WHY THE WORLD LOVES THE LEVEL SET METHOD (AND RELATED) METHODS

Stan Osher

Mathematics Department
UCLA
sjo@math.ucla.edu

and

Level Set Systems, Inc.
1058 Embury St.
Pacific Palisades, CA 90272-2501
sjo@levelset.com

Google: “Level Set Method”

$\approx$ 53,000,000 responses

Osher-Sethian original paper (1988)

Now cited $\approx$ 9,000 times (google scholar)

Book

S. Osher & R.P. Fedkiw

Level Set Methods and Dynamic Implicit Surfaces.
Springer-Verlag (2002)

Also

S. Osher & N. Paragios (Eds)

Geometric Level Set Methods in Imaging, Vision & Graphics

Springer-Verlag (2003)

Also
Course at SIGGRAPH (2004)
Los Angeles, August 2004

Applied Mathematical Sciences

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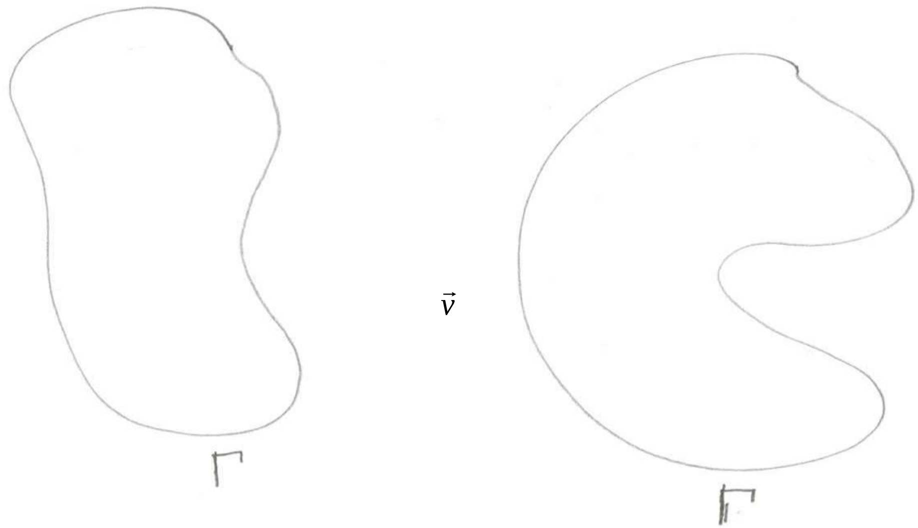
Stanley Osher
Ronald Fedkiw

Level Set Methods and Dynamic Implicit Surfaces

Springer

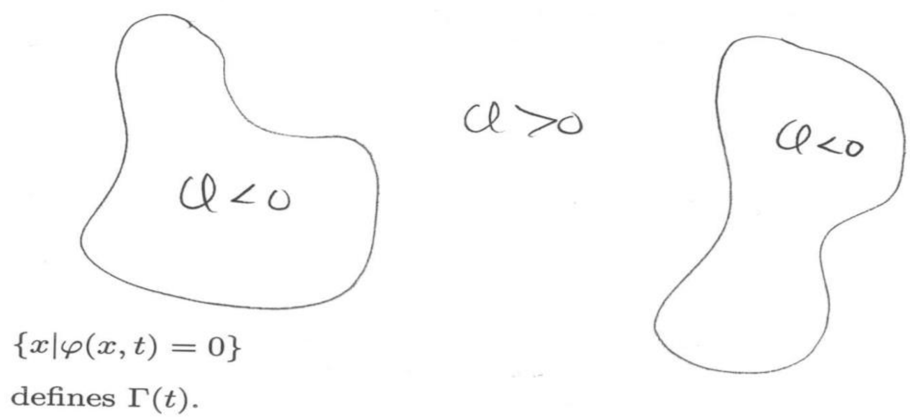
Given an interface in R^n, call it Γ, of codimension one,



Move it normal to itself under velocity $\vec{v}$

$\vec{v} = \vec{v}$ (x, geometry, external physics)

O & Sethian (1987)



Also: Unreferenced papers by

Dervieux, Thomassett, (1979, 1980).

Some of the key ideas in obscure proceedings.

Trivial fact

Zero level contour

$$\phi(x,t) = 0$$

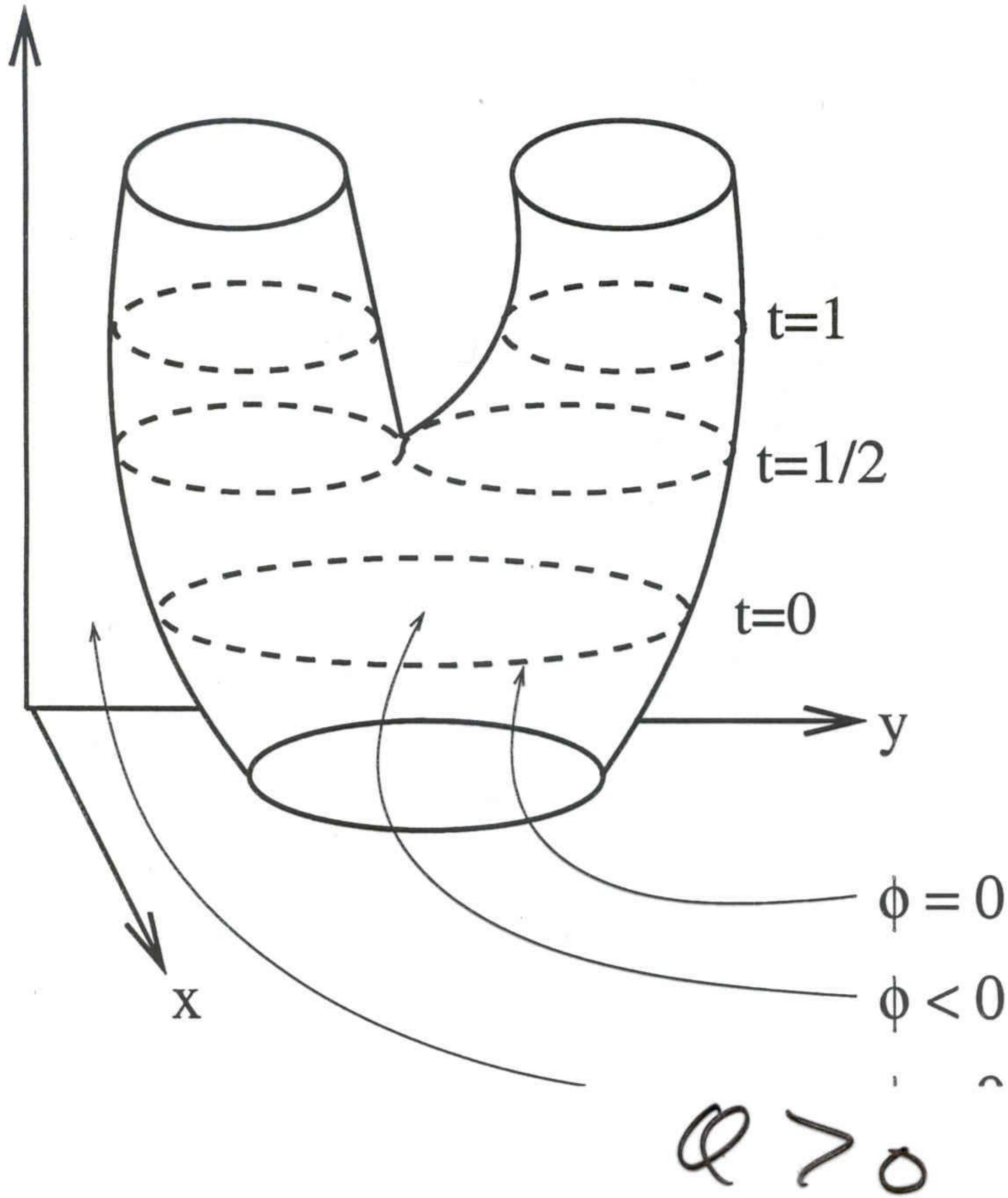
$$\frac{d}{dt}\phi(x(t),t) = 0$$

$$\varphi_t + \vec{v} \cdot \vec{\nabla}\varphi = 0$$

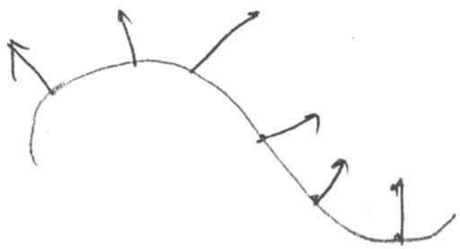
Normal to $\Gamma : \vec{n} = \dfrac{\vec{\nabla}\varphi}{|\nabla\varphi|}$

$$\varphi_t + v_n |\vec{\nabla}\varphi| = 0$$

$$v_n = \vec{v} \cdot \frac{\vec{\nabla}\varphi}{|\vec{\nabla}\varphi|}$$


t=1
t=1/2
t=0
y
x
φ = 0
φ < 0
φ > 0

Tracking:

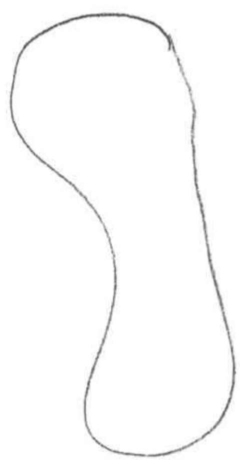


Merging is difficult

3D is difficult

Reparametrization needed

VOF



Advect $\chi(x) \equiv 1$ if $x \,\varepsilon\, \Omega$

$\equiv 0$ outside

\+ Merging ok

-- Spurious discontinuity

-- Hard to compute curvature.

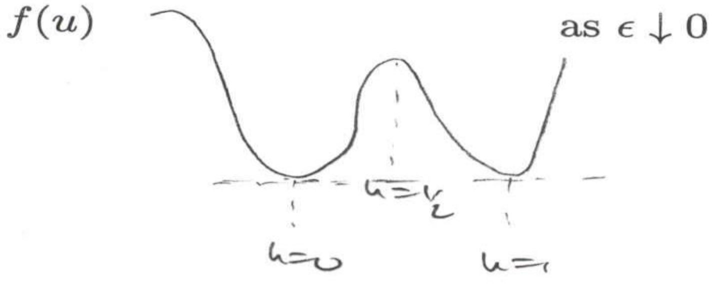
Phase field

e.g.

Mean Curvature

$$u_t = \Delta u - \frac{1}{\varepsilon} f_u$$

get curvature



interface $O(\varepsilon)$

width

But $\Delta x < \varepsilon$, otherwise

(Thm: MBO, phase field gives the wrong answer)

Need adaptive grid,

NO ε in our approach.

(1) Reinitialize

$\varphi \rightarrow$ signed distance to Γ (SSO).

(2) $v_n \rightarrow$ extends smoothly off of Γ (CMOS).

(3) Local level set (near interface) $|\varphi| < \varepsilon$.

Easy to implement

Near boundary singularities, 2 or 3D.

Also

$$v_n = v_n(\vec{n}), \qquad v_n(\vec{n}) = \gamma\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right)$$

$$\varphi_t + |\nabla\varphi|\,\gamma\left(\frac{\nabla\varphi}{|\nabla\varphi|}\right) = 0$$

High order accurate ENO schemes for HJ equations

(Kinks develop)

[OSe] [OSh]

Theoretical Justification

Viscosity solutions for scalar 2^{nd} order (or 1^{st} order)

Evolution eqns.

Motion by mean curvature e.g.

$$\varphi_t = |\nabla \varphi| \nabla \cdot \frac{\nabla \varphi}{|\nabla \varphi|}$$

ESS showed same as classical limit

$$u_t = \Delta u - \frac{1}{\varepsilon} f_u$$

$f =$ 

as $\varepsilon \downarrow 0$

Got e.g. motion of square by mean curvature.

Level Set Dictionary

1. $$\Gamma(t)\{x \mid \varphi(x,t)=0\}$$

$$\Omega(t) bdd \quad \text{by} \quad \Gamma(t)$$

$$\Omega(t)=\{x \mid \varphi(x,t)<0\}$$

2. Unit normal

$$\vec{n}=+\frac{\vec{\nabla}\varphi}{|\vec{\nabla}\varphi|}$$

3. Mean curvature

$$\kappa=\nabla\cdot\left(\frac{\vec{\nabla}\varphi}{|\vec{\nabla}\varphi|}\right)$$

4. Delta function on an interface

$$\delta(\varphi)\,|\nabla\varphi|$$

5. Characteristic function χ of Ω(*t*)

$$\chi = H(-\varphi)$$

$$H(x) \equiv 1 \quad \text{if} \quad x > 0$$

$$H(x) \equiv 0 \quad \text{if} \quad x < 0$$

6. Surface integral of *p*(*x*,*t*) over Γ

$$\int_{R^n} p(x,t)\,\delta(\varphi)\,|\nabla\varphi|\,dx$$

7. Volume integral of $p(x,t)$ over Ω

$$\int p(x,t)H(-\varphi)dx$$

8. Distance reinitialization $d(x,t)$ = signed distance to nearest point on Γ

$$|\nabla d| = 1 \quad d < 0 \quad \text{in} \quad \Omega, \quad d > 0 \quad \text{in} \quad \Omega^c$$

$$d = 0 \quad \text{on} \quad \Gamma$$

$$\frac{\partial \psi}{\partial \tau} + \text{sign}\,\varphi(|\nabla \psi| - 1) = 0$$

as $\tau \uparrow$ $\psi \to d$ <u>*very*</u> fast near $d = 0$.

9. Smooth extension of a quantity e.g. v_n on Γ, off of Γ.

Let $v_n = p(x,t)$

$$\frac{\partial q}{\partial \tau} + (\operatorname{sign} \varphi)\left(\frac{\nabla \varphi}{|\nabla \varphi|} \cdot \nabla q \right) = 0$$

$$q(x,0) = p(x,t)$$

very fast near $d = 0$.

10. Local level set method.

Solve PDE within $6\Delta x$ or so of $d = 0$.

11. Fast marching method: Tsitsiklis (1993)

Rediscovered by (1995): Helmsen P.C.D.,…, & Sethian

$$\varphi_t + v_n(\vec{x}) \,|\, \nabla \varphi \,| = 0$$
$$v_n > 0$$

Use heap-sort, Godunov's Hamiltonian (upwind, viscosity soln)

Solve in $O(N \log N)$

(First order accurate), jazzed up hyperbolic space Marching.

For this problem, probably fastest.

Although local level set more general & accurate.

For more complicated Hamiltonians

$$H(\vec{x}, \nabla\varphi) = c(\vec{x}) > 0$$

H convex in grad phi

Can do a simple *local* update $\begin{matrix} & 0 & \\ 0 & x & 0 \\ & 0 & \end{matrix}$ using a new

Formula of Tsai, et. al. (2001)

Sweep in pre-ordained directions. Converges rapidly. No heap sort. No large search and initialization regions.

Zhao: “convergence theorem” in special cases.

Now, with Kao, Jiang & O, can do a very simple sweeping method in very general cases.