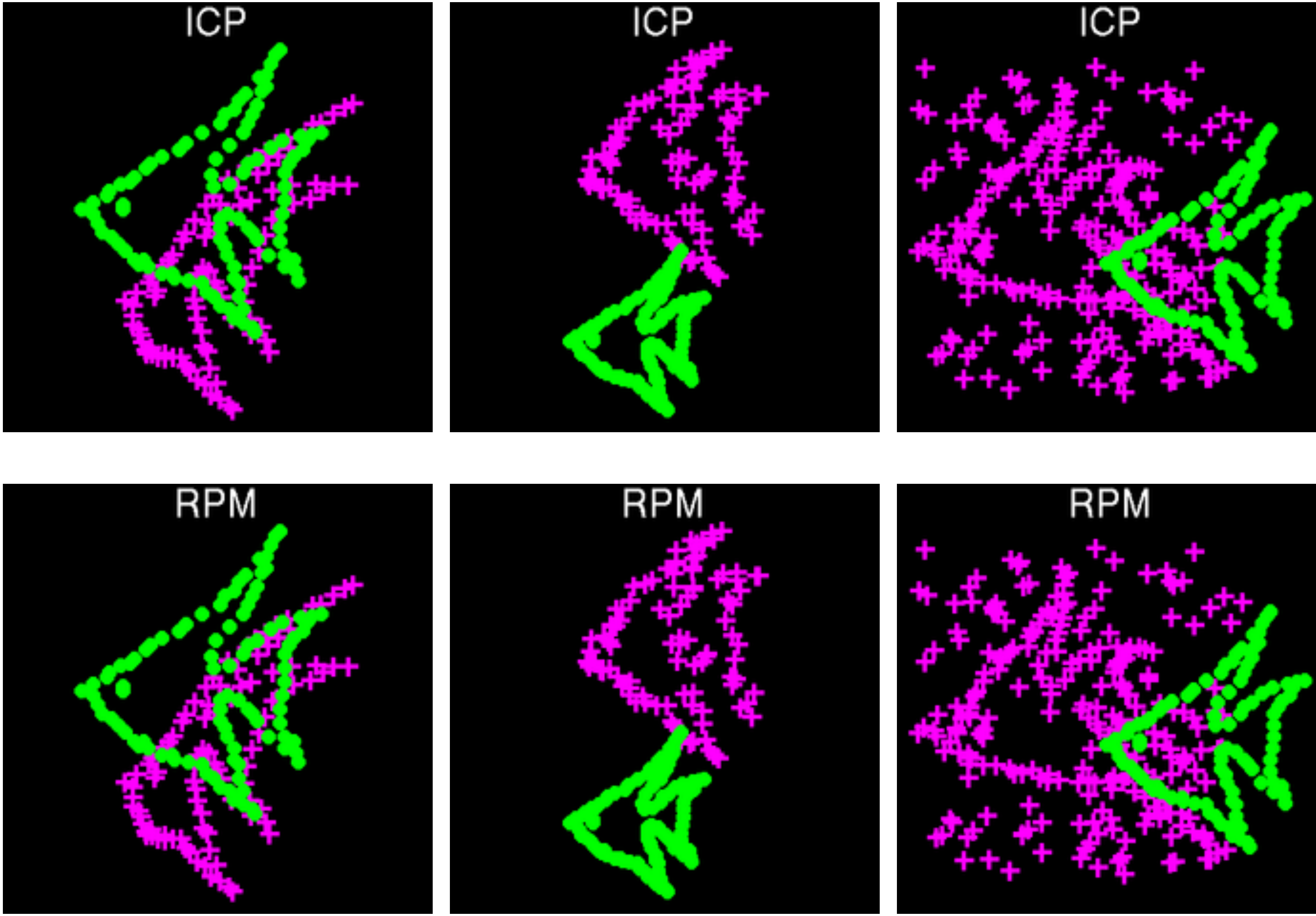
- Putting it all together:

$$E_{\text{RPM}}(M,A,c)=\sum_{a=1}^{K}\sum_{i=1}^{N}m_{ai}\,\|\,x_a-Av_i-\underbrace{\sum_{j=1}^{N}K(v_i,v_j)c_j}_{\text{Transformation}}\,\|^2+\lambda\,\text{trace}(\mathrm{c}^T Kc)-\zeta\sum_{a=1}^{K}\sum_{i=1}^{N}m_{ai}$$

Correspondence

Transformation

- Linear assignment + outliers for correspondence.
- Least-squares problem for transformation.
- Regularization and outlier parameters.
- TPS-RPM: Alternate between assignments and TPS.
- Iterative closest point (ICP): AVOID.


ICP
ICP
ICP
RPM
RPM
RPM

Implicit Correspondence

Density estimation and I-divergence minimization

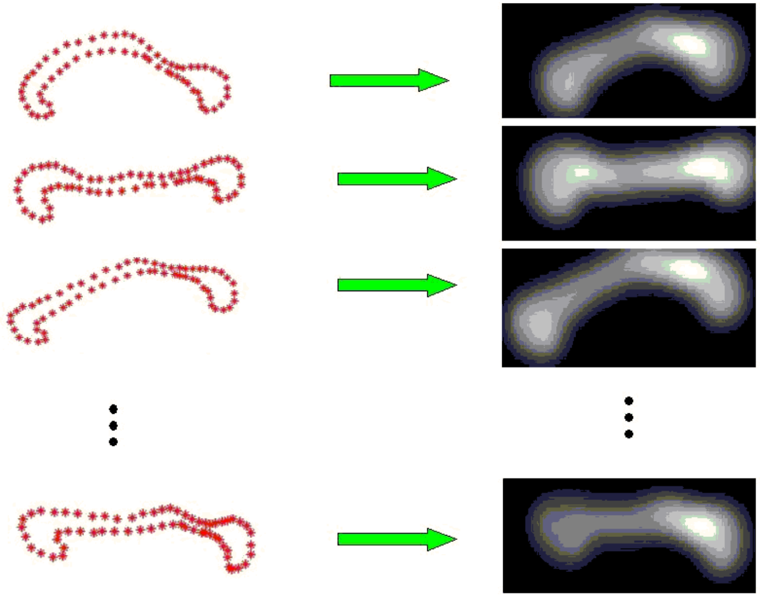
Shape atlas estimation

Gaussian Mixture Models

Mixture density used: $p(\mathbf{x}|\theta) = \frac{1}{K}\sum_{a=1}^{K} p(\mathbf{x}|\mu_a,\sigma)$ **where**

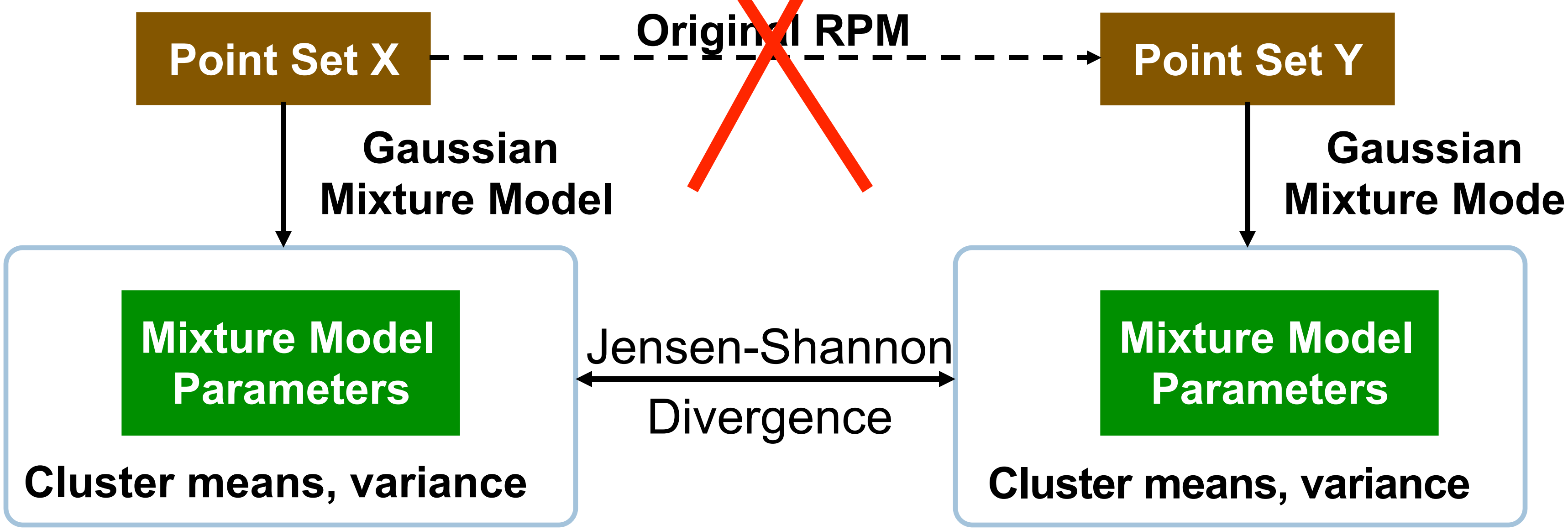
$$p(\mathbf{x}|\mu_a,\sigma) = \frac{1}{(2\pi\sigma^2)^{\frac{D}{2}}} \exp\{-\frac{1}{2\sigma^2}||\mathbf{x}-\mu_a||^2\}$$

multivariate Gaussian, isotropic covariance



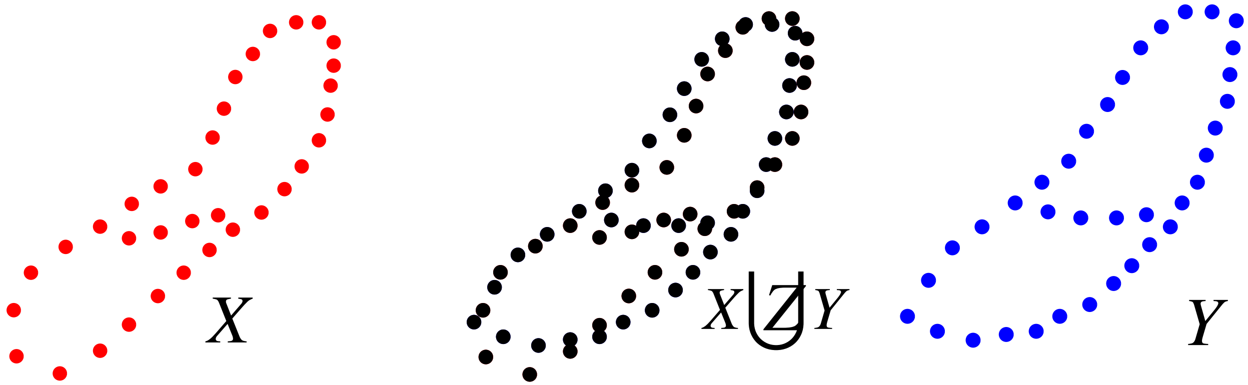
negative log-likelihood: $-\log p(X^{(1)}|\mu,\sigma) = -\frac{1}{K}\sum_{i=1}^{N_1}\log\sum_{a=1}^{K} p(X_i^{(1)}|\mu_a,\sigma)$

Implicit Correspondence: Density Distances



- Estimate Jensen-Shannon (JS) divergence using law of large numbers approach
- *Non-rigid* deformation estimation using thin-plate splines

Motivating the Jensen-Shannon divergence

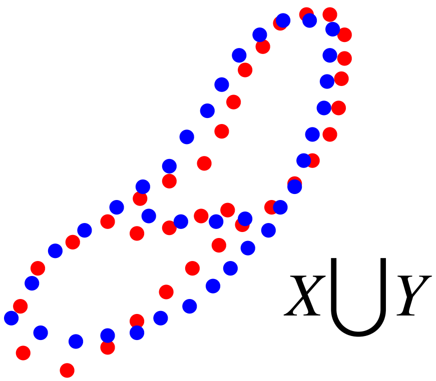


$$p(X|\theta^{(1)}) = \prod_{i=1}^{N_1}\sum_{a=1}^{K_1} p(X_i|\theta_a^{(1)})$$

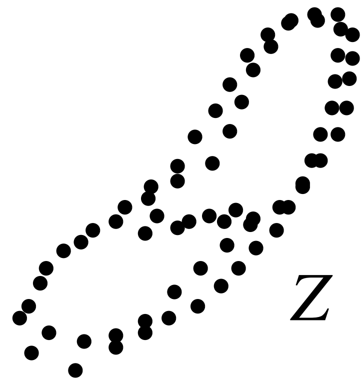
$$p(Y|\theta^{(2)}) = \prod_{j=1}^{N_2}\sum_{b=1}^{K_2} p(Y_j|\theta_b^{(2)})$$

$$p(Z = X\bigcup Y|\theta^{(1)},\theta^{(2)}) = \prod_{i=1}^{N_1+N_2}\left\{\frac{N_1}{N_1+N_2}\sum_{a=1}^{K_1} p(Z_i|\theta_a^{(1)}) + \frac{N_2}{N_1+N_2}\sum_{b=1}^{K_2} p(Z_i|\theta_b^{(2)})\right\}$$

Motivating the JS divergence (contd.)



Overlay with identity



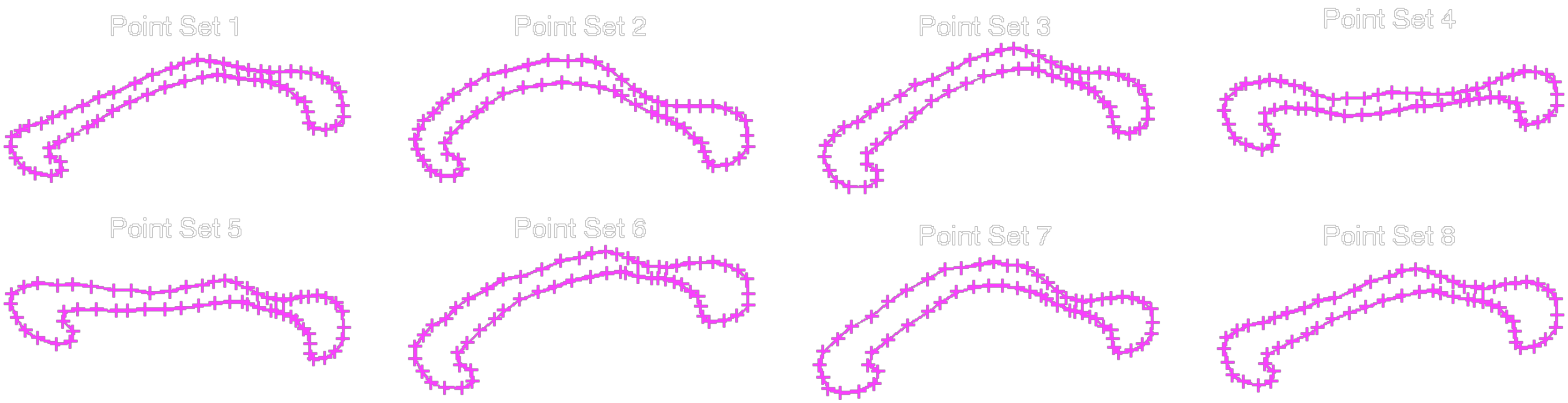
Overlay without identity

$$\log \Lambda = \log \frac{p(Z \mid \theta^{(1)}, \theta^{(2)})}{p(X \bigcup Y \mid \theta^{(1)}, \theta^{(2)})} = \frac{\prod_{i=1}^{N_1+N_2} \left\{ \frac{N_1}{N_1+N_2} \sum_{a=1}^{K_1} p(Z_i \mid \theta_a^{(1)}) + \frac{N_2}{N_1+N_2} \sum_{b=1}^{K_2} p(Z_i \mid \theta_b^{(2)}) \right\}}{\prod_{i=1}^{N_1} \sum_{a=1}^{K_1} p(X_i \mid \theta_a^{(1)}) \prod_{j=1}^{N_2} \sum_{b=1}^{K_2} p(Y_j \mid \theta_b^{(2)})}$$

Logarithm of likelihood ratio leads to JS divergence (using law of large numbers)

Estimating An Atlas

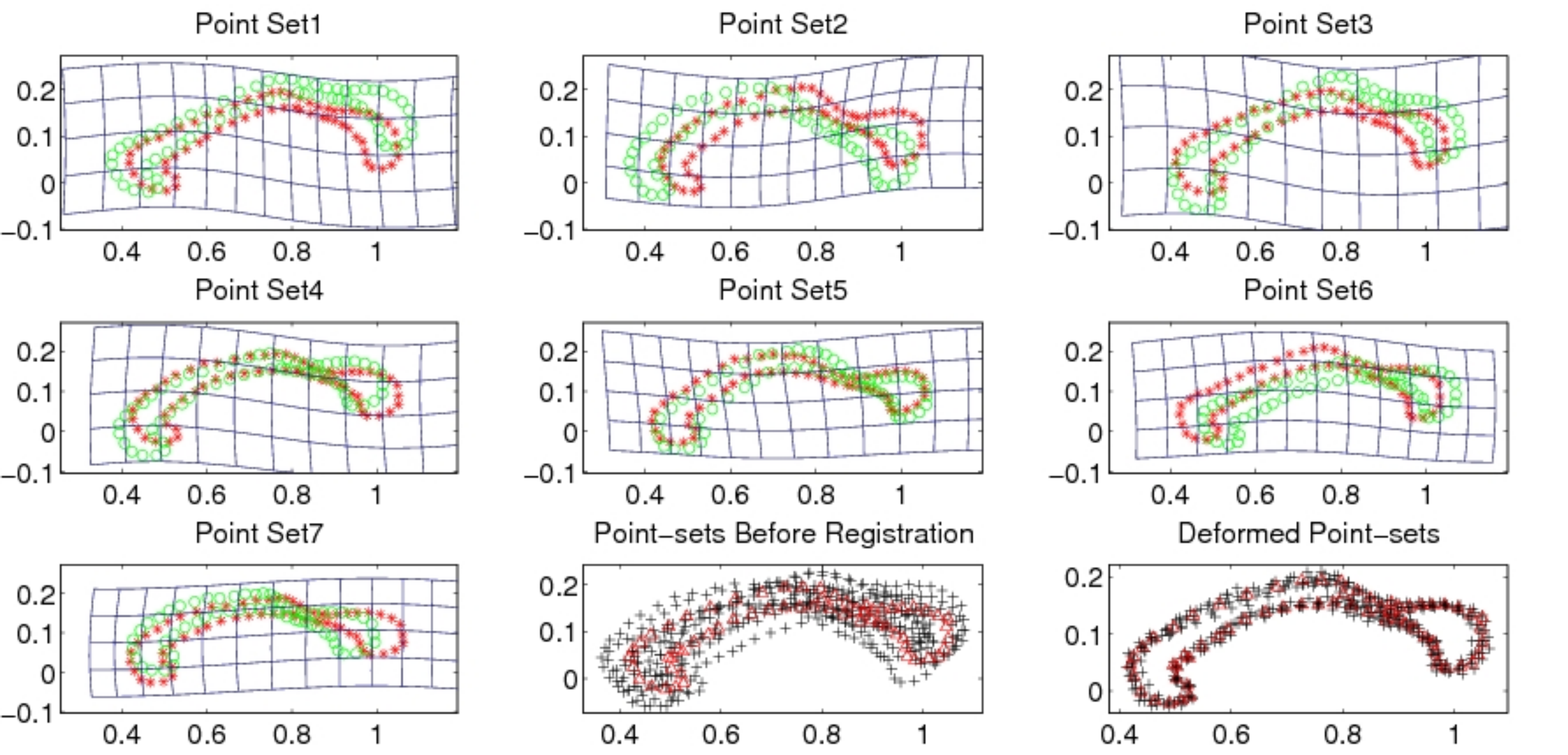
- Given multiple sample shape (sample point sets), compute the average shape for which the joint distance between the samples and the average is the shortest.



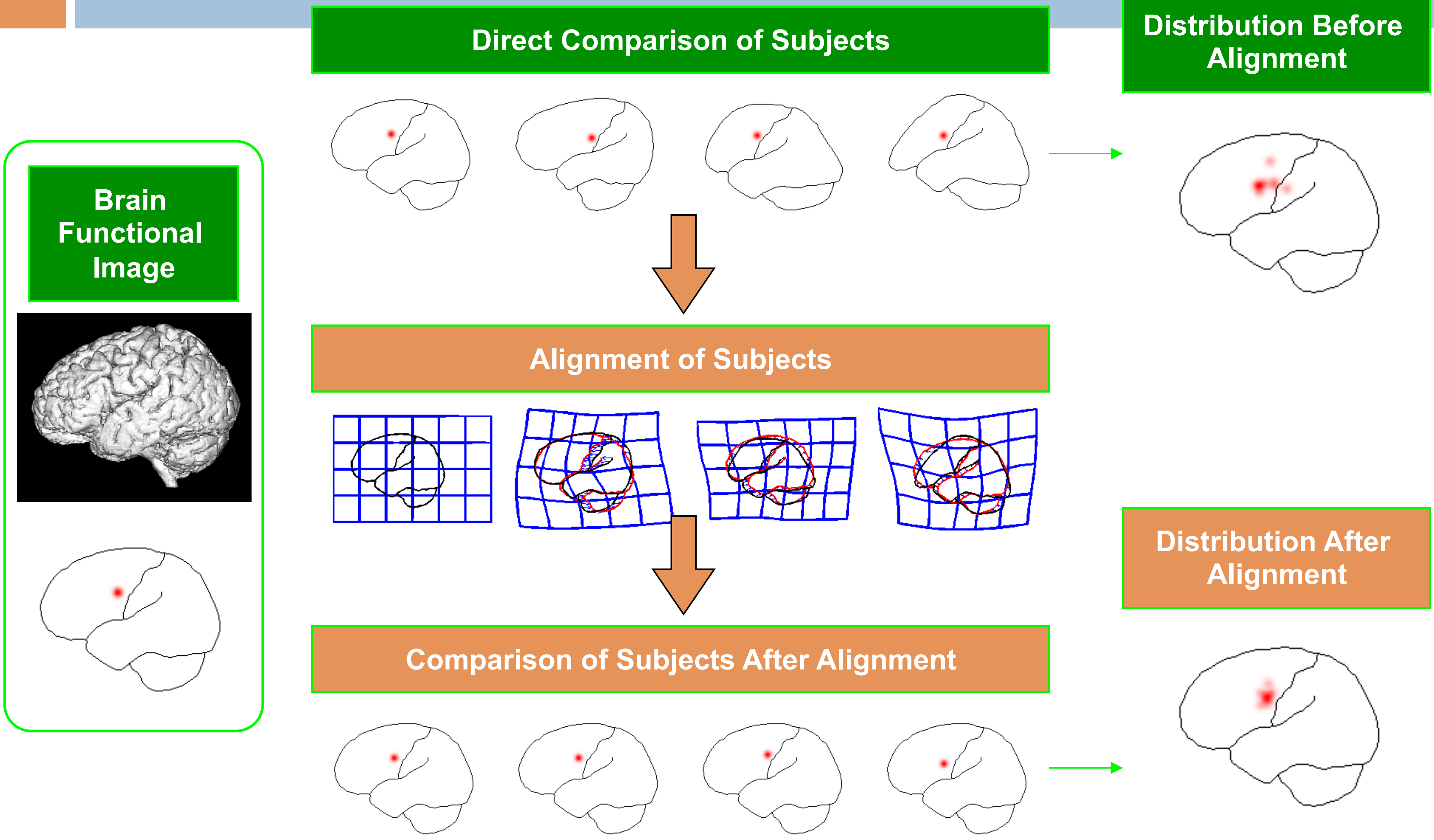
Average ?

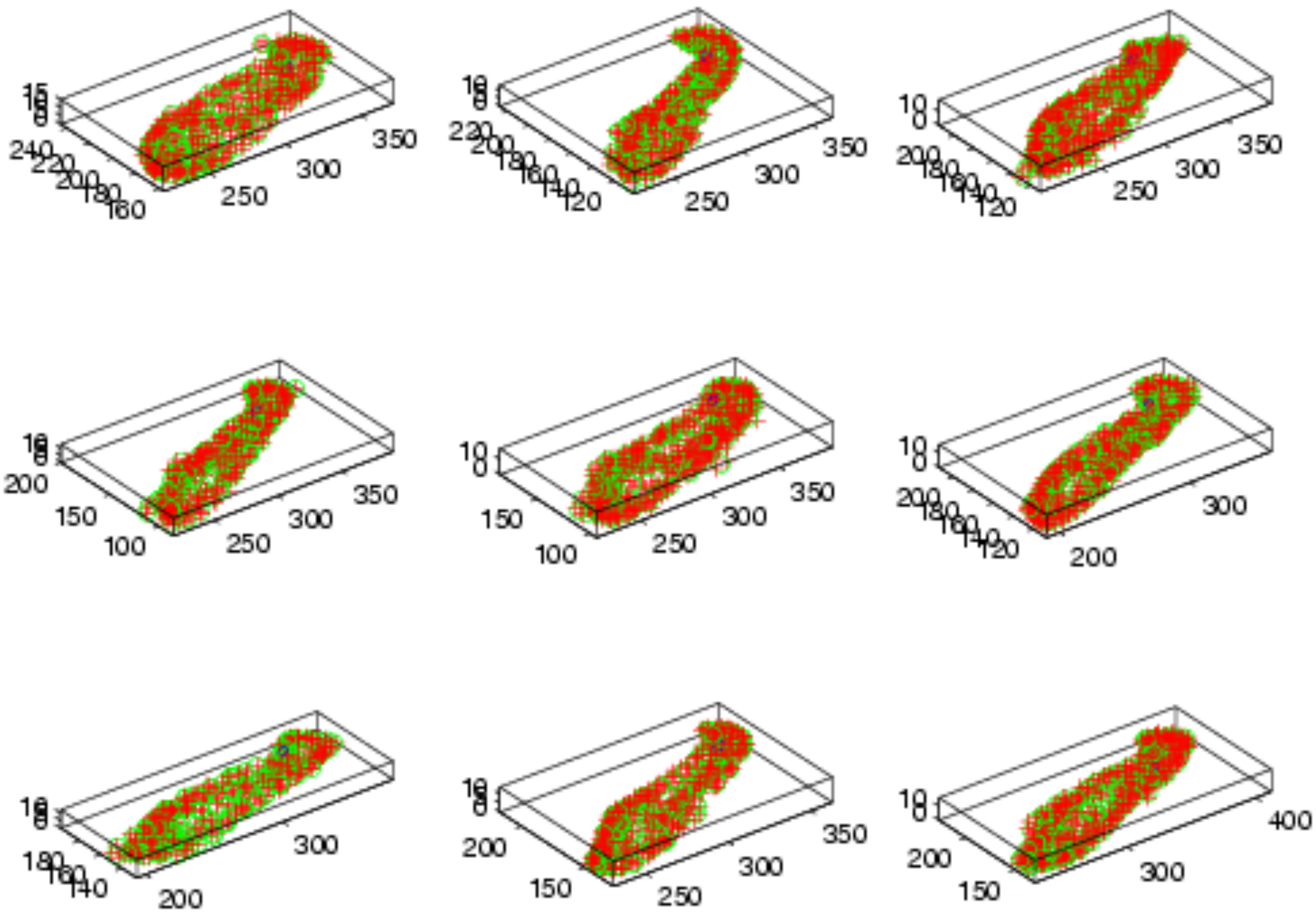
Difficult if the correspondences between the sample points are unknown.

Atlas estimation



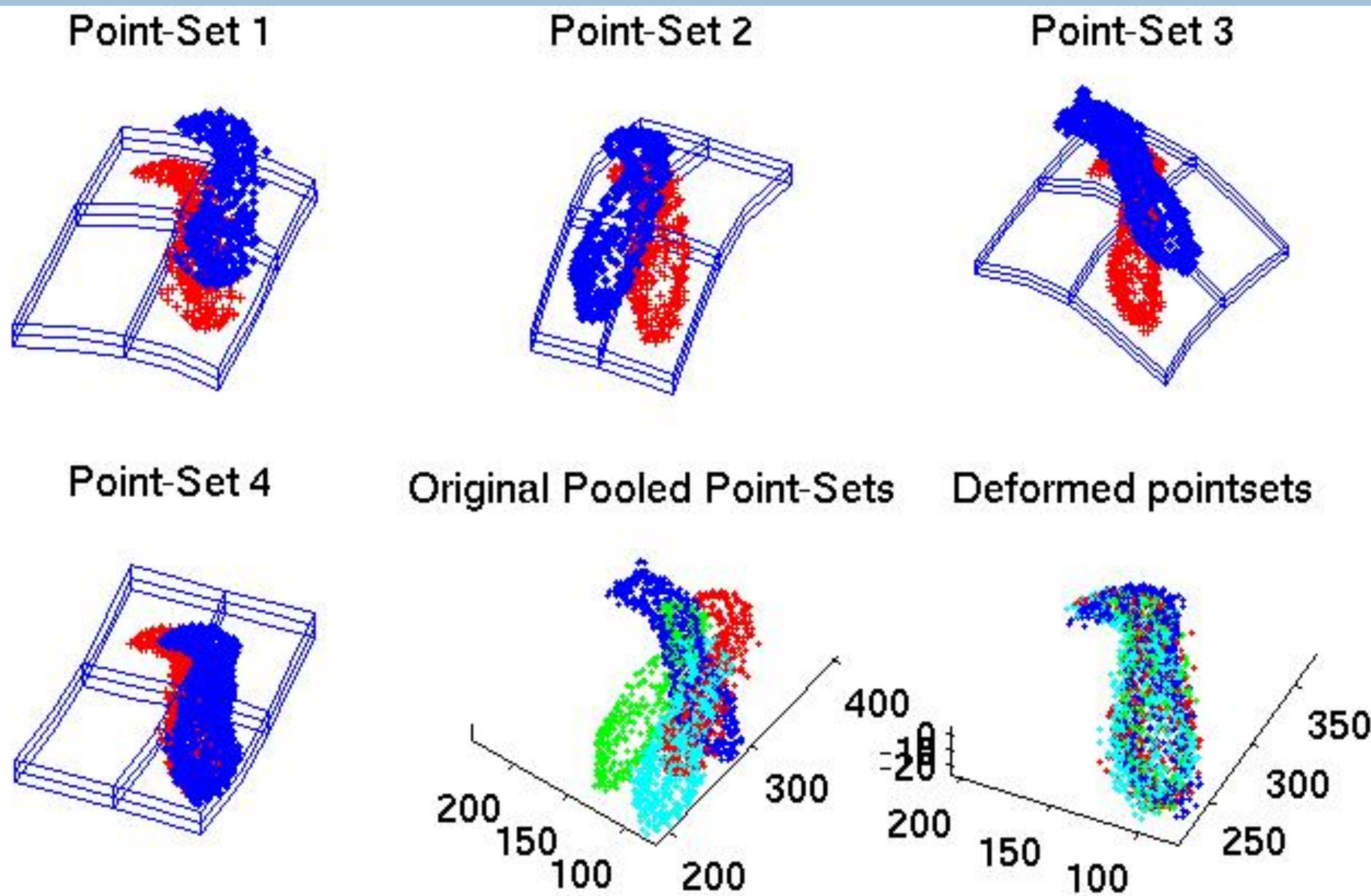
Brain Mapping motivation

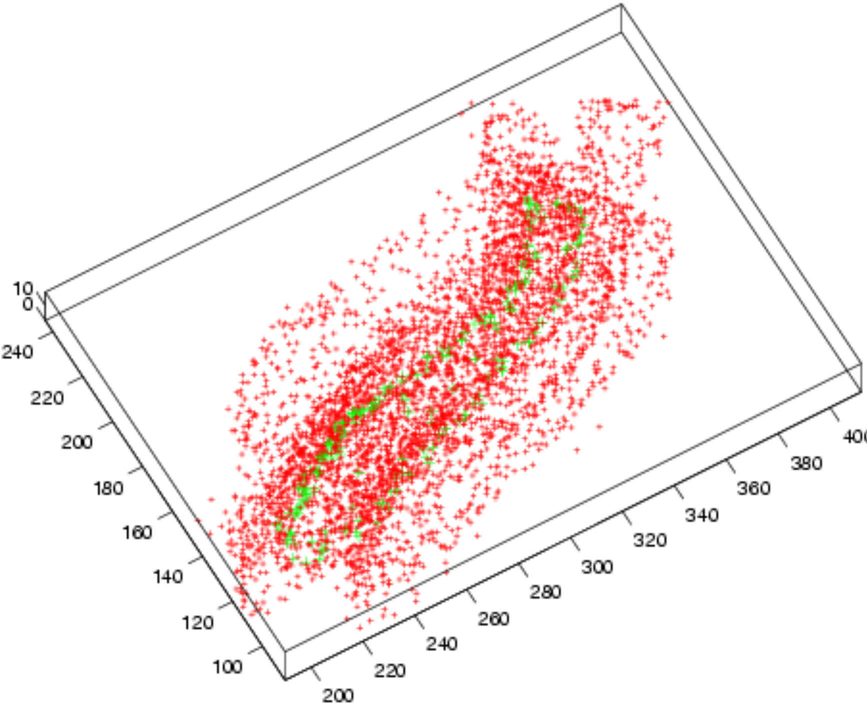




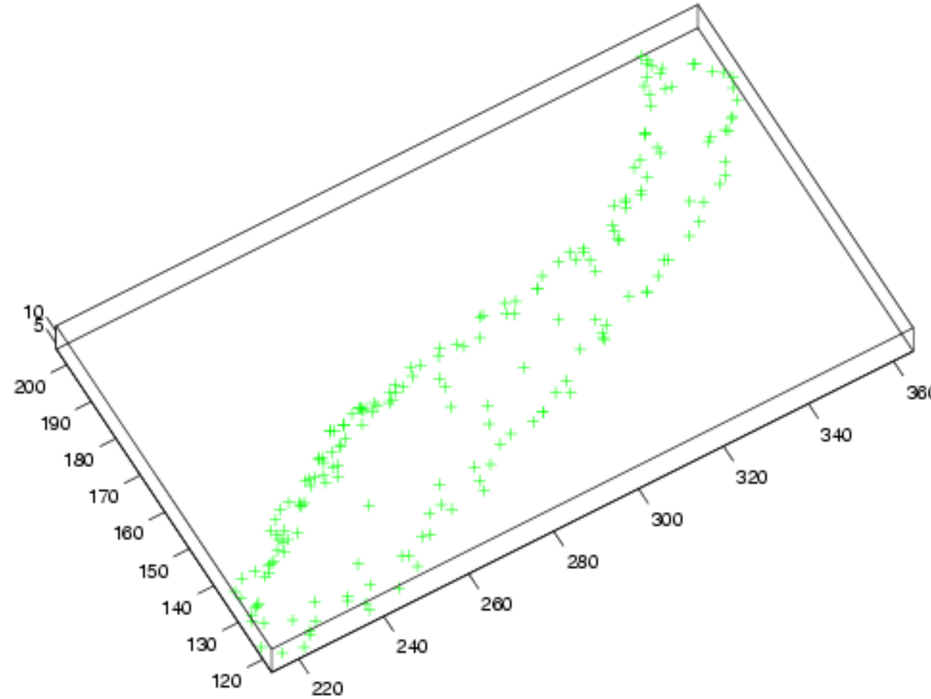
Nine 3D hippocampal point-sets

Atlas Estimation in 3D

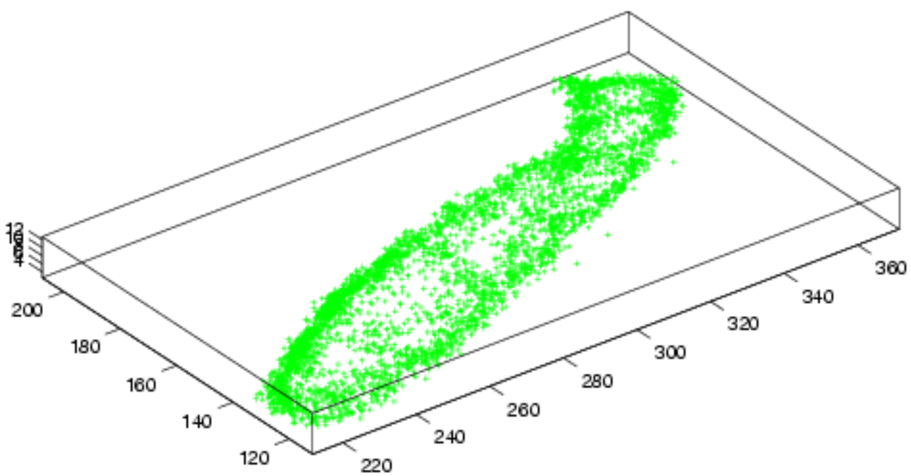




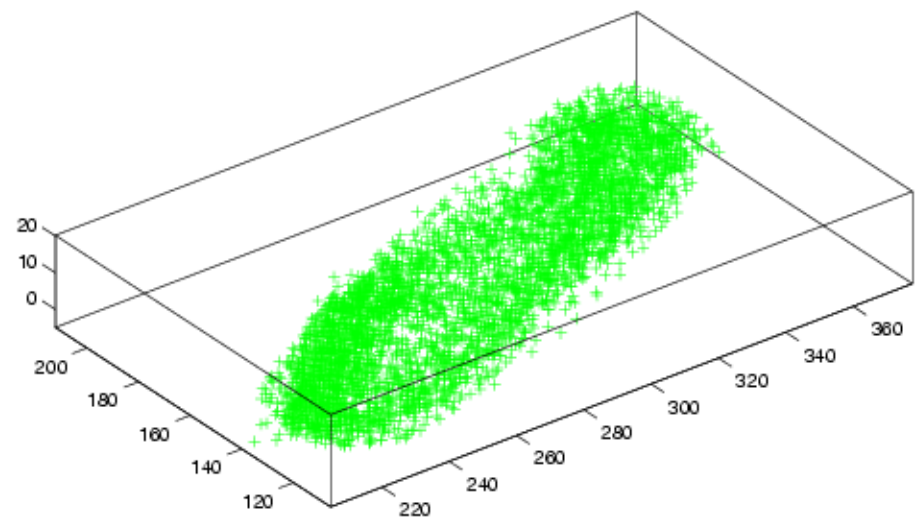
Overlay of point-sets



Estimated atlas

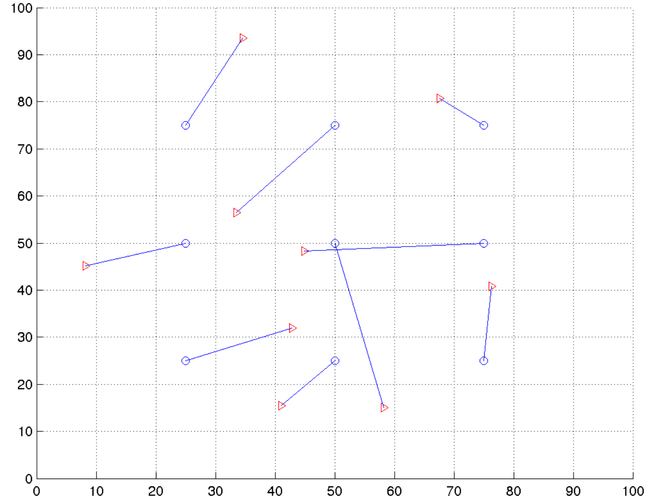


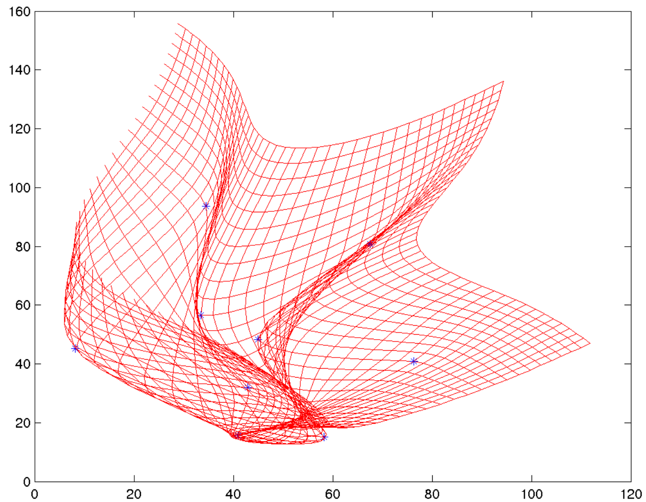
Point-sets warped into atlas space

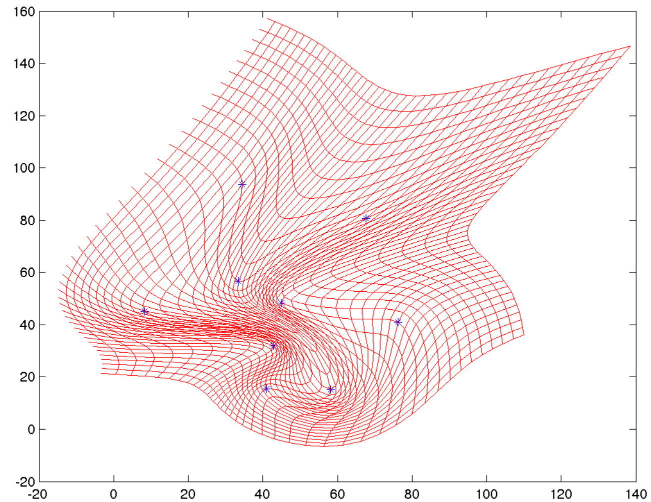


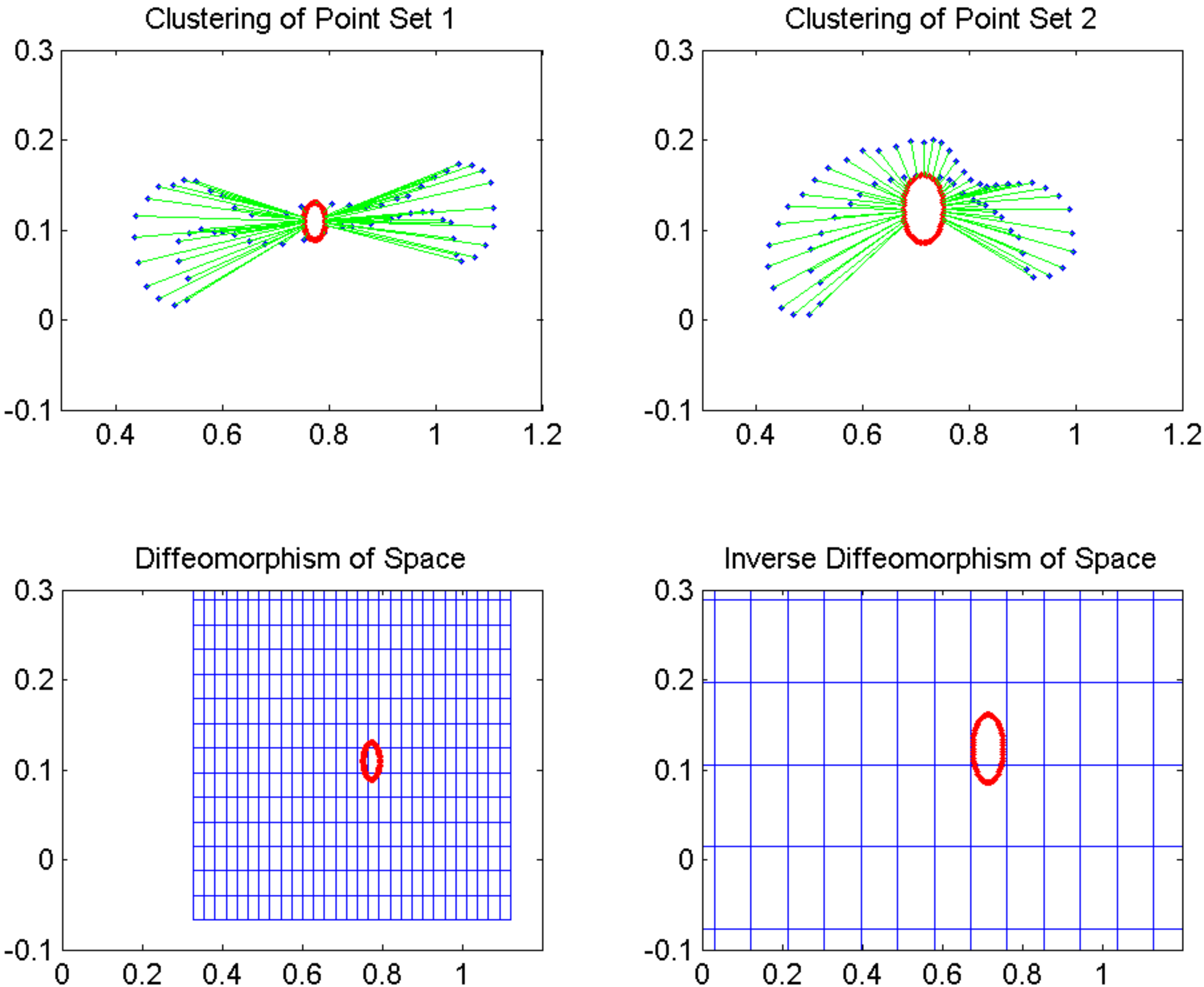
Affine warping

Diffeomorphisms



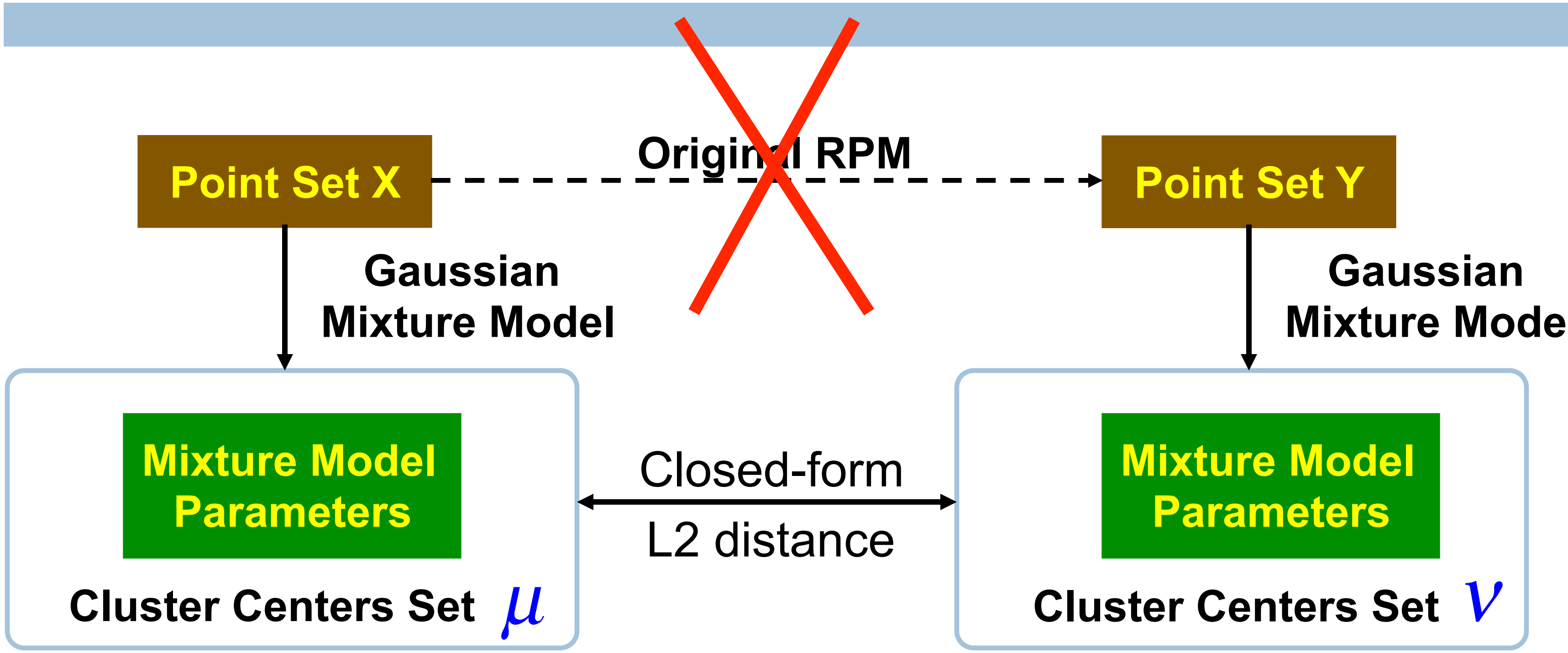






Clustering of Point Set 1
Clustering of Point Set 2
Diffeomorphism of Space
Inverse Diffeomorphism of Space
0.3
0.2
0.1
0
-0.1
0.4
0.6
0.8
1
1.2
0
0.2

Minimize distance between densities



- **Closed form L2 distance between Gaussian mixtures**
- **No correspondences at either the cluster or point level**

ℓ_2 distance between Gaussian mixtures

$$D[p(\mathbf{x}\,|\,\theta^{(1)}), p(\mathbf{x}\,|\,\theta^{(2)})] = \int_{\mathbb{R}^D} \left[p(\mathbf{x}\,|\,\theta^{(1)}) - p(\mathbf{x}\,|\,\theta^{(2)}) \right]^2 d\mathbf{x}$$

$$D[p(\mathbf{x}|\theta^{(1)}), p(\mathbf{x}|\theta^{(2)})] \propto -\sum_{a=1}^{K_1}\sum_{\alpha=1}^{K_2} \frac{2\exp\{-\frac{1}{2(\sigma^2+\xi^2)}||\mu_a-\nu_\alpha||^2\}}{K_1K_2(\sigma^2+\xi^2)^{\frac{3}{2}}} + \sum_{\alpha=1}^{K_2}\sum_{\beta=1}^{K_2} \frac{\exp\{-\frac{1}{4\xi^2}||\nu_\alpha-\nu_\beta||^2\}}{2^{\frac{3}{2}}K_2^2\xi^3}$$

Distance in closed form for Gaussian mixtures

$$E_{\ell_2}(A,c) = D[p(\mathbf{x}\,|\,\theta^{(1)}), p(\mathbf{x}\,|\,\tilde{\theta}^{(2)})] + \lambda\,\mathrm{trace}(c^T K c)$$

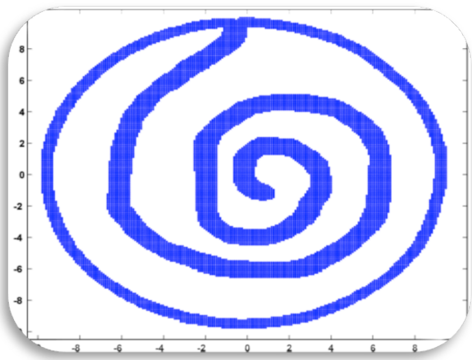
$\tilde{\theta}^{(2)}$ **is the warped version of** $\theta^{(2)}$

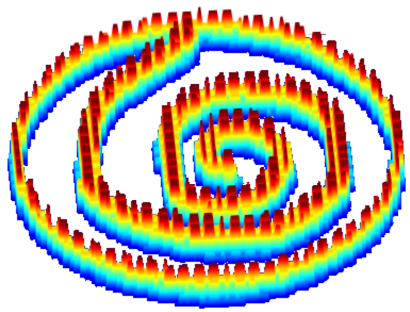
Nonlinear optimization on (*A,c*)

Shape L'Âne Rouge

A red donkey solves Klotski

Square-root densities

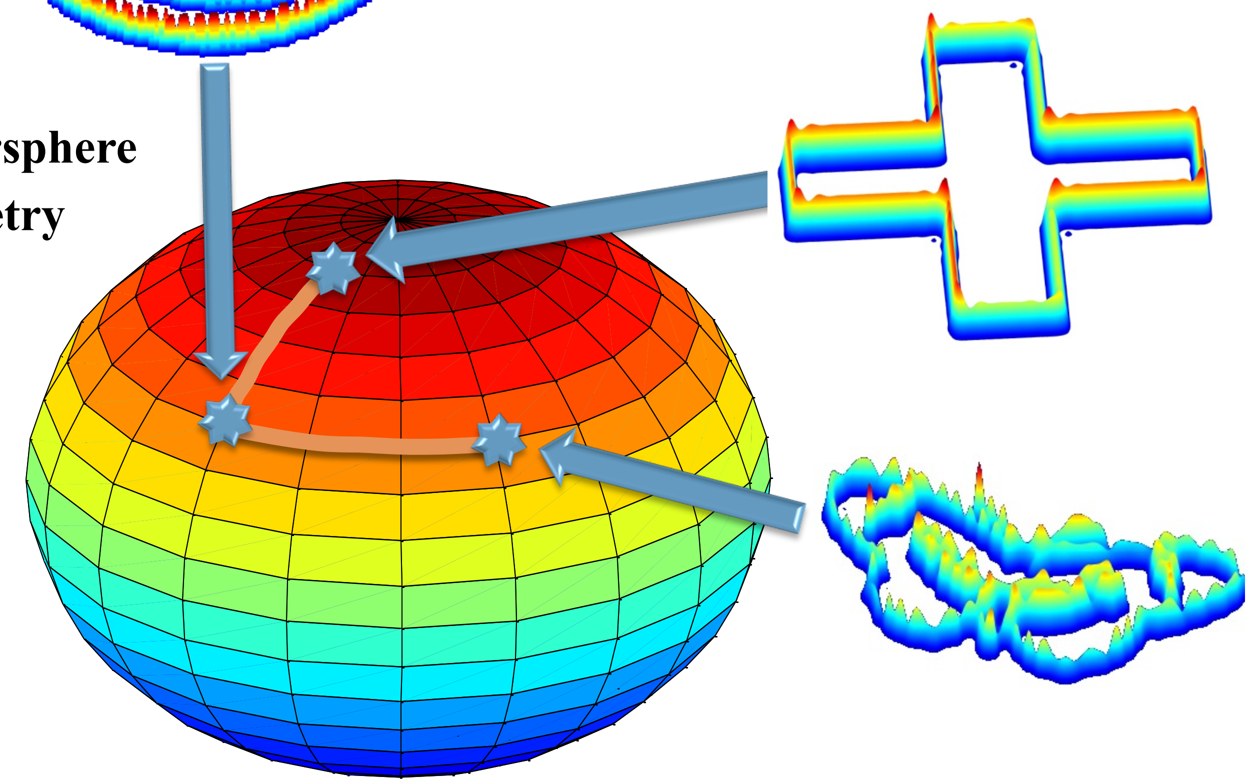




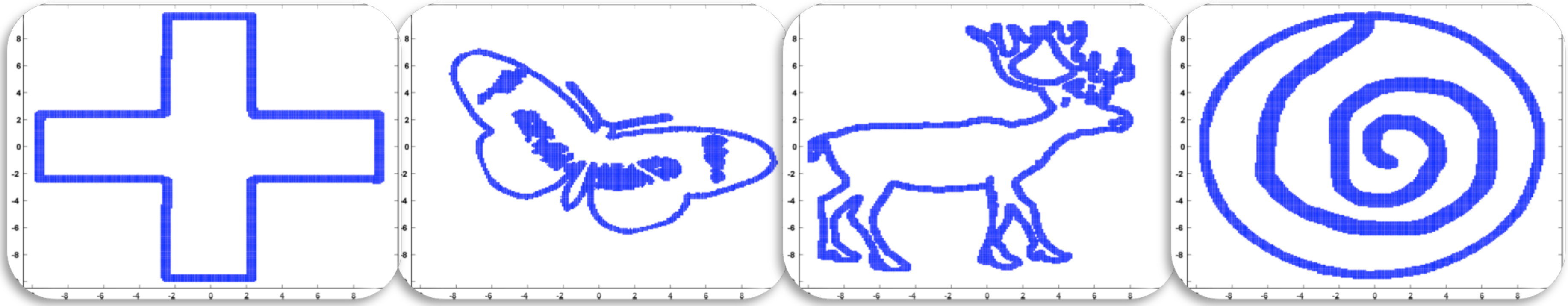
Wavelets

$$\sqrt{p(x)} = \sum_{k} \alpha_{j_0,k}\phi_{j_0,k}(x) + \sum_{j \geq j_0,k}^{\infty} \beta_{j,k}\psi_{j,k}(x)$$

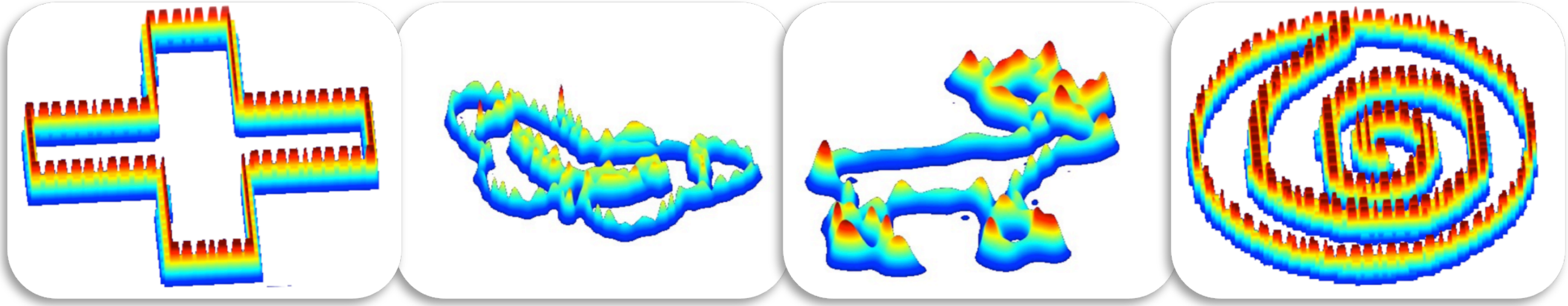
Shape is a point on hypersphere due to Fisher-Rao geometry



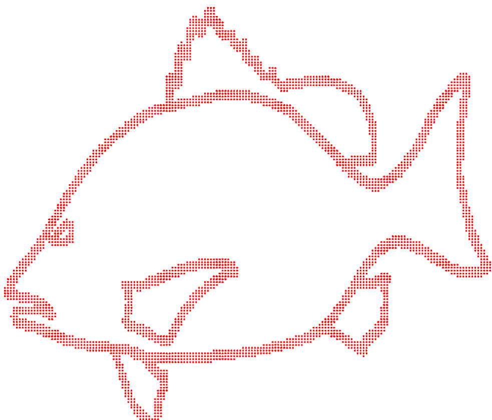
Shape L'Âne Rouge: Sliding Wavelets



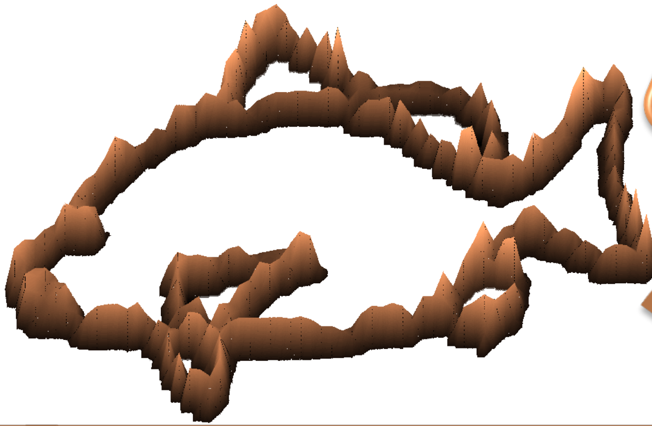
MPEG-7 Shape Densities



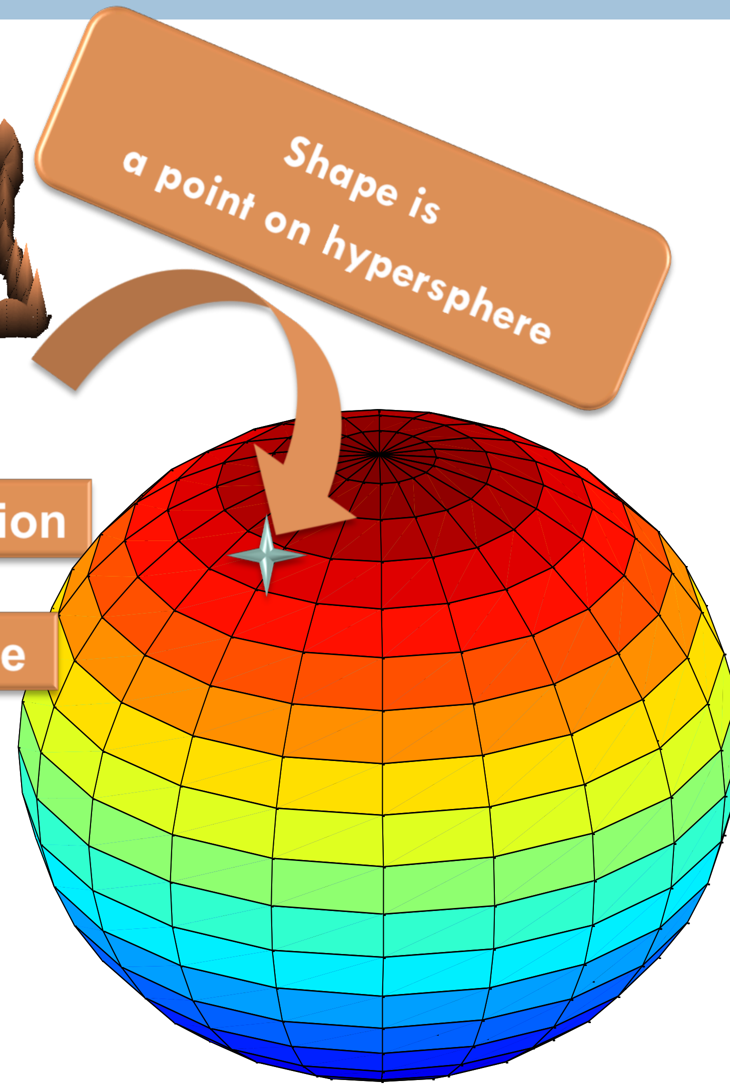
Geometry of Shape Matching



Point set representation



Wavelet density estimation



Fast Shape Similarity Using Hellinger Divergence

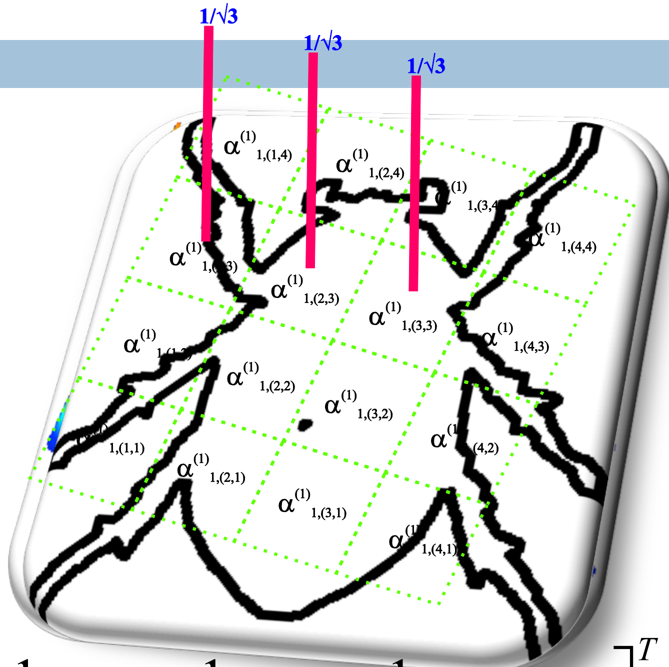
$$D(p_1 \| p_2) = \int \left(\sqrt{p(\mathbf{x} | \Theta_1)} - \sqrt{p(\mathbf{x} | \Theta_2)} \right)^2 d\mathbf{x}$$

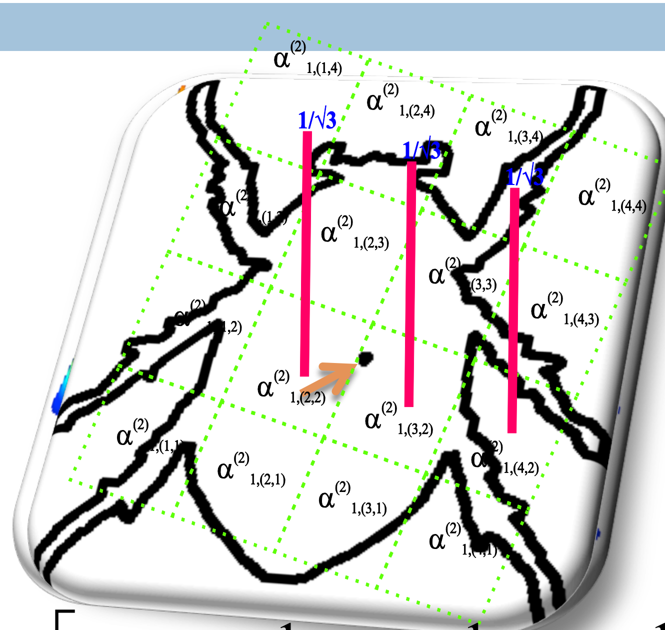
$$= 2 - 2\left(\Theta_1^T \Theta_2\right)$$

Or Geodesic Distance

$$D(p_1, p_2) = \cos^{-1}(\Theta_1^T \Theta_2)$$

Localized Alignment Via **Sliding**

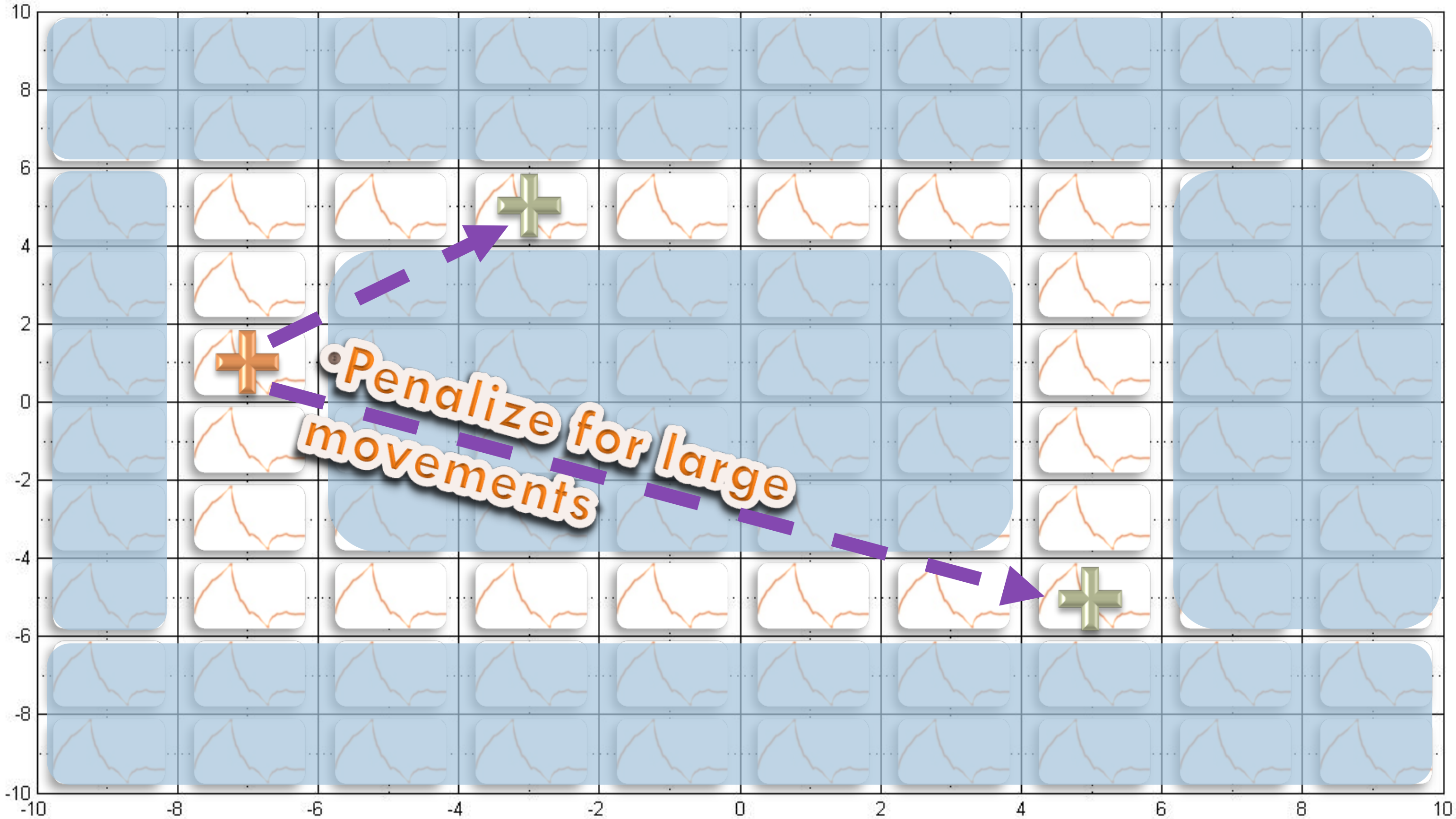




$$\left[00\frac{1}{\sqrt{3}}000\frac{1}{\sqrt{3}}000\frac{1}{\sqrt{3}}00000\right]^T \qquad \left[00000\frac{1}{\sqrt{3}}000\frac{1}{\sqrt{3}}000\frac{1}{\sqrt{3}}00\right]^T$$

- Local shape differences will cause coefficients to shift.
- Permutations ⇒ Translations
  - Slide coefficients back into alignment.

Penalize Excessive Sliding



- Location operator, $\mathbf{r}(j,k)$, gives centroid of each (j,k) basis.
- Sliding cost equal to square of Euclidean distance.

Sliding Objective

- Objective minimizes over penalized permutation assignments

$$E(\pi) = -\left(\sum_{j_0,k} \alpha^{(1)}_{j_0,k} \alpha^{(2)}_{j_0,\pi(k)} + \sum_{j>j_0,k} \beta^{(1)}_{j,k} \beta^{(2)}_{j,\pi(k)} \right) + \lambda \left[\sum_{j_0,k} \left\| \mathbf{r}(j_0,k) - \mathbf{r}(\pi(j_0,k)) \right\|^2 + \sum_{j,k} \left\| \mathbf{r}(j,k) - \mathbf{r}(\pi(j,k)) \right\|^2 \right]$$

Permutation

Location operator

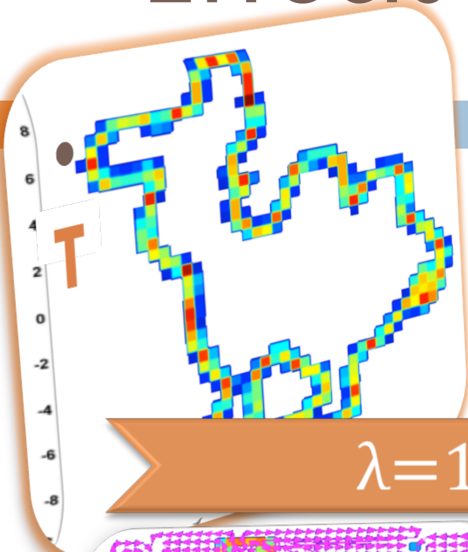
Penalty Weight

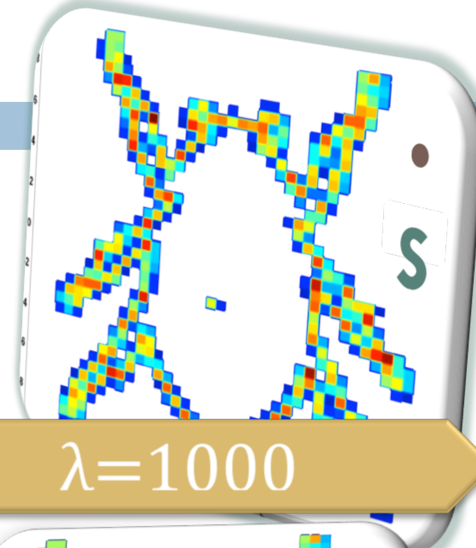
- Solve via linear assignment using cost matrix

$$C = \Theta_1 \Theta_2^T + \lambda D$$

  - where Θ_i is vectorized list of i^{th} shape's coefficients and D is the matrix of distances between basis locations.

Effects of λ

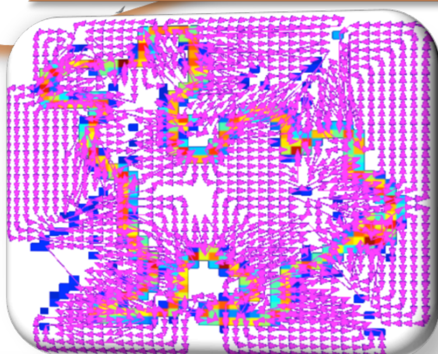


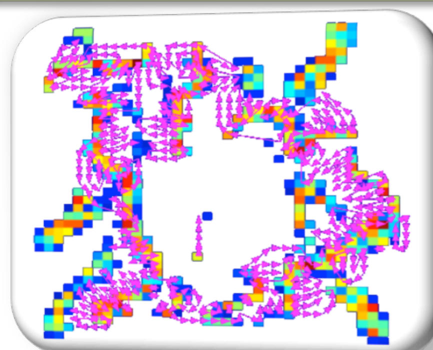


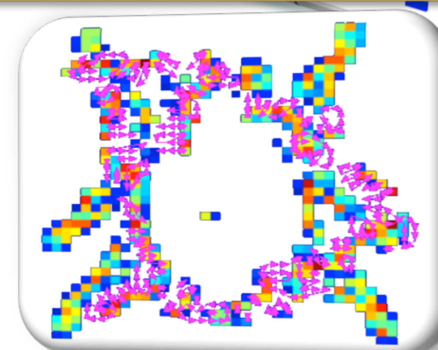
λ=10

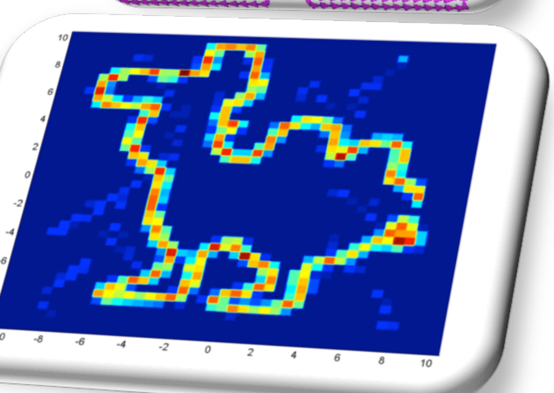
λ=500

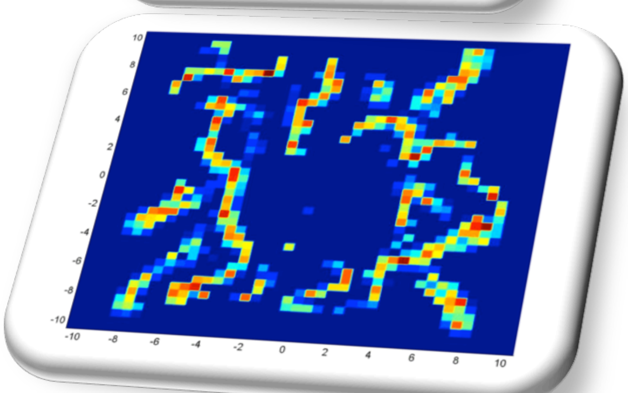
λ=1000

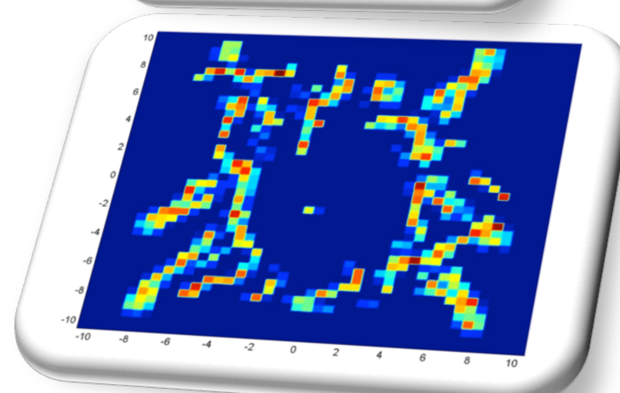












Discussion

- Implicit and explicit approaches to shape correspondence
- RPM: point-to-point matching
- JS divergence: density matching
- Diffeomorphic matching
- L2 distance: density matching in closed form
- Fast square-root wavelet density matching (Klotski)
- All approaches use *separate* regularization

The shape matching ecosystem

- Henry Baird thesis (‘84): Linear assignment
- Besl and McKay (‘92): Iterative Closest Point (ICP)
- Yuille and Grzywacz (‘88): Motion coherence theory
- Chui and Rangarajan (‘00): TPS-RPM
- Myronenko *et al*. (’09): Coherent Point Drift (CPD)
- Klotski and Earth Mover’s Distance (EMD): (’08 & earlier)
- Lee and Won (‘11): TPRL (topology preserving relax. lab.)
- L2 distance minimization: (Jian et al. ‘05, Yuille *et al*. ’13)
- Information-theoretic shape matching: Principe (‘12)
- Shapes and diffeomorphisms: Laurent Younes book