IPAM Graduate Summer School: Computer Vision, July 2013

Inverse modelling using optimization

to solve imaging tasks

Part III

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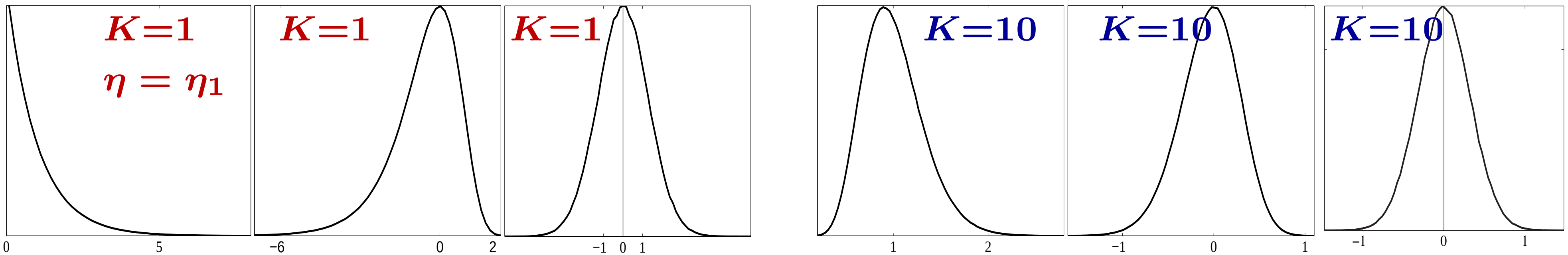
Multiplicative noise removal on Frame coefficients [Durand, Fadili, Nikolova 09]

Multiplicative noise arises in various active imaging systems e.g. synthetic aperture radar

- Original image: S_o
- One shot: $\Sigma_k = S_o \eta_k$
- Data: $\Sigma = \frac{1}{K}\sum_{k=1}^{K} \Sigma_k = S_o \frac{1}{K}\sum_{k=1}^{K} \eta_k = S_o\, \eta$ where $\mathrm{pdf}(\eta) =$ Gamma density
- Log-data: $v = \log \Sigma = \log S_o + \log \eta = u_0 + n$
- Frame Coefficients: $y = Wv = Wu_0 + Wn$ (W curvelets)



$\mathrm{pdf}(\eta) = \mathrm{pdf}(\eta_k)$ $\mathrm{pdf}(n)$ $\mathrm{pdf}\big(Wn\big)$ $\mathrm{pdf}(\eta)$ $\mathrm{pdf}(n)$ $\mathrm{pdf}\big(Wn\big)$

Question **39** Please comment the noise distribution of Wn

- Hard Thresholding: $y_T[i] = \begin{cases} 0 & \text{if } |y[i]| \leqslant T, \\ y[i] & \text{otherwise} \end{cases} \quad \forall i \in I, \quad T > 0$ (suboptimal).

$$I_1 = \{i \in I \,:\, |y[i]| > T\} \text{ and } I_0 = I \setminus I_1$$

- **Restored coefficients:** $\hat{x} = \arg\min_{x} \mathcal{F}_y(x)$ ($\ell_1 - \mathrm{TV}$ energy)

$$\mathcal{F}_y(x) = \lambda_0 \sum_{i \in I_0} \big|x[i]\big| + \lambda_1 \sum_{i \in I_1} \big|x[i] - y[i]\big| + \|\widetilde{W} x\|_{\mathrm{TV}}$$

$\hat{S} = B \exp\left(\widetilde{W}\hat{x}\right)$, where $\widetilde{W}$ left inverse, B bias correction

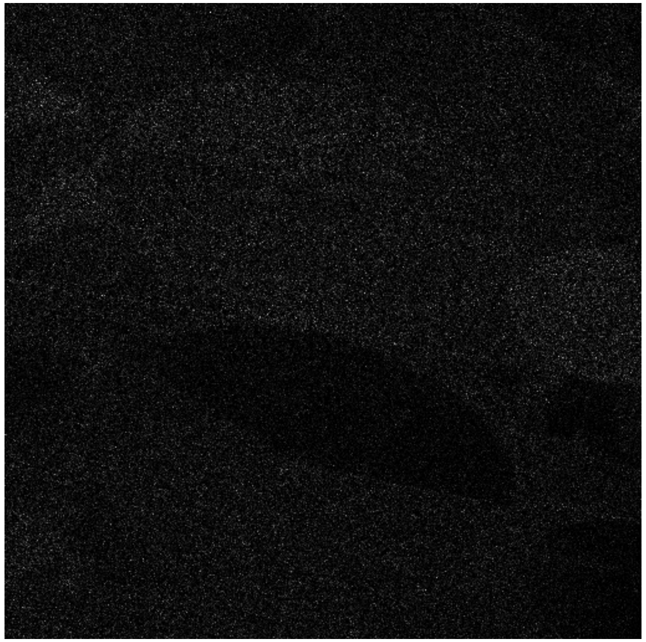
Question **40** Explain the job the minimizer $\hat{x}$ of $\mathcal{F}_y$ should do.

Question **41** What is the difference with the model on pp. 35-36 and why it is needed?

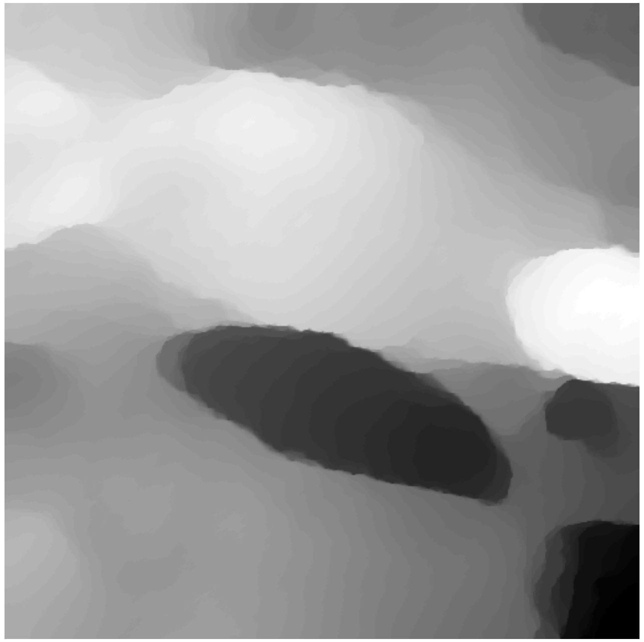
Some comparisons

- **BS** [Chesneau,Fadili,Starck 08]: Block-Stein thresholds the curvelet coefficients, $\approx$ minimax(large class of images with additive noises), optimal threshold $\mathfrak{T} = 4.50524$
- **AA** [Aubert,Aujol 08]: $\Psi = -$ Log-Likelihood(Σ), $\Phi = \mathrm{TV}(\Sigma)$ (i.e. $\mathcal{F}_v \equiv$ MAP for Σ)
- **SO** [Shi,Osher 08]: relaxed inverse scale-space for $\mathcal{F}_v(u) = \|v - u\|_2^2 + \beta \mathrm{TV}(u)$ $\approx$ MAP(u)

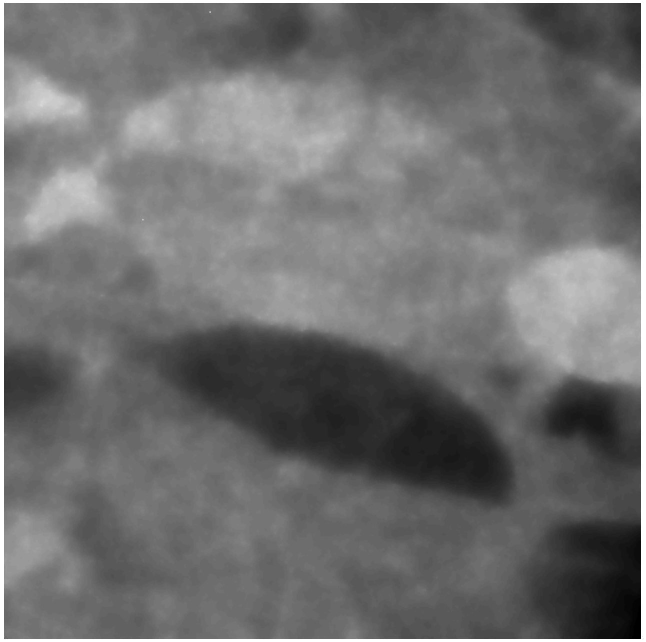
  Stopping rule: $k^* = \max\{k \in \mathrm{N} : \mathrm{Var}(u^{(k)} - u_o) \geqslant \mathrm{Var}(n)\}$.



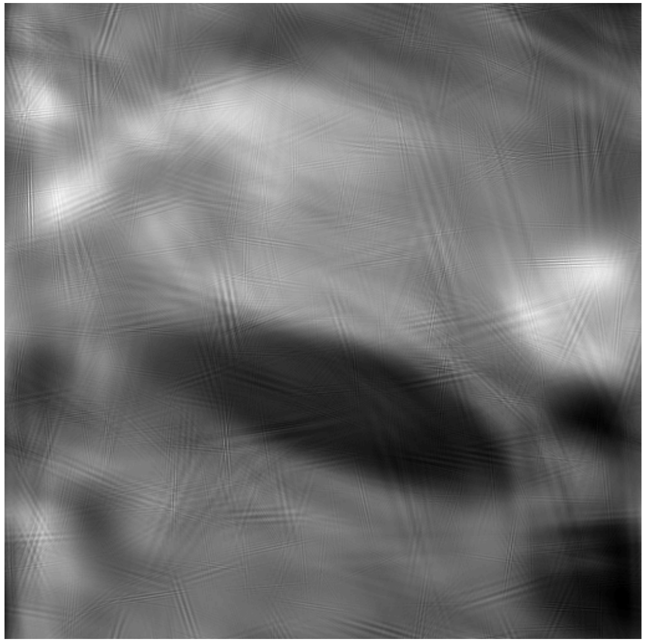
Noisy Fields $K = 1$ (512×512)



SO: PSNR=9.59, MAE=196

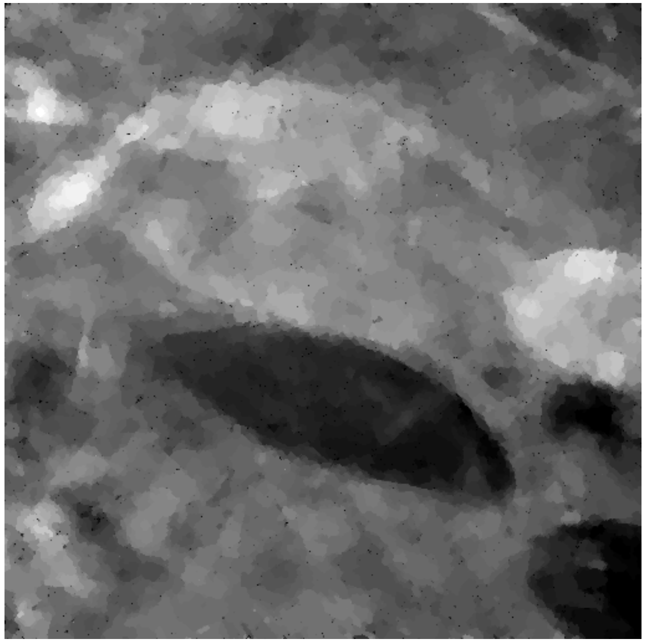


AA: PSNR=15.74, MAE=76.66

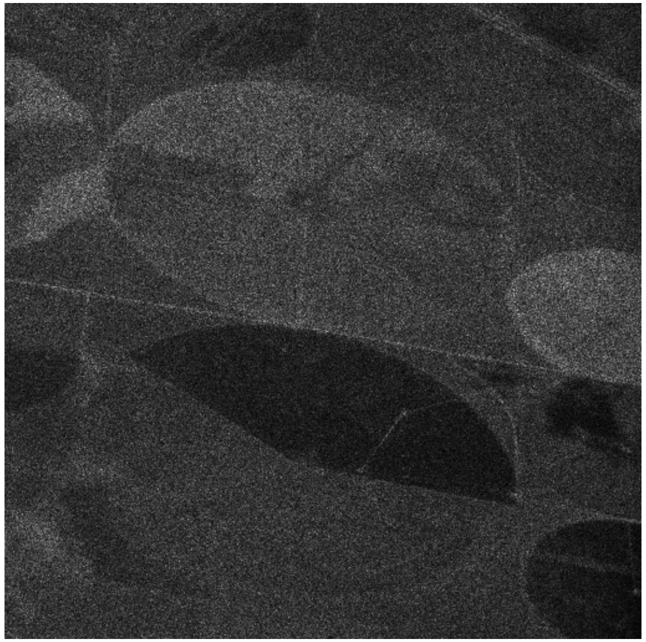


BS: PSNR=22.52, MAE=35.22

Fields (original)

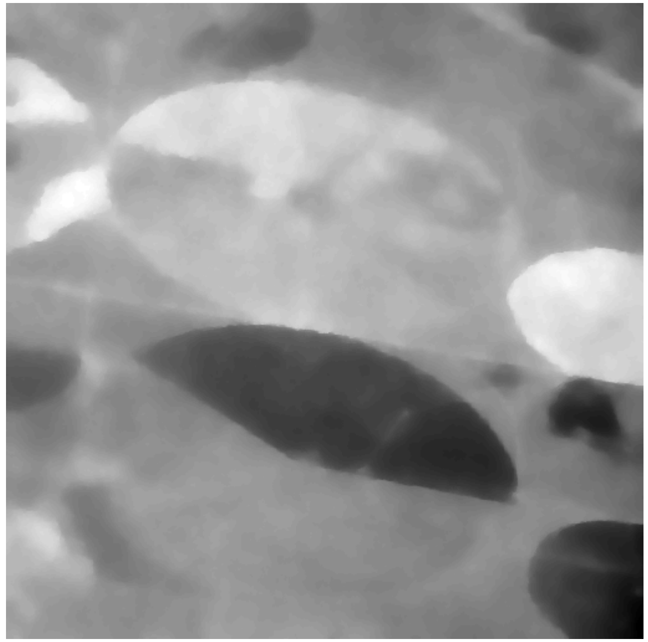


Our: PSNR=22.89, MAE=33.67

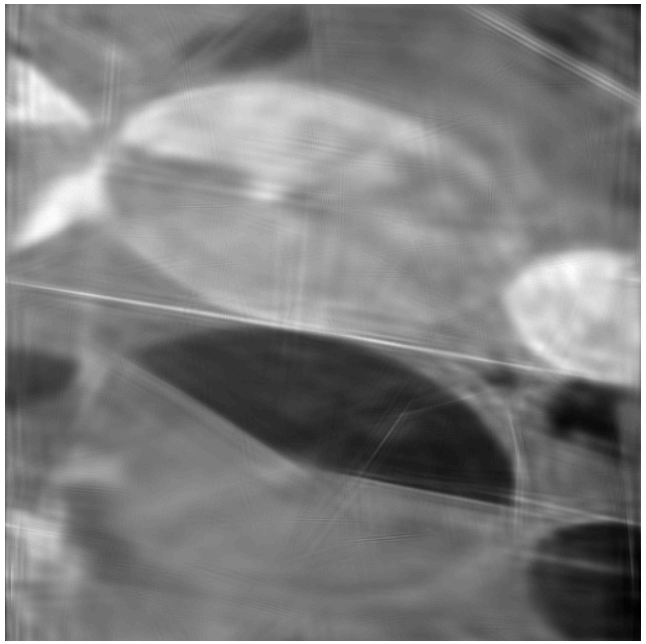


Noisy $K = 10$

SO: PSNR=25.36, MAE=25.14

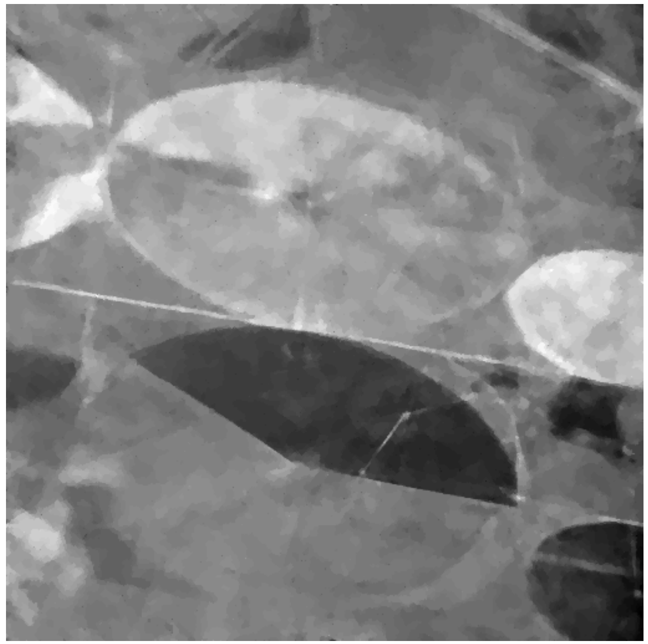


AA: PSNR=17.13, MAE=65.40



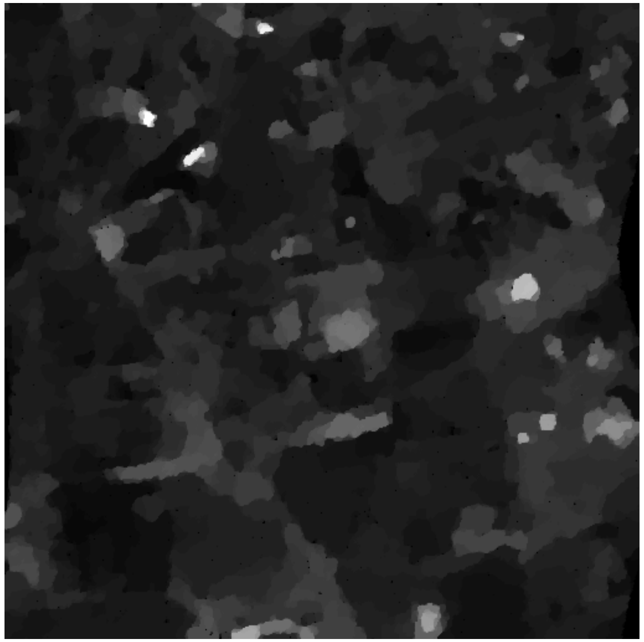
BS: PSNR=27.24, MAE=19.61

Fields (original)



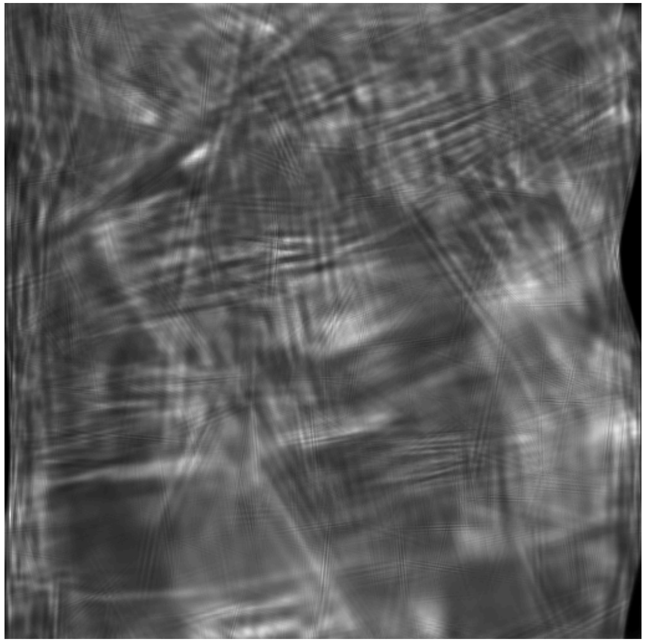
Our: PSNR=28.04, MAE=18.19

Noisy City $K = 1$ (512×512)



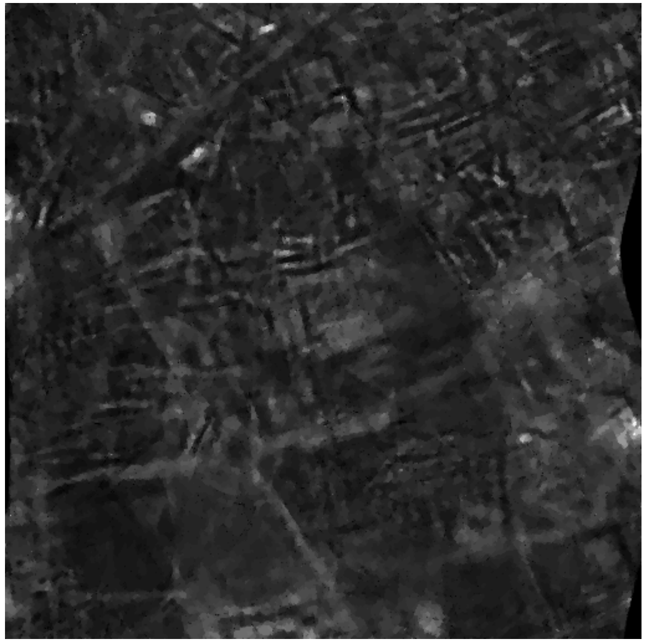
SO: PSNR=18.39, MAE=24.08

AA: PSNR=22.18, MAE=13.71



BS: PSNR=22.25, MAE=13.96

City (original)

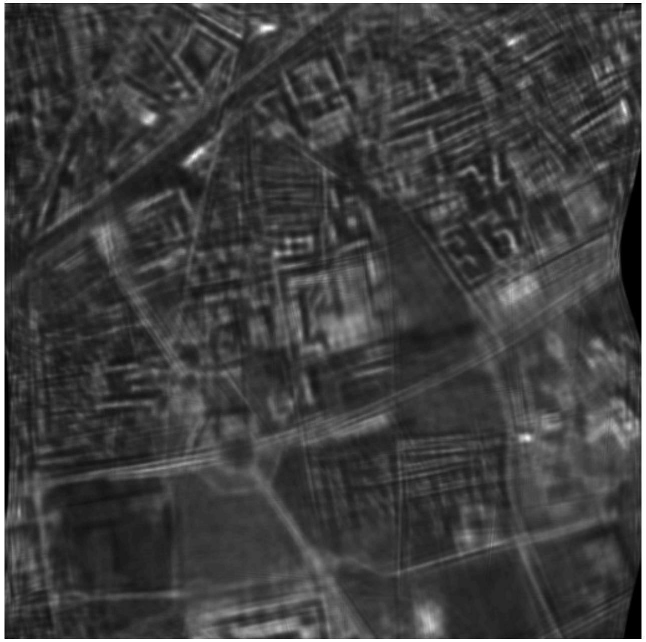


Our: PSNR=22.64, MAE=13.39

Noisy $K = 4$

SO: PSNR=24.40, MAE=10.76

AA: PSNR=24.55, MAE=10.06



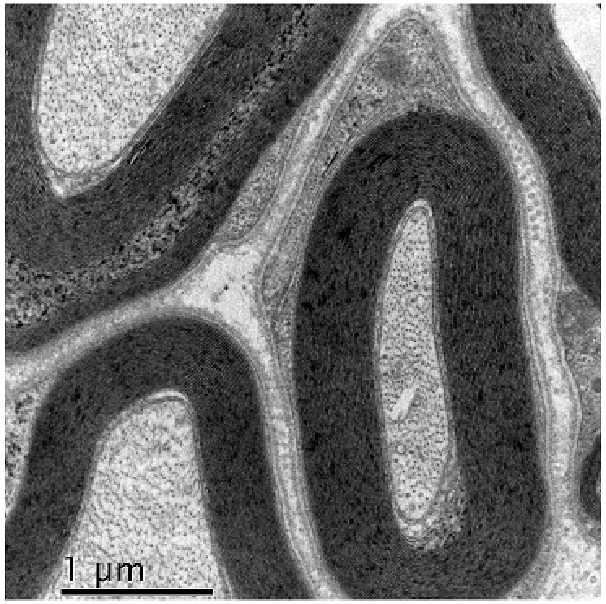
BS: PSNR=24.92, MAE=9.87

City (original)

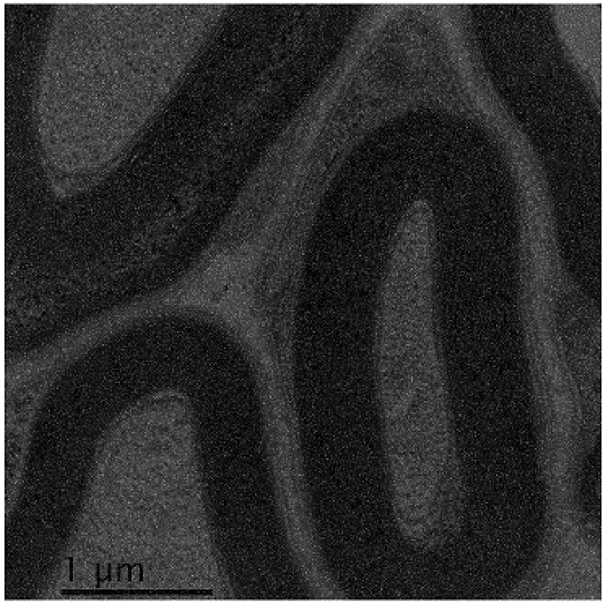
Our: PSNR=25.84, MAE=9.09

C. Clason, B. Jin, K. Kunisch
"Duality-based splitting for fast $\ell_1 - \mathrm{TV}$ image restoration", 2012,
http://math.uni-graz.at/optcon/projects/clason3/

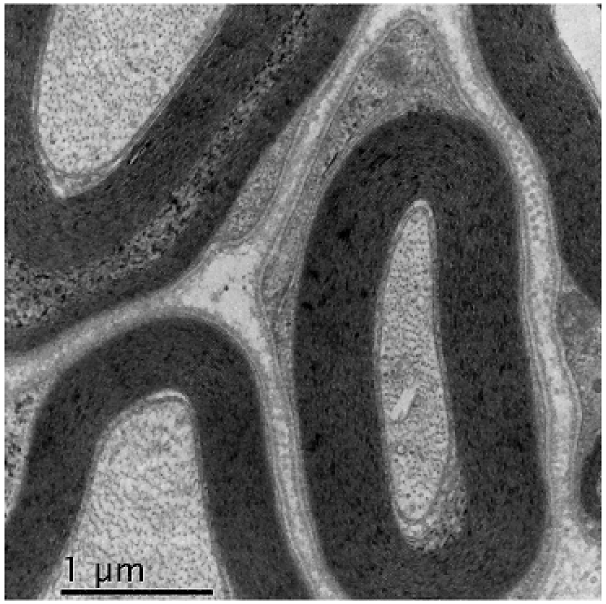
Scanning transmission electron microscopy (2048×2048 image)



true image



noisy image



restoration

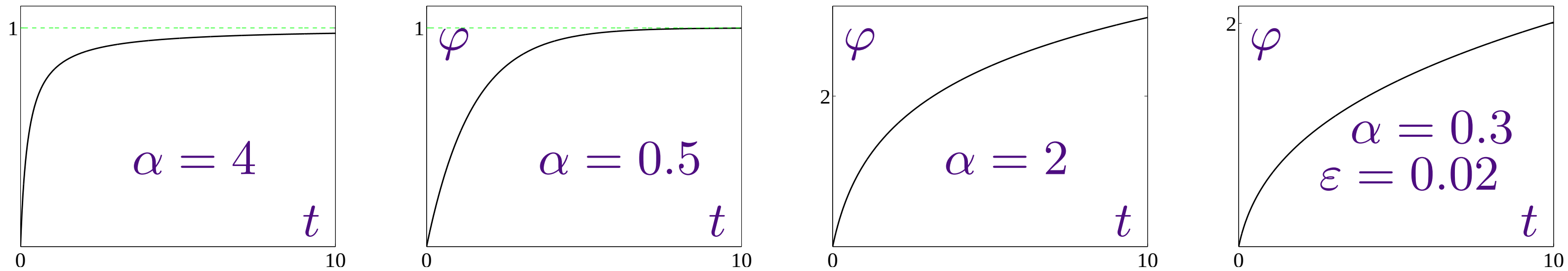
ℓ_1 data-fidelity with concave regularization [Nikolova, Ng, Tam 12]

$$\mathcal{F}_v(u) = \sum_{i\in I} \big|a_i u - v[i]\big| + \beta \sum_{j\in J} \varphi(\|G_j u\|_2), \quad \varphi'(0^+) > 0, \; \varphi''(t) < 0, \; \forall t \geqslant 0$$

$$I = \{1, \cdots, q\}, \quad J = \{1, \cdots, r\}$$

φ is strictly concave on $[0, +\infty)$.

$$\varphi(t) \;\Big\|\; \frac{\alpha\, t}{\alpha\, t + 1} \;\Big|\; 1 - \alpha^t, \; \alpha \in (0,1) \;\Big|\; \ln(\alpha t + 1) \;\Big|\; (t+\varepsilon)^\alpha, \; \alpha \in (0,1), \; \varepsilon > 0 \;\Big|\; (\cdots)$$

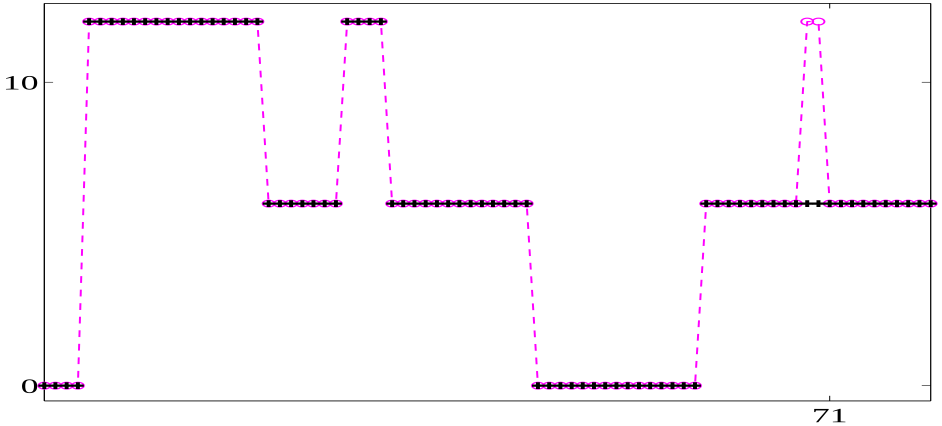


Motivation

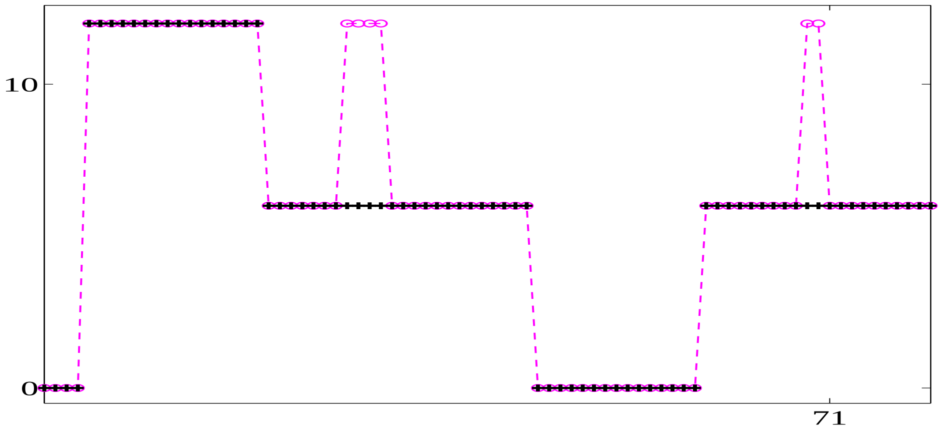
- This family of objective functions has never been considered before
- $\mathcal{F}_v$ can be seen as an extension of $L1 - \mathrm{TV}$
- $\hat{u}$—(local) minimizer of $\mathcal{F}_v$ $\overset{?}{\Longrightarrow}$ many i, j such that $a_i \hat{u} = v[i]$ and $G_j \hat{u} = 0$

Minimizers of $\mathcal{F}_v(u) = \|u - v\|_1 + \beta \sum_{i=1}^{p-1} \varphi(|u[i+1] - u[i]|)$

$\varphi(t) = \frac{\alpha t}{\alpha t + 1}$ for $\alpha = 4$

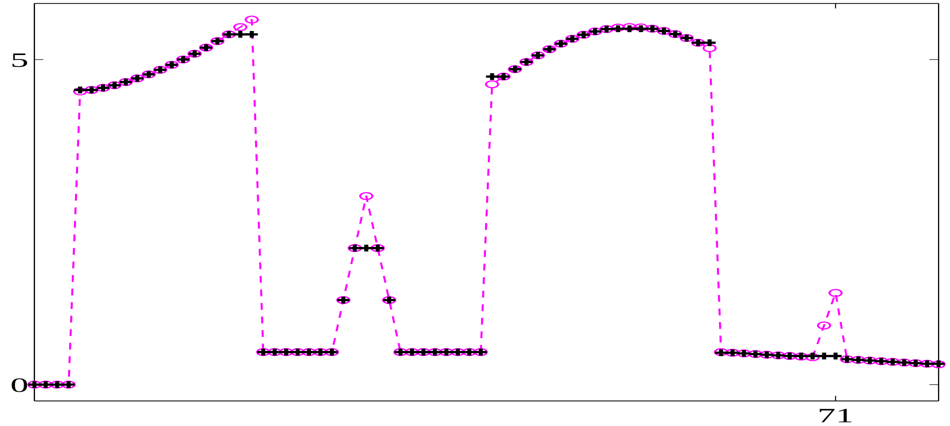


$\beta \in \{78, \cdots, 156\}$

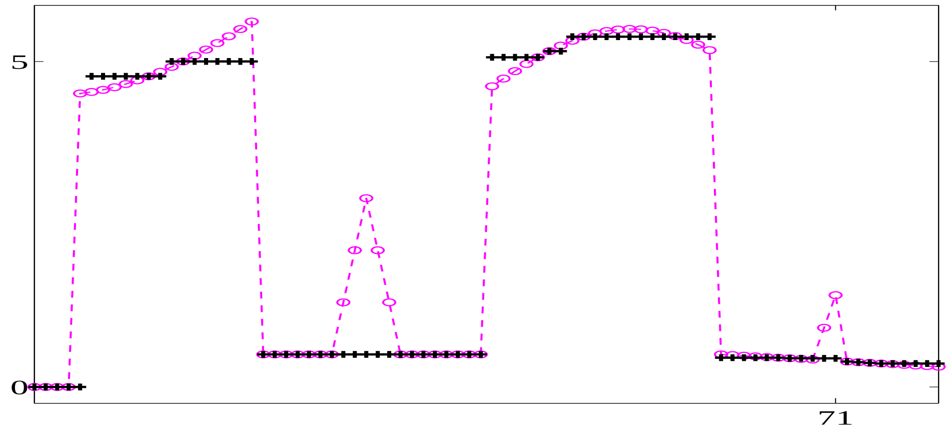


$\beta \in \{157, \cdots, 400\}$

$\varphi(t) = \ln(\alpha t + 1)$ for $\alpha = 2$

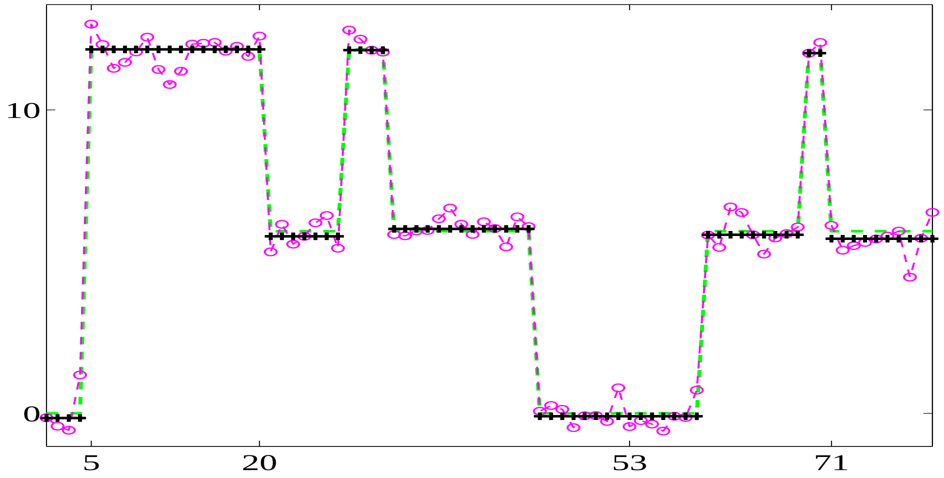


$\beta \in 0.1 \times \{10, \cdots, 14\}$

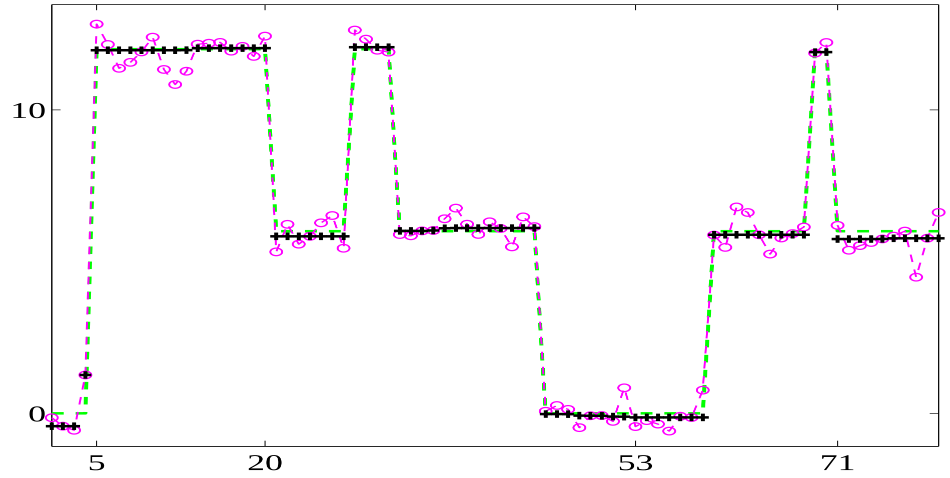


$\beta \in 0.1 \times \{16, \cdots, 30\}$

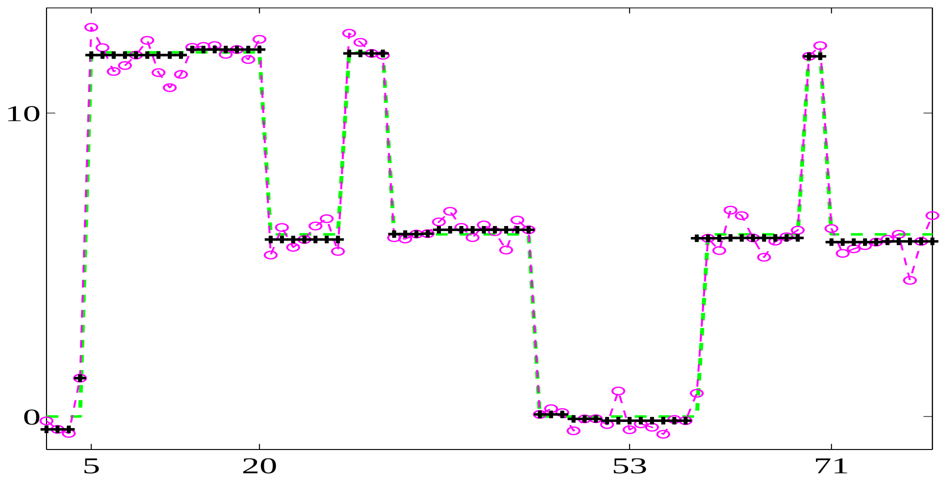
Data samples (ooo), Minimizer samples $\hat{u}[i]$ (+++).



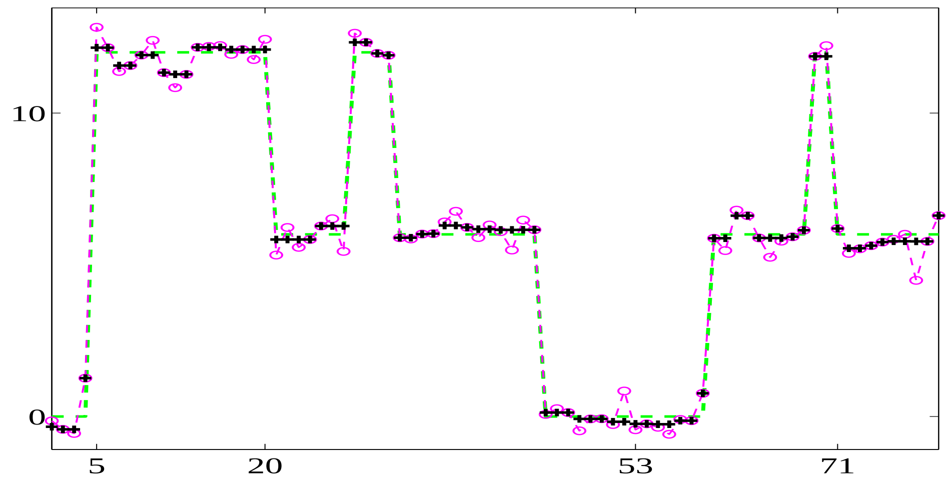
(a) $\varphi(t) = \frac{\alpha\, t}{\alpha\, t+1}$, $\alpha = 4$, $\beta = 3$



(b) $\varphi(t) = 1 - \alpha^t$, $\alpha = 0.1$, $\beta = 2.5$



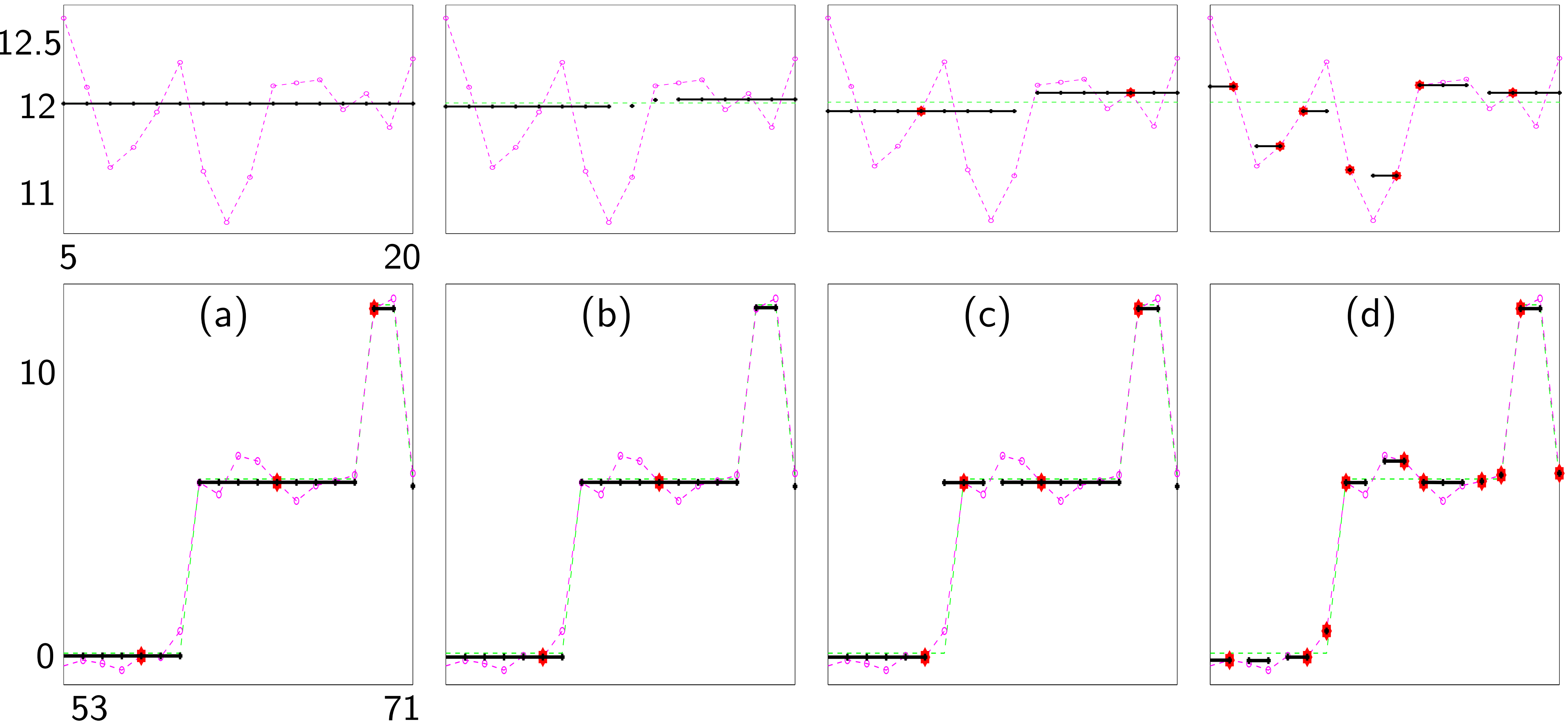
(c) $\varphi(t) = \ln(\alpha t + 1)$, $\alpha = 2$, $\beta = 1.3$



(d) $\varphi(t) = (t + 0.1)^{\alpha}$, $\alpha = 0.5$, $\beta = 1.4$

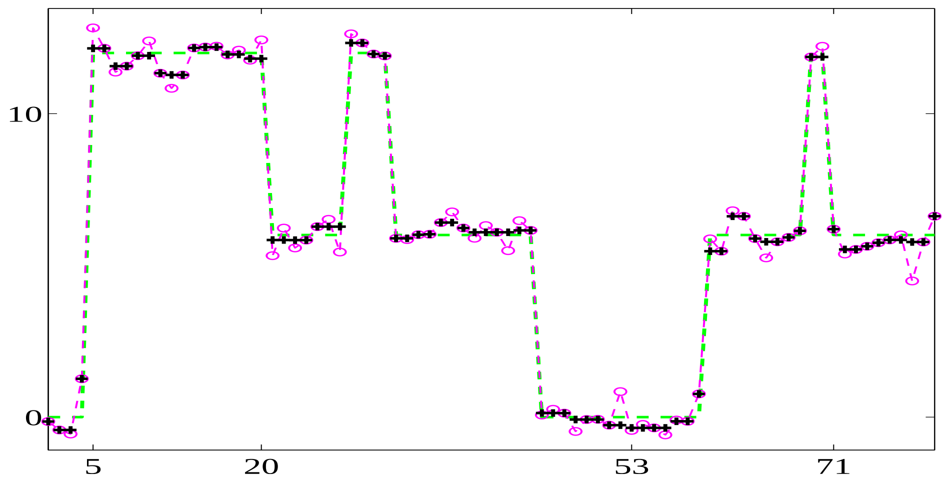
Denoising: Data samples (ooo) are corrupted with Gaussian noise. Minimizer samples $\hat{u}[i]$ (+++). Original (– – –). $\boldsymbol{\beta}$—the <u>largest</u> value so that the gate at 71 survives.

Zooms



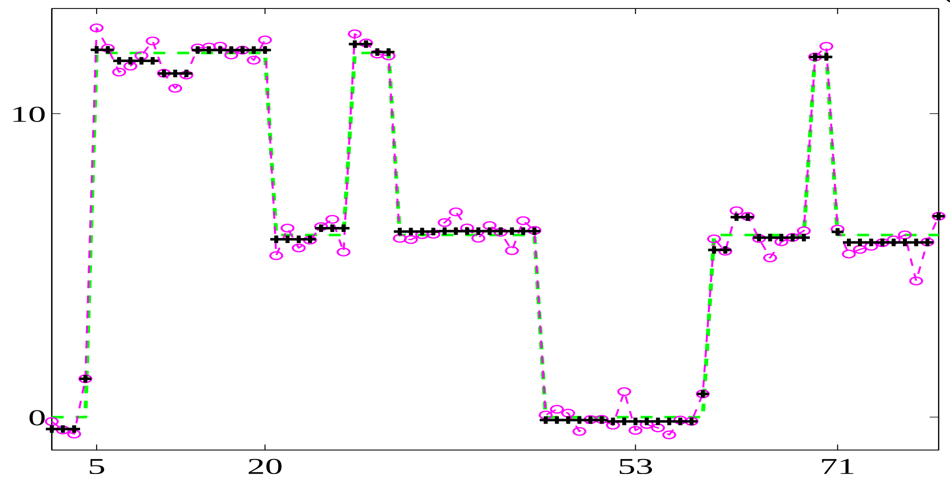
Constant pieces—solid black line.

Data points $v[i]$ fitted exactly by the minimizer $\hat{u}$ (♦).



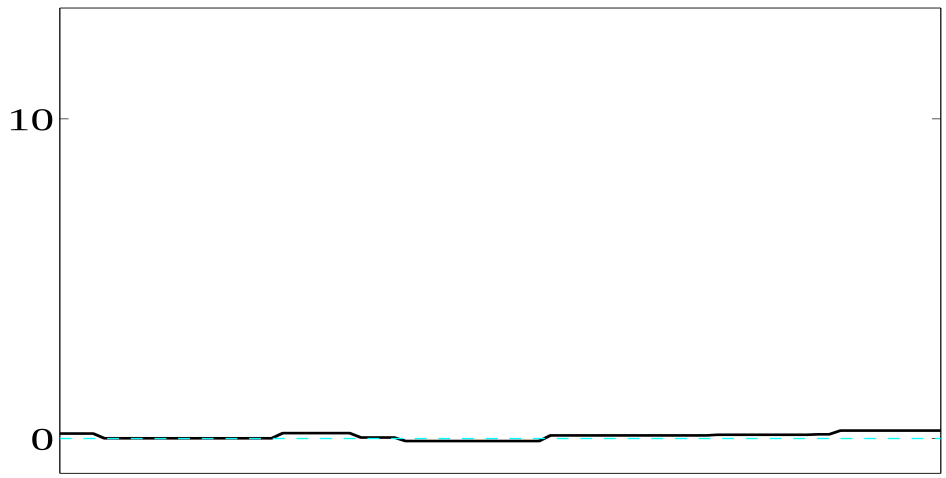
$\varphi(t) = t$, $\beta = 0.8 \quad (\ell_1 - \mathrm{TV})$

the convex relaxation of $\mathcal{F}_v$



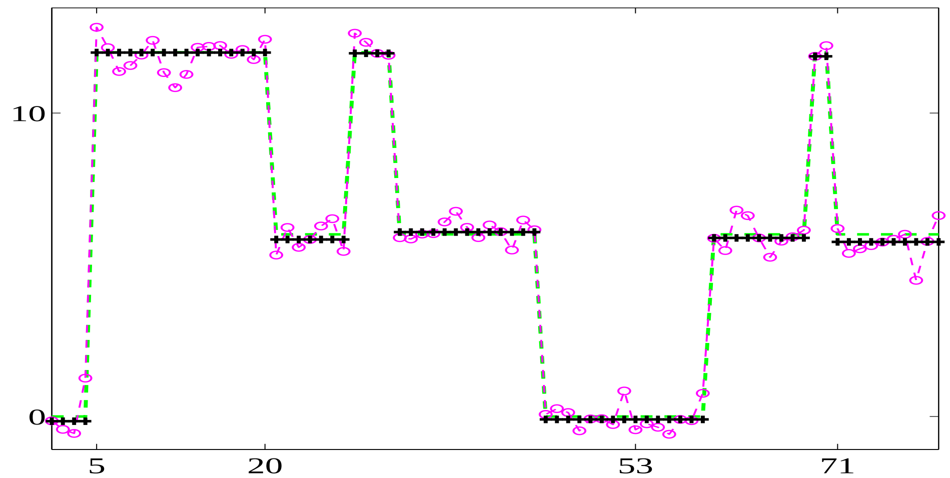
the minimizer for $\varphi(t) = \frac{\alpha\, t}{\alpha\, t+1}$, $\alpha = 4$, $\beta = 3$

closest to $(\ell_1 - \mathrm{TV})$



error for $\varphi(t) = \frac{\alpha\, t}{\alpha\, t+1}$, $\alpha = 4$, $\beta = 3$

$\|\text{original} - \hat{\boldsymbol{u}}\|_\infty = 0.24$



$\varphi(t) = \frac{\alpha\, t}{\alpha\, t+1}$, $\alpha = 4$, $\beta = 3$

original $\in [0, 12]$, data $v \in [-0.6, 12.9]$

On the figures, $\hat{u}$ are global minimizers of $\mathcal{F}_v$ (Viterbi algorithm)

Question **42** Can you sketch the main properties of the minimizers of $\mathcal{F}_v$?

Question **43** What seems being the role of the asymptotic of φ?

Numerical evidence:

critical values $\beta_1, \cdots, \beta_n$ such that

- $\beta \in [\beta_i, \beta_{i+1}) \Rightarrow$ the minimizer remains unchanged
- $\beta \geqslant \beta_{i+1} \quad \Rightarrow$ the minimizer is simplified

Result proven (under conditions) for the minimizers of $L_1 - \mathrm{TV}$ [Chan, Esedoglu 2005]

Given $v \in \mathbb{R}$ consider the function

$$\mathcal{F}_v(u) = |u - v| + \beta\varphi(|u|) \ \text{ for } \ \varphi(u) = \frac{\alpha u}{1 + \alpha u} \quad u \in \mathbb{R}, \quad \beta > 0$$

Question **44** Does $\mathcal{F}_v$ have a global minimizer for any v? Explain.

Question **45** Determine $\varphi''(u)$ for $u \in \mathbb{R} \setminus \{0\}$.

Question **46** Show that $\forall\, v \in \mathbb{R}$, any minimizer $\hat{u}$ of $\mathcal{F}_v$ obeys $\hat{u} \in \{0, v\}$.

The reminder on p. 50 can help.

Question **47** Can you extend this result to the other φ on p. 62?

- $\mathcal{F}_v$ does have global minimizers, for any $\{a_i\}$, for any v and for any $\beta > 0$.

- Let $\hat{u}$ be a (local) minimizer of $\mathcal{F}_v$. Set

$$\begin{aligned} \widehat{I}_0 &= \{i \in I \; : \; a_i\hat{u} = v[i]\} \\ \widehat{J}_0 &= \{j \in J \; : \; G_j\hat{u} = 0\} \end{aligned}$$

$\hat{u}$ is the **unique** point solving the liner system

$$\begin{cases} a_i\hat{u} = v[i] & \forall i \in \widehat{I}_0 \\ G_j\hat{u} = 0 & \forall j \in \widehat{J}_0 \end{cases}$$

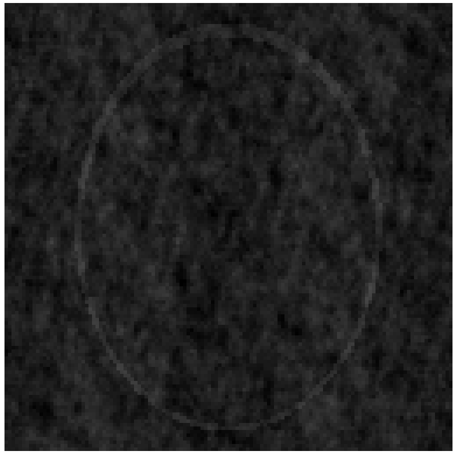
Each pixel of a (local) minimizer $\hat{u}$ of $\mathcal{F}_v$ is involved in (at least) one equation $a_i\hat{u} = v[i]$, or in (at least) one equation $G_j\hat{u} = 0$, or in both types of equations.

- "Contrast invariance" of (local) minimizers

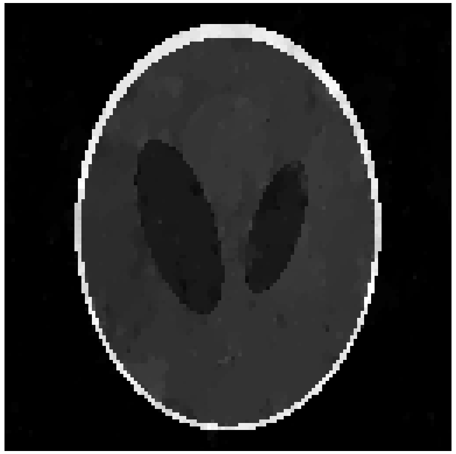
- The matrix with rows $\left(a_i, \forall i \in \widehat{I}_0, \;\; G_j, \forall j \in \widehat{J}_0\right)$ has **full column rank**

- All (local) minimizers of $\mathcal{F}_v$ are **strict**

MR Image Reconstruction from Highly Undersampled Data



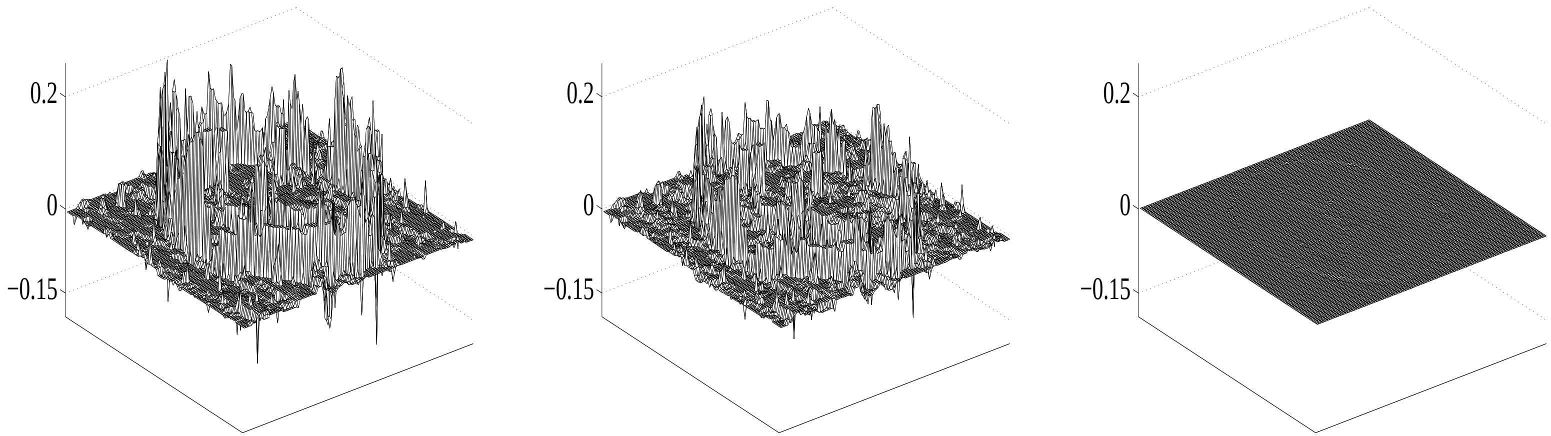
0-filling Fourier



$\|\cdot\|_2^2$+TV

$\|\cdot\|_1$+TV

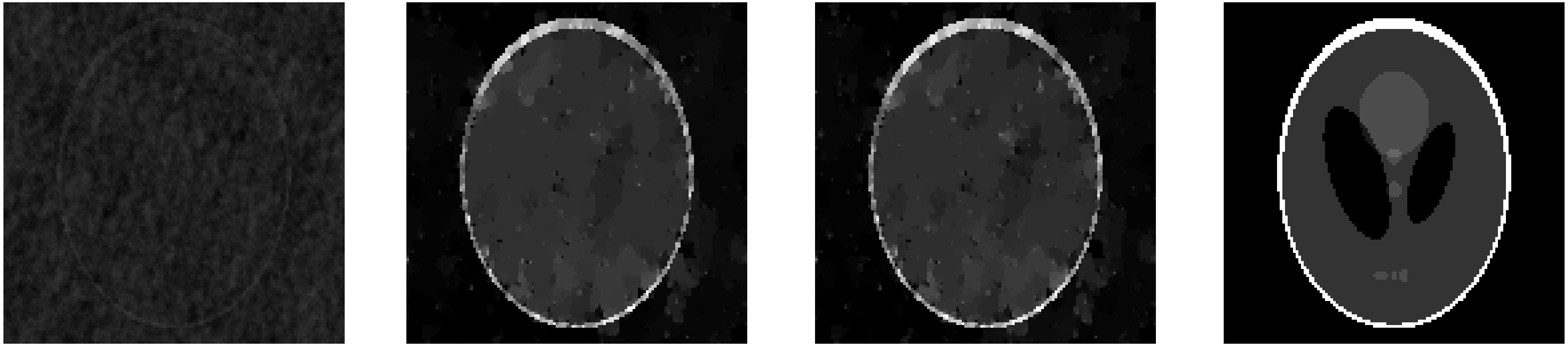
Our method



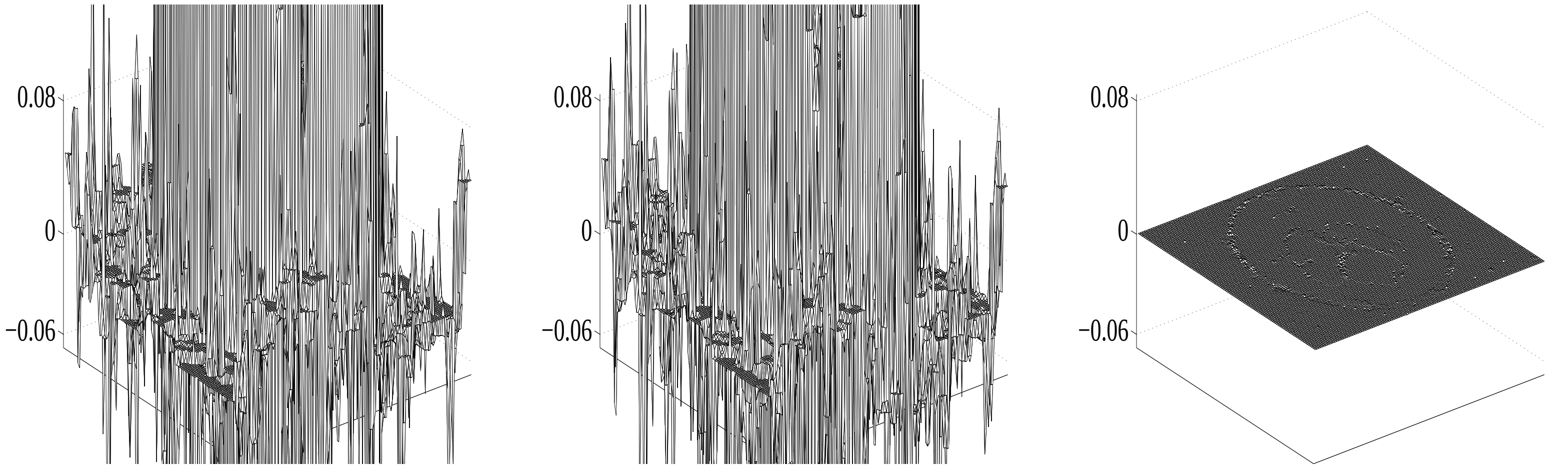
Reconstructed images from **7% noisy** randomly selected samples in the k-space.

Our method for $\varphi(t) = \dfrac{\alpha t}{\alpha t + 1}$.

MR Image Reconstruction from Highly Undersampled Data



0-filling Fourier | $\|\cdot\|_2^2$+TV | $\|\cdot\|_1$+TV | Our method



Reconstructed images from **5% noisy** randomly selected samples in the k-space.

Our method for $\varphi(t) = \dfrac{\alpha t}{\alpha t + 1}$.

Cartoon

Observed

ℓ_1-TV

Our method, $\varphi(t) = \frac{\alpha t}{\alpha t + 1}$

7. Fully smoothed ℓ_1 – TV

$$\mathcal{F}_v(u) = \Psi(u,v) + \beta\Phi(u), \quad \beta > 0$$

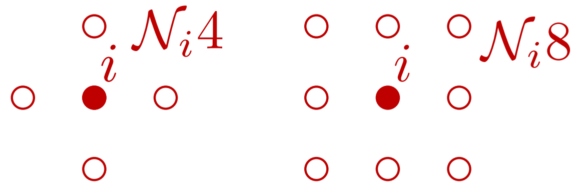
$$\Psi(u,v) = \sum_{i=1}^{p} \psi(u[i] - v[i]) \quad \text{and} \quad \Phi(u) = \sum_i \varphi(|G_i u|)$$

$\psi(\cdot) \doteq \psi(\cdot, \alpha_1)$

$\varphi(\cdot) \doteq \varphi(\cdot, \alpha_2)$

$(\alpha_1, \alpha_2) > 0$

$\{G_i \in \mathbb{R}^{1\times p}\}$ – forward discretization:



$\mathcal{N}4$ Only vertical and horizontal differences;

$\mathcal{N}8$ Diagonal differences are added.

(ψ, φ) belong to the family of functions $\theta(\cdot, \alpha) : \mathbb{R} \to \mathbb{R}$ satisfying

H 1 For any $\alpha > 0$ fixed, $\theta(\cdot, \alpha)$ is $\mathcal{C}^{s\geqslant 2}$-continuous, even and $\theta''(t,\alpha) > 0$, $\forall\, t \in \mathbb{R}$.

H 2 For any $\alpha > 0$ fixed, $|\theta'(t,\alpha)| < 1$ and for $t > 0$ fixed, it is strictly decreasing in $\alpha > 0$

$$\alpha > 0 \quad \Rightarrow \quad \lim_{t\to\infty} \theta'(t,\alpha) = 1$$

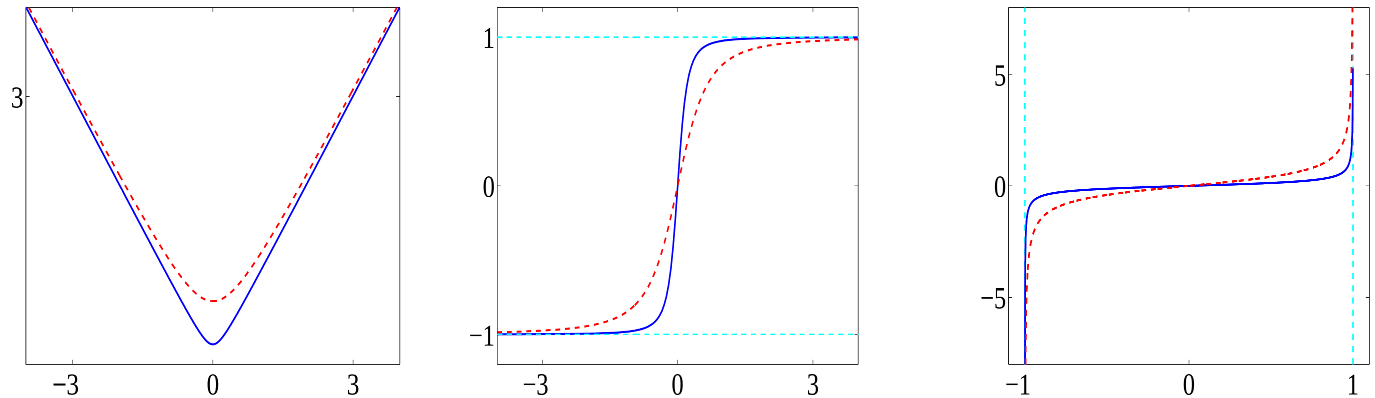
$$\theta'(t,\alpha) \doteq \frac{d}{dt}\theta(t,\alpha)$$

$$t \in \mathbb{R} \quad \Rightarrow \quad \lim_{\alpha\to 0} \theta'(t,\alpha) = 1 \quad \text{and} \quad \lim_{\alpha\to\infty} \theta'(t,\alpha) = 0\ .$$

$\Rightarrow$ $\mathcal{F}_v$ is a fully smoothed ℓ_1 – TV energy.

	θ	θ'
f1	$\sqrt{t^2+\alpha}$	$\dfrac{t}{\sqrt{t^2+\alpha}}$
f2	$\alpha \log\left(\cosh\left(\dfrac{t}{\alpha}\right)\right)$	$\tanh\left(\dfrac{t}{\alpha}\right)$
f3	$\lvert t\rvert - \alpha \log\left(1+\dfrac{\lvert t\rvert}{\alpha}\right)$	$\dfrac{t}{\alpha+\lvert t\rvert}$

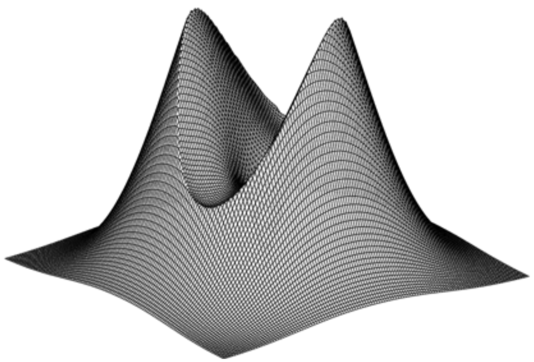
Choices for $\theta(\cdot,\alpha)$ obeying H1 and H2. When $\alpha \searrow 0$, $\theta(\cdot,\alpha)$ becomes stiff near the origin.



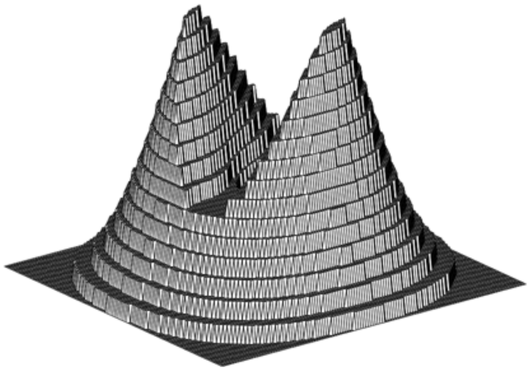
$\theta(t) = \sqrt{t^2+\alpha}$ $\qquad$ $\theta'(t) = \frac{t}{\sqrt{t^2+\alpha}}$ $\qquad$ $\left(\theta'\right)^{-1}(y) = y\sqrt{\frac{\alpha}{1-y^2}}$

Plots of f1 for $\boldsymbol{\alpha = 0.05}$ (——) and for $\boldsymbol{\alpha = 0.5}$ (− − −).

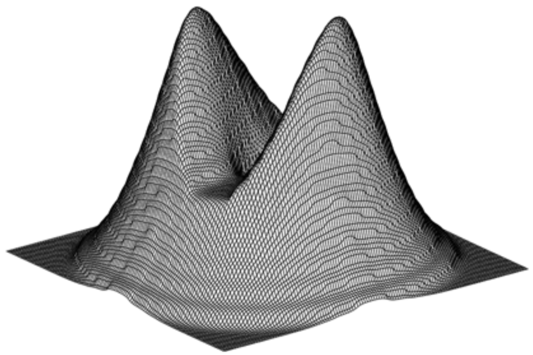
The minimizers $\hat{u}$ of $\mathcal{F}_v$ can decrease the quantization noise



Real-valued original



v quantized on $\{0, \cdots, 15\}$



Restored $\hat{u}$

[Nikolova, Wen, Chan 12]

- For any $\beta > 0$, $\mathcal{F}_v(\mathbb{R}^p)$ has a unique minimizer function $\mathcal{U} : \mathbb{R}^p \to \mathbb{R}^p$ which is $\mathcal{C}^{s-1}$.

Define $\mathcal{G} \doteq \bigcup_{i=1}^{p} \bigcup_{j=1}^{p} \left\{ g \in \mathbb{R}^{1\times p} \ : \ g[i] = -g[j] = 1, \ i \neq j, \ g[k] = 0 \text{ if } k \notin \{i, j\} \right\}$

All difference operators G_i belong to $\mathcal{G}$.

$$N_{\mathcal{G}} \doteq \bigcup_{g\in\mathcal{G}} \left\{ v \in \mathbb{R}^p \ : \ g\,\mathcal{U}(v) = 0 \right\} \quad \text{and} \quad N_I \doteq \bigcup_{i=1}^{p} \bigcup_{j=1}^{p} \left\{ v \in \mathbb{R}^p \ : \ \mathcal{U}_i(v) = v[j] \right\}$$

Question **48** How to interpret the sets $N_{\mathcal{G}}$ and N_I?

- The sets $N_{\mathcal{G}}$ and N_I are closed in $\mathbb{R}^p$ and obey

$$\mathbb{L}^p(N_{\mathcal{G}}) = 0 \quad \text{and} \quad \mathbb{L}^p(N_I) = 0$$

The property is true for any $\beta > 0$ and $(\alpha_1, \alpha_2) > 0$.

- $\mathbb{R}^p \setminus (N_{\mathcal{G}} \cup N_I)$ is open and dense in $\mathbb{R}^p$.

  The elements of $(N_{\mathcal{G}} \cup N_I)$ are highly exceptional in $\mathbb{R}^p$.

- The minimizers $\hat{u}$ of $\mathcal{F}_v$ generically satisfy $\hat{u}[i] \neq \hat{u}[j]$ for any (i, j) such that $i \neq j$ and $\hat{u}[i] \neq v[j]$ for any (i, j).

The minimizers $\hat{u}$ of $\mathcal{F}_v$ have pixel values that are different from each other and different from any data pixel.

Question **49** Describe the precise consequences if $\ell_1 - \mathrm{TV}$ is approximated by a smooth function like $\mathcal{F}_v$.

Recall the illustration on p. 21 and the results in section 3 (p. 22) and section 4 (p. 29).

Further... [Bauss, Nikolova, Steidl 13]

- For any $\alpha_1 > 0$ fixed, there is an inverse function $\boldsymbol{(\psi')^{-1}(\cdot,\alpha_1)} : (-1,1) \to \mathbb{R}$ which is odd, $\mathcal{C}^{s-1}$ and strictly increasing.

  $\alpha_1 \mapsto (\psi')^{-1}(y,\alpha_1)$ is also strictly increasing on $(0,+\infty)$, for any $y \in (0,1)$.

- Set $\boldsymbol{\eta} := \|\boldsymbol{G}\|_1$. Then

$$\boldsymbol{\beta\eta < 1 \quad \Rightarrow \quad \|\hat{u} - v\|_\infty \leqslant (\psi')^{-1}(\beta\eta, \alpha_1) \quad \forall\, v \in \mathbb{R}^p}$$

- Also, $\boldsymbol{\|\hat{u} - v\|_\infty \nearrow (\psi')^{-1}(\beta\eta,\alpha_1)}$ as $\boldsymbol{\alpha_2 \searrow 0}$.

We have a full control on the bound $\|\hat{u} - v\|_\infty$.

Question **50** Can you suggest applications where the properties of $\mathcal{F}_v$ are important?

Exact histogram specification

- v – input digital gray value $m \times n$ image / stored as an $p \doteq mn$ vector
- $v[i] \in \{0, \cdots, L-1\} \quad \forall\, i \in \{1, \cdots, p\}$ 8-bit image $\Rightarrow \quad L = 256$
- Histogram of v: $H_v[k] = \frac{1}{p}\#\big\{v[i] = k \;:\; i \in \{1, \cdots, p\}\big\} \quad \forall\, k \in \{0, \cdots, L-1\}$
- Target histogram: $\zeta = (\zeta[1], \cdots, \zeta[L])$
- Goal of histogram specification (HS): convert v into $\hat{u}$ so that $H_{\hat{u}} = \zeta$

  order the pixels in v: $i \prec j$ if $v[i] < v[j]$

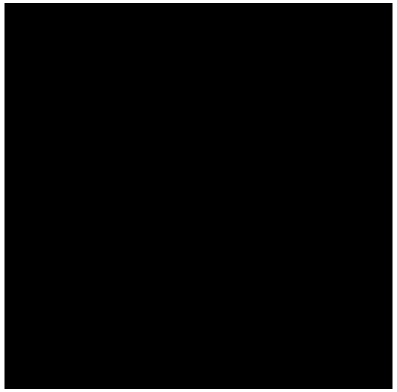
$$\underbrace{i_1 \prec i_2 \prec \cdots \prec i_{\zeta[1]}}_{\zeta[1]} \prec \cdots \prec \underbrace{i_{p-\zeta[L]+1} \prec \cdots \prec i_p}_{\zeta[L-1]}$$

- Ill-posed problem for digital (quantized) images since $p \gg L$
- An issue: obtain a meaningful total strict ordering of all pixels in v

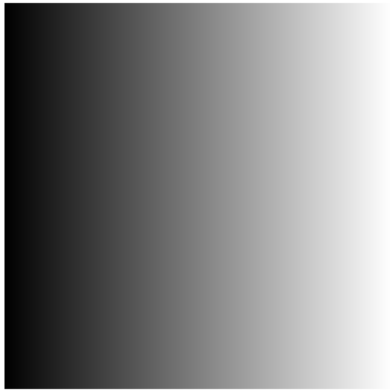
Histogram equalization is a particular case of HS where $\zeta[k] = p/L \quad \forall\, k \in \{0, \cdots L-1\}$

Histogram Equalization using Matlab sorting

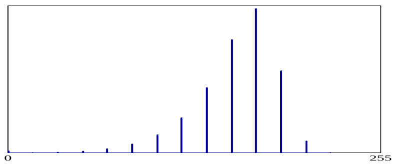
Original black



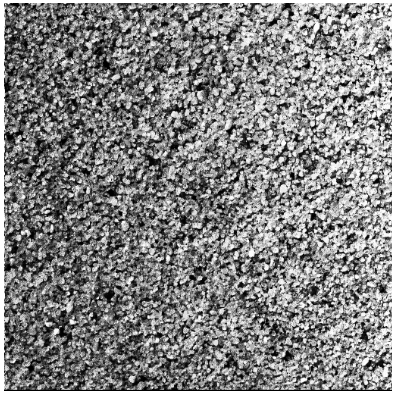
Matlab “sort”



Sand

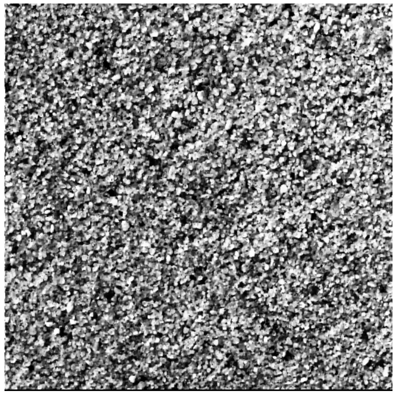


Matlab “sort”



Uniform $[0, \cdots, 255]$

Our ordering



Uniform $[0, \cdots, 255]$

Modern sorting algorithms

For any pixel $v[i]$, extract K auxiliary information, $a_k[i]$, $k \in \{1, \cdots, K$, from v. Set $a_0 := v$. Then

$$i \prec j \quad \text{if} \quad v[i] \leqslant v[j] \;\text{ and }\; a_k[i] < a_k[j] \;\text{ for some }\; k \in \{0, \cdots, K\}.$$

Local Mean Algorithm (LM) [Coltuc, Bolon, Chassery 06]

– If two pixels are equal and their local mean is the same, take a larger neighborhood.

– The procedure smooths edges and sorting often fails.

Wavelet Approach (WA) [Wan, Shi 07]

– Use wavelet coefficients from different subbands to order the pixels.

– Heavy and high level of failure.

Specialized variational approach (SVA) [Nikolova, Wen and Chan 12]

– Minimize $\mathcal{F}_v$ for a parameter choice yielding $\|\hat{u} - v\|_\infty \lessapprox 0.1$.

– Almost no failure, faithful order and fast algorithm. [Nikolova 13]

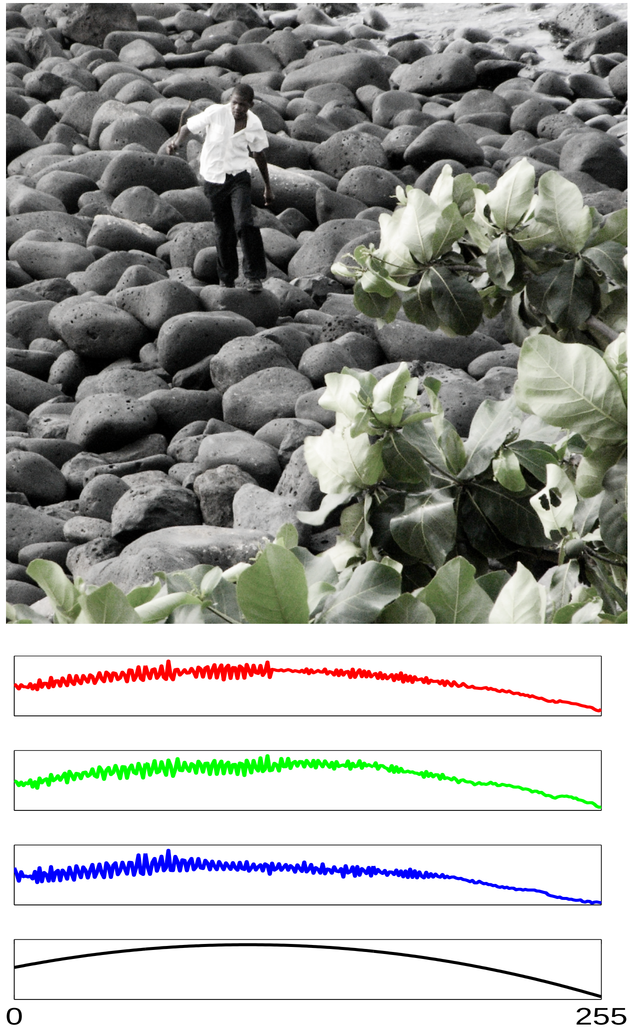
Some results using $\mathcal{F}_v$ for color histogram specification

New fast color assignment algorithm. [Nikolova 13]

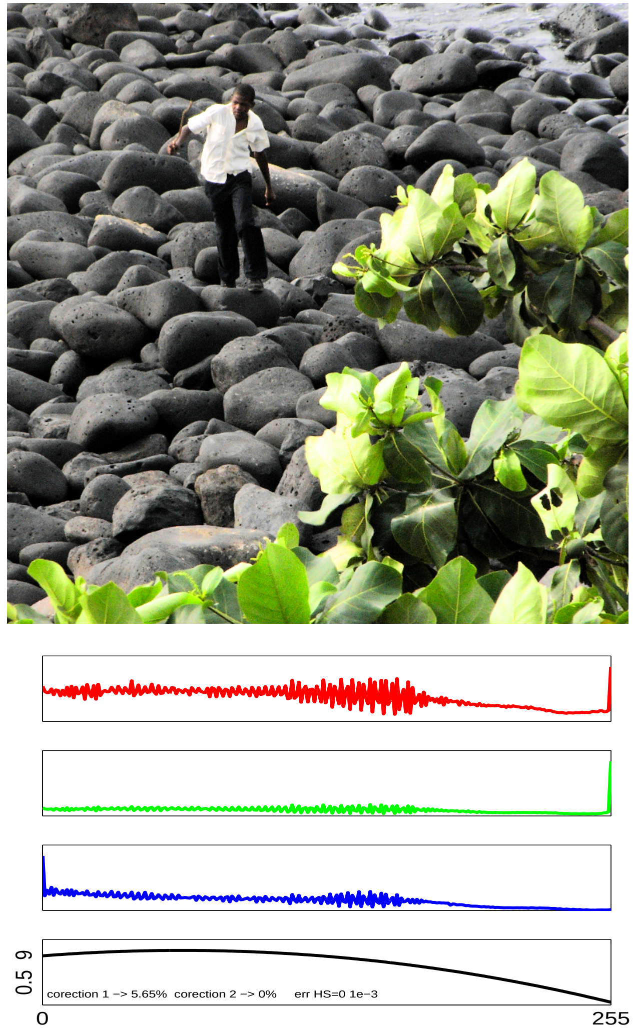
Comparison with the method of [Han, Yang, Lee 11]

Original image

HS by [Han, Yang, Lee 11]

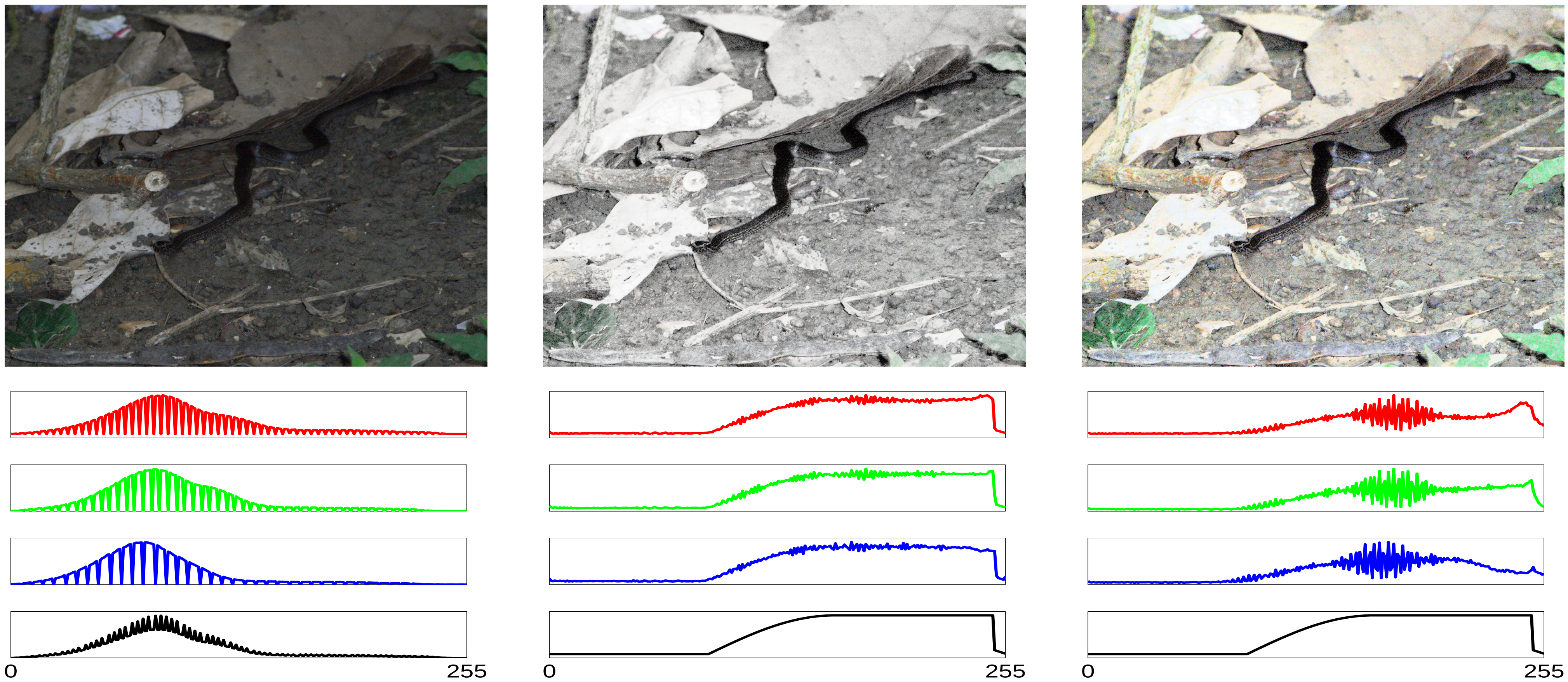


HS - ours



Original image – $(800 \times 800 \times 3)$.

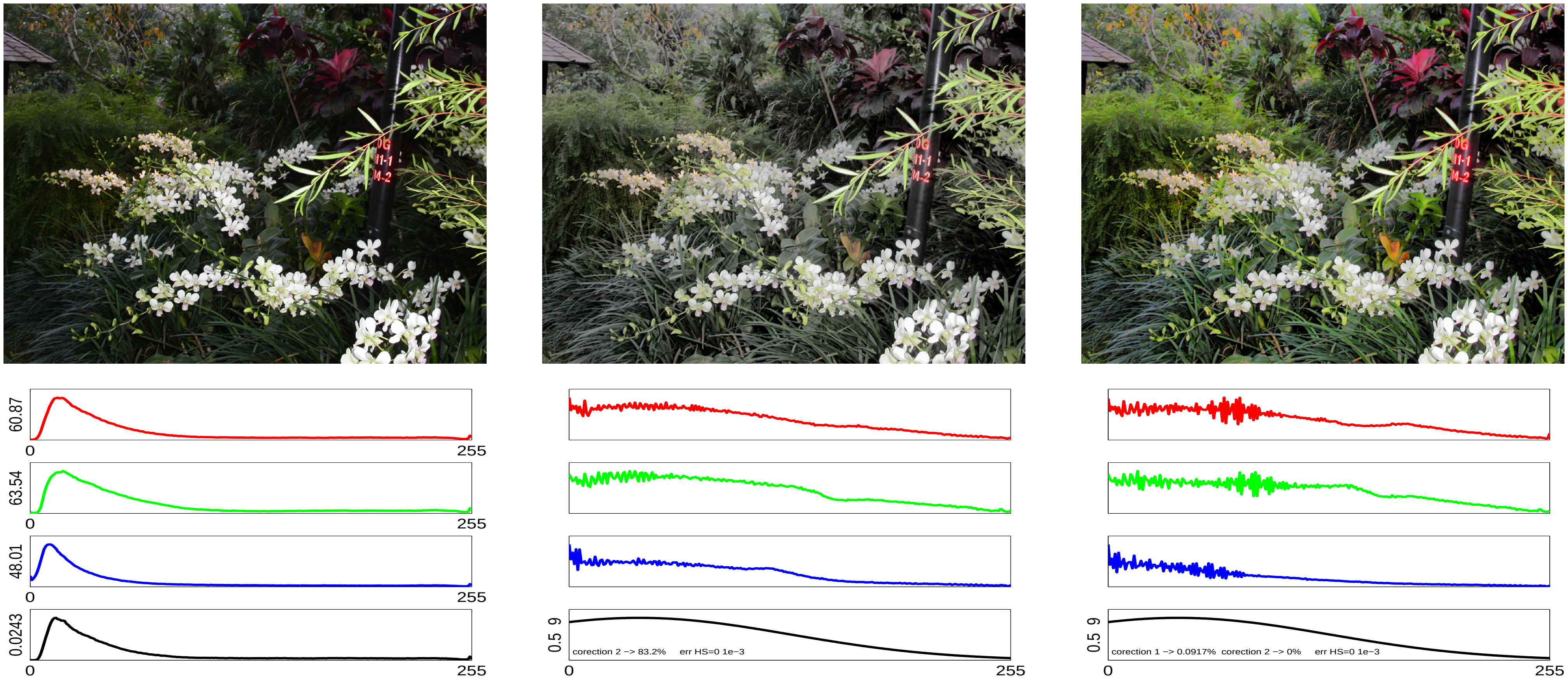
Original image | HS by [Han, Yang, Lee 11] | HS - ours



Original image – $1000 \times 1000 \times 3$.

Goal – enhance the snake.

Original image | HS by [Han, Yang, Lee 11] | HS - ours



Original image – $768 \times 1024 \times 3$.

Goal – remove the flash effect.

Knowledge on the features of the minimizers enables new energies yielding appropriate solutions to be conceived

`` We're in Act I of a digital revolution.''

Jay Cassidy (film editor at Mathematical Technologies Inc.)

Thank you!