Tutorial on Conformal Geometry for Computer Vision

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Thanks

Thanks for the invitation.

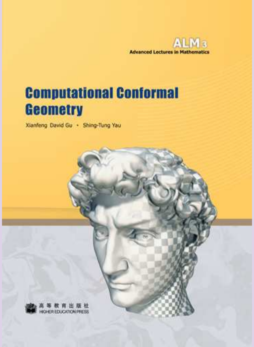
Collaborators

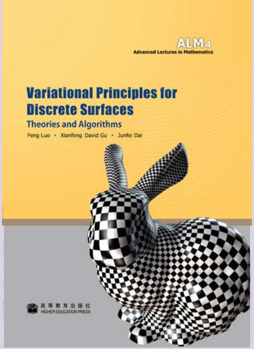
The work is collaborated with Shing-Tung Yau, Feng Luo, Ronald Lok Ming Lui and many other mathematicians, computer scientists and medical doctors.

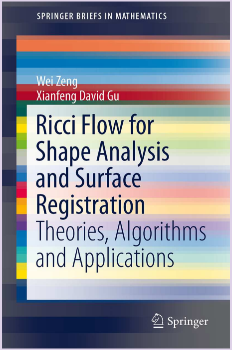
Learning Materials

Books

The theory, algorithms and sample code can be found in the following books.







You can find them in the book store.

Lecture Notes

1. Detailed lecture notes can be found at:

   http://www.cs.sunysb.edu/˜gu/lectures/index.html

2. Binary code and demos can be found at:

   http://www.cs.sunysb.edu/˜gu/software/index.html

3. Source code and data sets:

   http://www.cs.sunysb.edu/˜gu/software/index.html

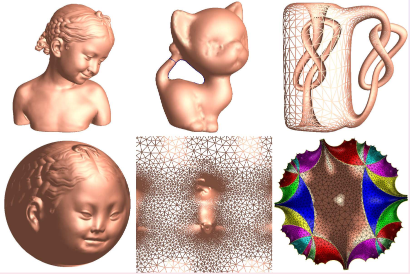
Complete Tools

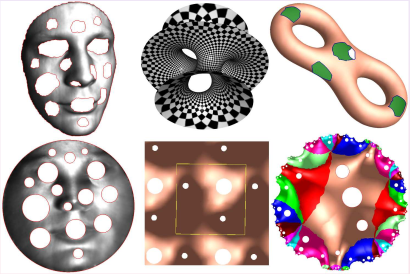
Computational Conformal Geometry Library

1. Compute conformal mappings for surfaces with arbitrary topologies
2. Compute conformal modules for surfaces with arbitrary topologies
3. Compute Riemannian metrics with prescribed curvatures
4. Compute quasi-conformal mappings by solving Beltrami equation

Source Code Library

Please email me gu@cs.sunysb.edu for updated code library on computational conformal geometry.





Recent Papers

- L. M. Lui, W. Zeng, S.-T. Yau and X. Gu. Shape Analysis of Planar Multiply-connected Objects using Conformal Welding. IEEE Transactions on Pattern Analysis and Machine Intelligence (IEEE TPAMI), 2013. To appear.
- W. Zeng, R. Shi, Y. Wang, S.-T. Yau and X. Gu. Teichmüller Shape Descriptor and Its Application to Alzheimer's Disease Study. International Journal of Computer Vision (IJCV), 2012.
- L. M. Lui, T. W. Wong, W. Zeng, X. Gu, P. M. Thompson, T. F. Chan and S.-T. Yau. Optimization of Surface Registrations Using Beltrami Holomorphic Flow. Journal of Scientific Computing (JSC), 50(3):557-585, 2012.
- L. M. Lui, T. W. Wong, W. Zeng, X. Gu, P. M. Thompson, T. F. Chan and S.-T. Yau. Detecting Shape Deformations Using Yamabe Flow and Beltrami Coefficients. Journal of Inverse Problems and Imaging (IPI), 4(2):311-333, 2010.
- W. Zeng, D. Samaras and X. Gu. Ricci Flow for 3D Shape Analysis. IEEE Transactions on Pattern Analysis and Machine Intelligence (IEEE TPAMI), 32(4): 662-677, 2010.

Recent Papers

- W. Zeng, J. Marino, K.C. Gurijala, X. Gu and A. Kaufman. Supine and Prone Colon Registration Using Quasi-Conformal Mapping. IEEE Transactions on Visualization and Computer Graphics (IEEE TVCG), 16(6): 1348-1357, 2010.
- M. Jin, W. Zeng, F. Luo and X. Gu. Computing Teichmüller Shape Space. IEEE Transactions on Visualization and Computer Graphics (IEEE TVCG), 15(3): 504-517, 2009.
- W. Zeng, L. M. Lui, X. Gu and S.-T. Yau. Shape Analysis by Conformal Modules. International Journal of Methods and Applications of Analysis (MAA), 15(4): 539-556, 2008.
- R. Shi, W. Zeng, Z. Su, Hanna Damasio, Zhonglin Lv, Y. Wang, S.-T. Yau and X. Gu. Hyperbolic Harmonic Mapping for Constrained Brain Surface Registration. IEEE Conference on Computer Vision and Pattern Recognition (CVPR13), Jun 23-28, 2013, Portland, Oregon, USA. (Oral)
- Z. Su, W. Zeng, R. Shi, Y. Wang, J. Sun and X. Gu. Area Preserving Brain Mapping. IEEE Conference on Computer Vision and Pattern Recognition (CVPR13), Jun 23-28, 2013, Portland, Oregon, USA.

Recent Papers

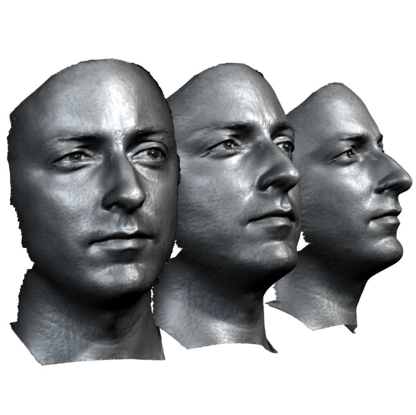
- W. Zeng and X. Gu. Registration for 3D Surfaces with Large Deformations Using Quasi-Conformal Curvature Flow. IEEE Conference on Computer Vision and Pattern Recognition (CVPR11), Jun 20-25, 2011, Colorado Springs, Colorado, USA.
- L. M. Lui, W. Zeng, T.F. Chan, S.-T. Yau and X. Gu. Shape Representation of Planar Objects with Arbitrary Topologies Using Conformal Geometry. The 11th European Conference on Computer Vision (ECCV10), Sep 5-11, 2010, Crete, Greece.
- W. Zeng, Y. Zeng, Y. Wang, X. T. Yin, X. Gu and D. Samaras. 3D Non-rigid Surface Matching and Registration Based on Holomorphic Differentials. The 10th European Conference on Computer Vision (ECCV08), Oct 12-18, 2008, Marseille, France. (Oral)
- R. Shi, W. Zeng, Z. Su, Y. Wang, Hanna Damasio, Zhonglin Lv, S.-T. Yau and X. Gu. Hyperbolic Harmonic Brain Surface Registration with Curvature-based Landmark Matching. International Conference on Information Processing in Medical Imaging (IPMI13), Jun 28-Jul 3, 2013, Asilomar, California, USA.

3D Geometric Data Acquisition

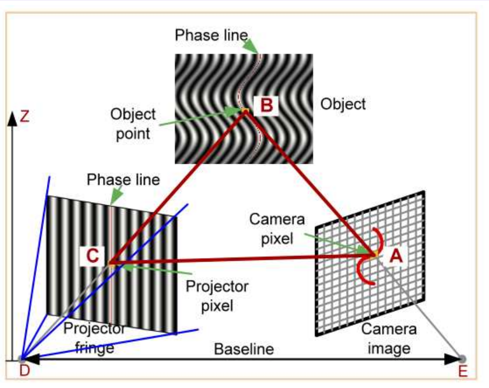
3D Geometric Data Acquisition

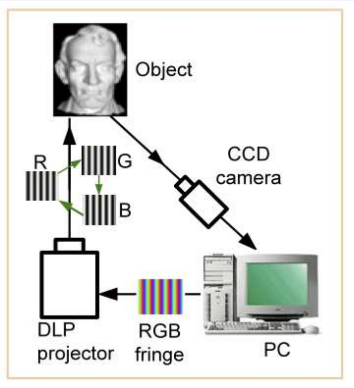
3D Scanning

3D scanning technology becomes mature, it is easier to obtain surface data.

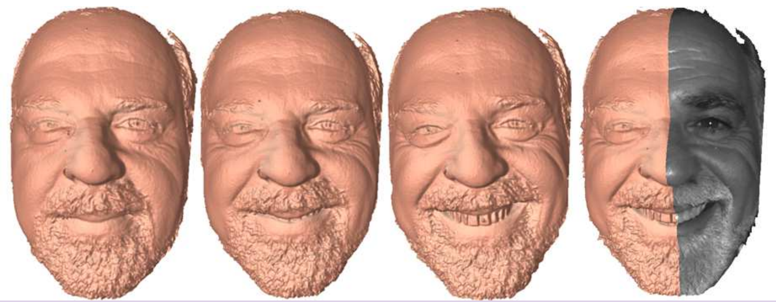


System Layout

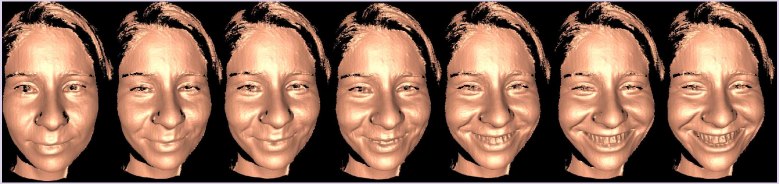




3D Scanning Results



3D Scanning Results



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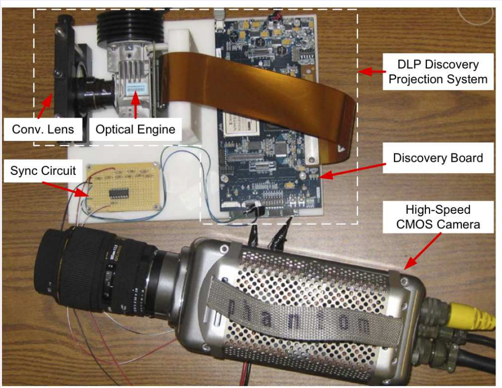
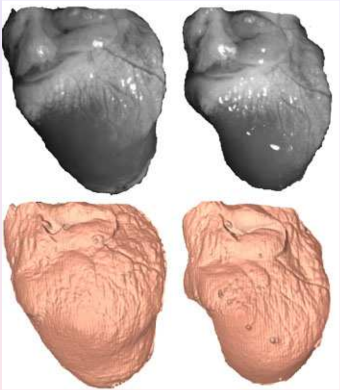




3D Geometric Data Acquisition

Geometric Informatics

Our group has developed high speed 3D scanner, which can capture dynamic surfaces 180 frames per second.

Motivation

With huge amount real time 3D geometric data, it is critical to develop methods to efficiently and accurately process them.

Central Tasks

1. Shape Space: Consider all shapes in real world, how to classify them, measure distances among them, deform from one to the other.
2. Mapping Space: Study all the mappings between two given shapes, how to find the optimal mapping.

Solution

Methods based on differential geometry: mathematical rigor, efficiency, efficacy.

Klein's Program

Klein's Erlangen Program

Different geometries study the invariants under different transformation groups.

Geometries

- Topology - homeomorphisms
- Conformal Geometry - Conformal Transformations
- Riemannian Geometry - Isometries
- Differential Geometry - Rigid Motion

Motivation

Conformal Geometric methods have merits:

1. Unification: All the surfaces in real life can be eventually unified to one of three canonical shapes, the sphere, the plane or the hyperbolic disk.
2. Dimension Reduction: All 3D geometric processing problems are converted to 2D image processing problems.
3. Information Preservation: All the deformation preserves the intrinsic geometric information.
4. General Transformation: Capable of modeling all the mappings among surfaces.

Conformal Geometry

In Pure Mathematics

Conformal geometry is the intersection of complex analysis, algebraic topology, Riemann surface theory, algebraic curves, Teichmüller theory, differential geometry, partial differential equation and so on. It has intrinsic relation to theoretic physics as well.

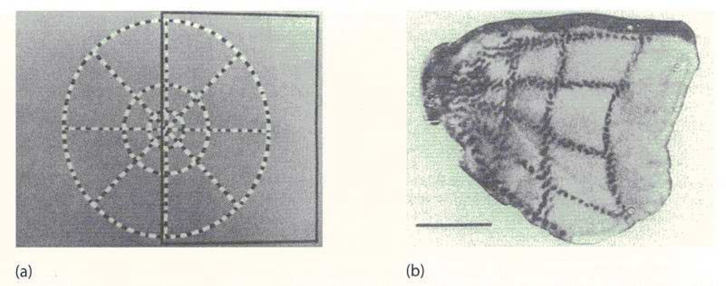
In Applied Mathematics

Computational complex function theory has been developed, and applied in mechanics, electro-magnetic field design, fluid dynamics and areospace design and so on. However, most existing methods focus on the conformal mappings between planar domains.

Fundamental Problems

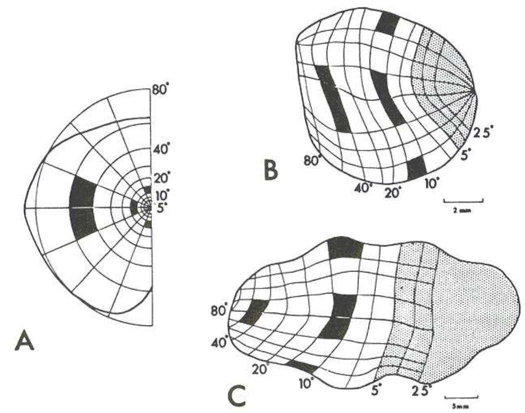
1. Given a Riemannian metric on a surface with an arbitrary topology, determine the corresponding conformal structure.
2. Compute the complete conformal invariants (conformal modules), which are the coordinates of the surface in the Teichmuller shape space.
3. Fix the conformal structure, find the simplest Riemannian metric among all possible Riemannian metrics
4. Given desired Gaussian curvature, compute the corresponding Riemannian metric.
5. Given the distortion between two conformal structures, compute the quasi-conformal mapping.
6. Compute the extremal quasi-conformal maps.
7. Conformal welding, glue surfaces with various conformal modules, compute the conformal module of the glued surface.

Retinotopic Mapping



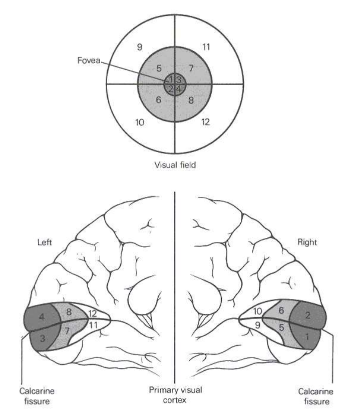
The retinotopic mapping is clearly demonstrated by an experiment (Tootell et al, 1982).

Retinotopic Mapping



Connolly and Van Essen 1984 showed the mapping.

Retinotopic Mapping



Retinotopic mapping is conformal.

Basic Concepts

Curve Curvature

Let $\gamma:[0,1]\to\mathbb{R}^3$ is a curve,

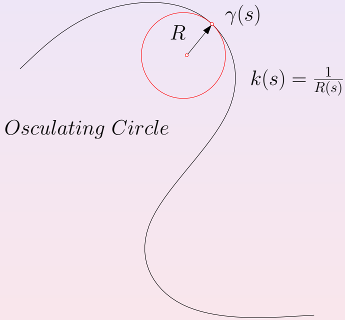
$$\gamma(t)=(x(t),y(t),z(t))$$

The arc length is

$$s(t)=\int_0^t\sqrt{\dot{\gamma}(\tau),\dot{\gamma}(\tau)}d\tau$$

Reparameterize the curve using arc length parameter. The curvature is given by

$$k(s)=\frac{d^2\gamma(s)}{ds^2}$$



Let Ω be a planar domain, $\gamma : \Omega \to \mathbb{R}^3$ is a surface patch,

$$\gamma(u,v) = (x(u,v), y(u,v), z(u,v))$$

a regular parameterization

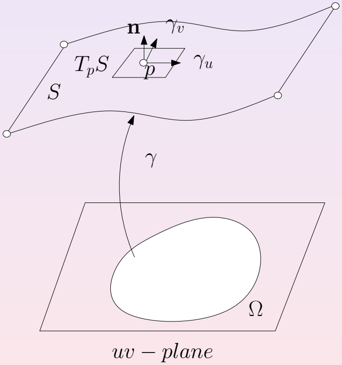
$$\frac{\partial \gamma}{\partial u} \times \frac{\partial \gamma}{\partial v} \neq 0, everywhere$$

The tangent vector is

$$d\gamma = \gamma_u du + \gamma_v dv,$$

The normal is

$$\mathbf{n} = \frac{\gamma_u \times \gamma_v}{|\gamma_u \times \gamma_v|}$$

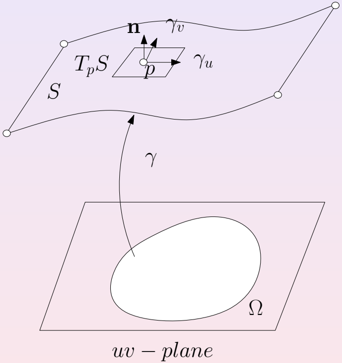


The first fundamental form is

$$I = \langle d\gamma, d\gamma \rangle$$

The second fundamental form is

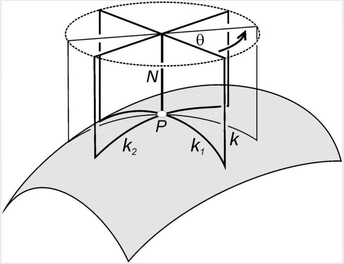
$$II = \langle d^2\gamma, \mathbf{n} \rangle = -\langle d\gamma, d\mathbf{n} \rangle$$



Given a tangent vector $d\gamma$ at point p, the plane $span\{d\gamma, \mathbf{n}\}$ intersects the surface, the curvature of the intersection curve at p is called the the normal curvature

$$k_n = \frac{II}{I}$$

By changing the tangent direction, the max, min normal curvatures are principle curvatures,the corresponding directions are principle directions.

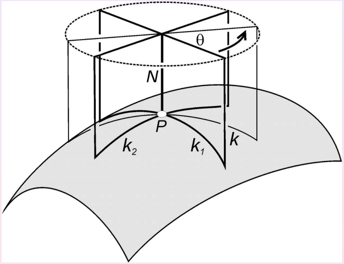


The product of the principle curvatures are called Gaussian curvature

$$K = k_1 k_2 = \frac{det(II)}{det(I)}$$

Theorem (Gauss)

The Gaussian curvature is solely determined by the first fundamental form, therefore it is intrinsic.

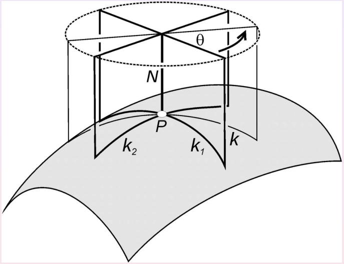


Theorem (Gauss-Bonnet)

The total Gaussian curvature is a topological invariant

$$\int_S K dA + \int_{\partial S} k_g ds = 2\pi\chi(S),$$

where ∂S is the boundary of the surface, k_g the geodesic curvature, $\chi(S)$ is the Euler characteristics number of S.



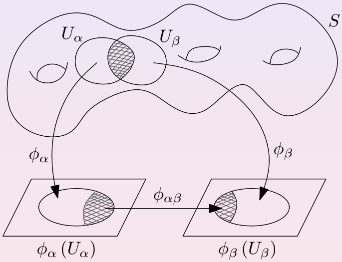
Manifold

Definition (Manifold)

S is a topological space, for each point $p \in M$, there is a neighborhood U and a homeomorphism $\phi : U \to \mathbb{R}^n$, which maps U to an open set in $\mathbb{R}^n$.

(U, ϕ) is a local chart. Two charts overlaps, then each point $p \in U_\alpha \cap U_\beta$ has two local parameters $\phi_\alpha(p)$ and $\phi_\beta(p)$, this induces the chart transition map $\phi_{\alpha\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ such that

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}.$$

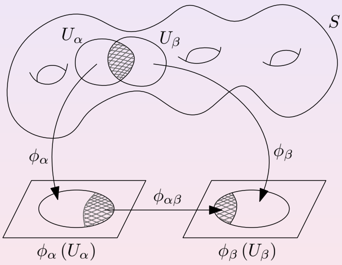


Definition (Riemannian metric)

Given a topological manifold, at each point p, one gives an inner product $\mathbf{g}(p)$ on the tangent space T_pM, then $\mathbf{g}$ is a Riemannian metric on M.

Choose a local chart (U_α, ϕ_α) with parameters (u_α, v_α), the metric tensor is represented as a symmetric positive definite matrix,

$$\mathbf{g}_\alpha = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix}$$

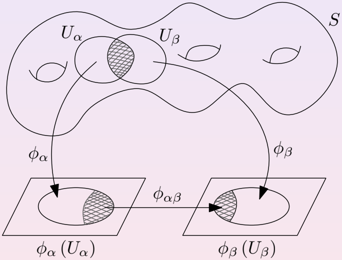


Change to another local chart (U_β, ϕ_β) with parameters (u_β, v_β), the metric tensor satisfies

$$\mathbf{g}_\alpha = D\phi_{\alpha\beta}^T \mathbf{g}_\beta D\phi_{\alpha\beta}$$

where $D\phi_{\alpha\beta}$ is the Jacobian matrix of $\phi_{\alpha\beta}$.

$$D\phi_{\alpha\beta} = \begin{pmatrix} \frac{\partial u_\beta}{\partial u_\alpha} & \frac{\partial u_\beta}{\partial v_\alpha} \\ \frac{\partial v_\beta}{\partial u_\alpha} & \frac{\partial v_\beta}{\partial v_\alpha} \end{pmatrix}$$



Riemannian Metric

Riemannian metric defines inner product

$$\langle d\gamma, \delta\gamma \rangle_{\mathbf{g}} = \begin{pmatrix} du & dv \end{pmatrix} \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} \delta u \\ \delta v \end{pmatrix}$$

to measure lengths

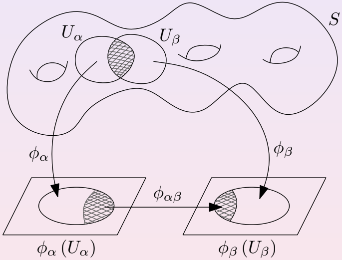
$$|d\gamma|_{\mathbf{g}} = \sqrt{\langle d\gamma, d\gamma \rangle_{\mathbf{g}}}$$

angles

$$\frac{\langle d\gamma, \delta\gamma \rangle_{\mathbf{g}}}{|d\gamma|_{\mathbf{g}} |\delta\gamma|_{\mathbf{g}}}$$

area

$$dA = (g_{11}g_{22} - g_{12}g_{21})dudv.$$



Pull back metric

Suppose $\varphi : (M, \mathbf{g}) \to (N, \mathbf{h})$ be a mapping, with local presentation

$$\varphi : (x, y) \to (u, v),$$

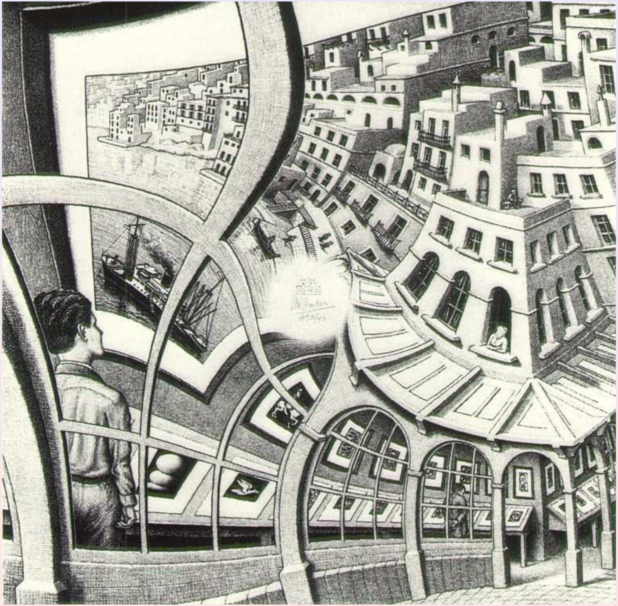
The pull back metric by φ is given by

$$\varphi^* \mathbf{h} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}^T \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Suppose $\gamma \subset M$ is a path on the source, $\varphi(\gamma)$ is a path on the target. We measure the length of $\varphi(\gamma)$ using $\mathbf{h}$ on N, this gives the pull back metric induced by φ on M.

Conformal Mapping

M.C.Escher's Print Gallery



Office Table

Office Table

Nine Golden Fish

Nine Golden Fish

Definition (Holomorphic Function)

Suppose $f:\mathbb{C}\to\mathbb{C}$ is a complex function, $f: x+iy \to u(x,y)+iv(x,y)$, if f satisfies Riemann-Cauchy equation

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},$$

then f is a holomorphic function.

Denote

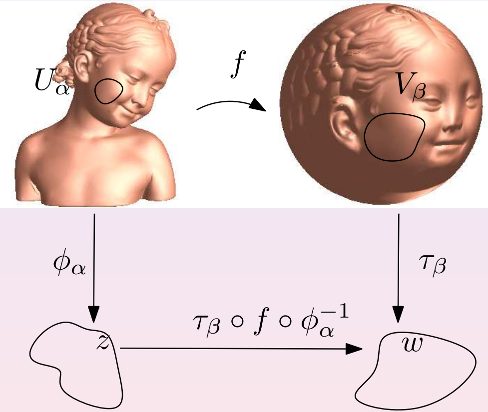
$$dz=dx+idy, d\bar{z}=dx-idy, \frac{\partial}{\partial z}=\frac{1}{2}(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}), \frac{\partial}{\partial \bar{z}}=\frac{1}{2}(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y})$$

then if $\frac{\partial f}{\partial \bar{z}}=0$, then f is holomorphic.

Definition (biholomorphic Function)

Suppose $f:\mathbb{C}\to\mathbb{C}$ is invertible, both f and f^{-1} are holomorphic, then then f is a biholomorphic function.

$S_1 \subset \{(U_\alpha, \phi_\alpha)\}$ $S_2 \subset \{(V_\beta, \tau_\beta)\}$



The restriction of the mapping on each local chart is biholomorphic, then the mapping is conformal.

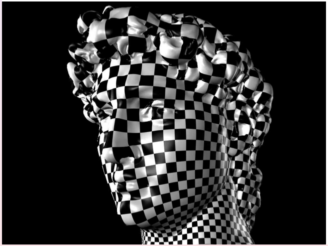
Conformal Mapping

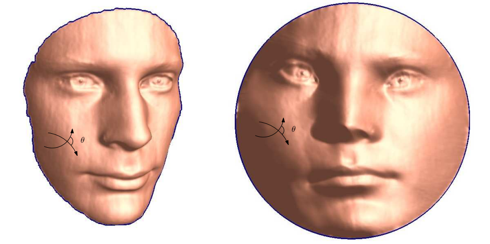
Definition (Conformal Map)

Let $\phi : (S_1, \mathbf{g}_1) \to (S_2, \mathbf{g}_2)$ is a homeomorphism, ϕ is conformal if and only if

$$\phi^* \mathbf{g}_2 = e^{2u} \mathbf{g}_1.$$

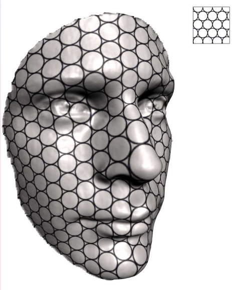
Conformal Mapping preserves angles.





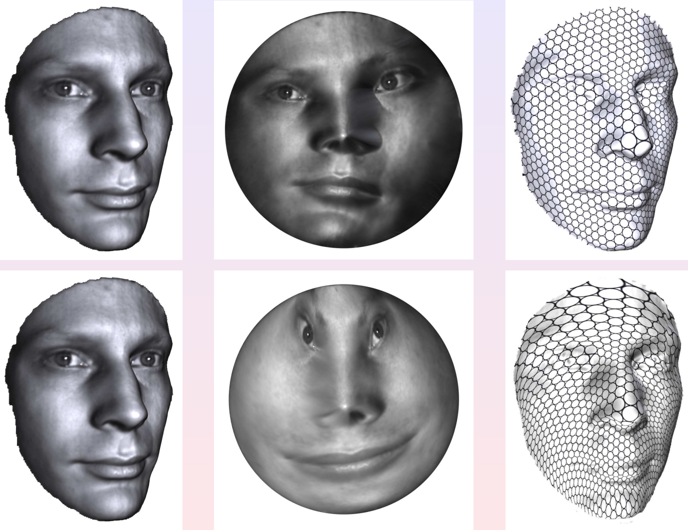
Conformal maps Properties

Map a circle field on the surface to a circle field on the plane.



Quasi-Conformal Map

Diffeomorphisms: maps ellipse field to circle field.



Conformal Structure

Intuition

1. A conformal structure is a equipment on a surface, such that the angles can be measured, but the lengths, areas can not.
2. A Riemannian metric naturally induces a conformal structure.
3. Different Riemannian metrics can induce a same conformal structure.

Manifold

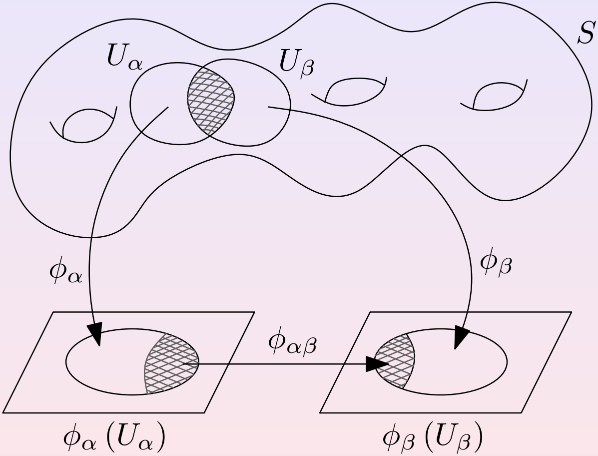
Definition (Manifold)

M is a topological space, $\{U_\alpha\}$ $\alpha \in I$ is an open covering of M, $M \subset \cup_\alpha U_\alpha$. For each U_α, $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ is a homeomorphism. The pair (U_α, ϕ_α) is a chart. Suppose $U_\alpha \cap U_\beta \neq \emptyset$, the transition function $\phi_{\alpha\beta} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is smooth

$$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$$

then M is called a smooth manifold, $\{(U_\alpha, \phi_\alpha)\}$ is called an atlas.

Manifold



S
U_α
U_β
ϕ_α
ϕ_β
$\phi_{\alpha\beta}$
$\phi_\alpha(U_\alpha)$
$\phi_\beta(U_\beta)$

Conformal Atlas

Definition (Conformal Atlas)

Suppose S is a topological surface, (2 dimensional manifold), $\mathfrak{A}$ is an atlas, such that all the chart transition functions $\phi_{\alpha\beta}:\mathbb{C}\to\mathbb{C}$ are bi-holomorphic, then A is called a conformal atlas.

Definition (Compatible Conformal Atlas)

Suppose S is a topological surface, (2 dimensional manifold), $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are two conformal atlases. If their union $A_1 \cup A_2$ is still a conformal atlas, we say $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are compatible.

The compatible relation among conformal atlases is an equivalence relation.

Definition (Conformal Structure)

Suppose S is a topological surface, consider all the conformal atlases on M, classified by the compatible relation

$$\{all\ conformal\ atlas\}/\sim$$

each equivalence class is called a conformal structure.

In other words, each maximal conformal atlas is a conformal structure.

Relation between Riemannian metric and Conformal Structure

Definition (Isothermal coordinates)

Suppose (S,g) is a metric surface, (U_α,ϕ_α) is a coordinate chart, (x,y) are local parameters, if

$$g = e^{2u}(dx^2+dy^2),$$

then we say (x,y) are isothermal coordinates.

Theorem

Suppose S is a compact metric surface, for each point p, there exits a local coordinate chart (U,ϕ), such that $p\in U$, and the local coordinates are isothermal.

Corollary

For any compact metric surface, there exists a natural conformal structure.

Definition (Riemann surface)

A topological surface with a conformal structure is called a Riemann surface.

Theorem

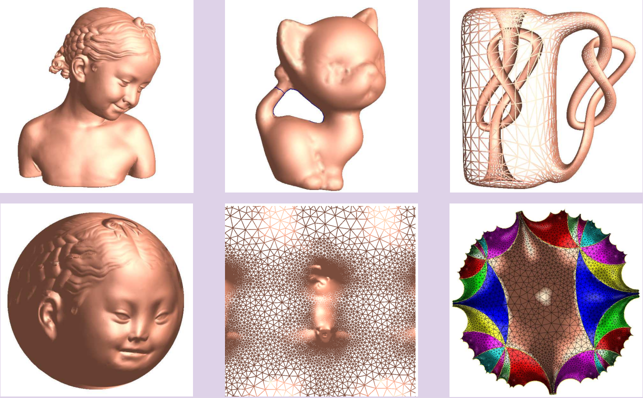
All compact orientable metric surfaces are Riemann surfaces.

Uniformization

Conformal Canonical Representations

Theorem (Poincaré Uniformization Theorem)

Let $(\Sigma, \mathbf{g})$ be a compact 2-dimensional Riemannian manifold. Then there is a metric $\tilde{\mathbf{g}} = e^{2\lambda}\mathbf{g}$ conformal to $\mathbf{g}$ which has constant Gauss curvature.



Spherical Euclidean Hyperbolic

Definition (Circle Domain)

A domain in the Riemann sphere $\hat{\mathbb{C}}$ is called a circle domain if every connected component of its boundary is either a circle or a point.

Theorem

Any domain Ω in $\hat{\mathbb{C}}$, whose boundary $\partial\Omega$ has at most countably many components, is conformally homeomorphic to a circle domain Ω^* in $\hat{\mathbb{C}}$. Moreover Ω^* is unique upto Möbius transformations, and every conformal automorphism of Ω^* is the restriction of a Möbius transformation.

Conformal Canonical Representations

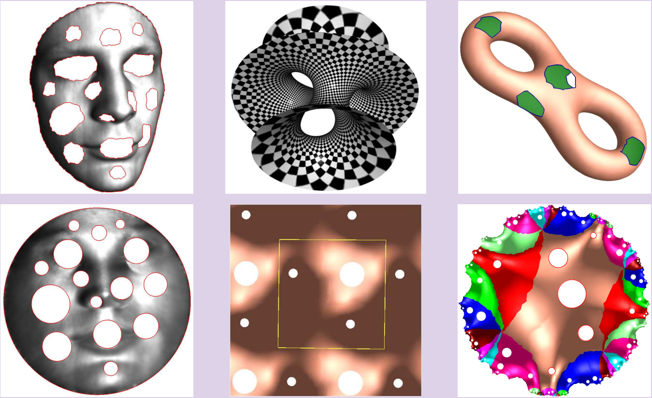
Definition (Circle Domain in a Riemann Surface)

A circle domain in a Riemann surface is a domain, whose complement's connected components are all closed geometric disks and points. Here a geometric disk means a topological disk, whose lifts in the universal cover or the Riemann surface (which is $\mathbb{H}^2$, $\mathbb{R}^2$ or $\mathbb{S}^2$ are round.

Theorem

Let Ω be an open Riemann surface with finite genus and at most countably many ends. Then there is a closed Riemann surface R^* such that Ω is conformally homeomorphic to a circle domain Ω^* in R^*. More over, the pair (R^*, Ω^*) is unique up to conformal homeomorphism.

Uniformization of Open Surfaces



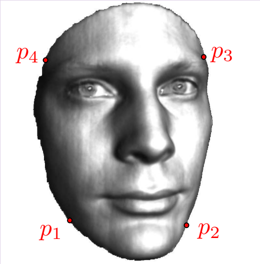
Spherical Euclidean Hyperbolic

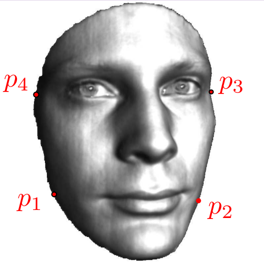
Conformal Canonical Representation

Simply Connected Domains

Conformal Canonical Forms

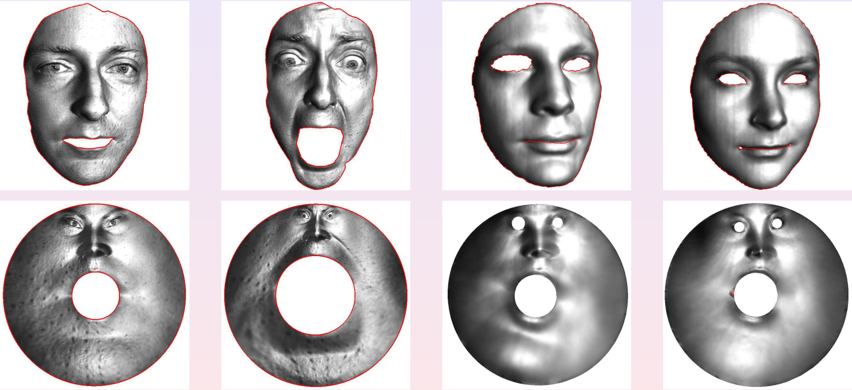
Topological Quadrilateral





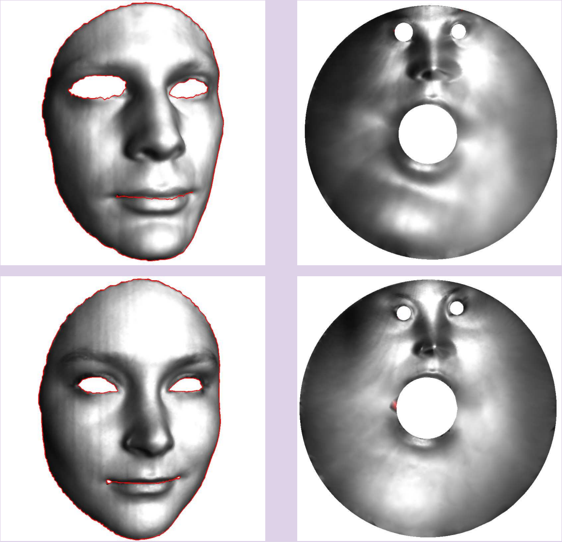
Conformal Canonical Forms

Multiply Connected Domains



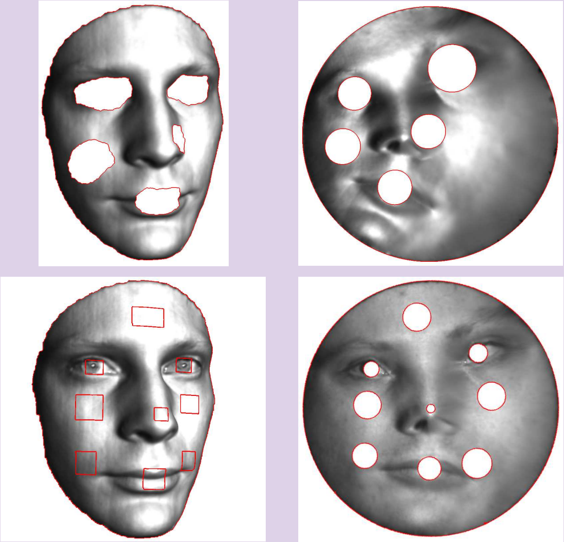
Conformal Canonical Forms

Multiply Connected Domains



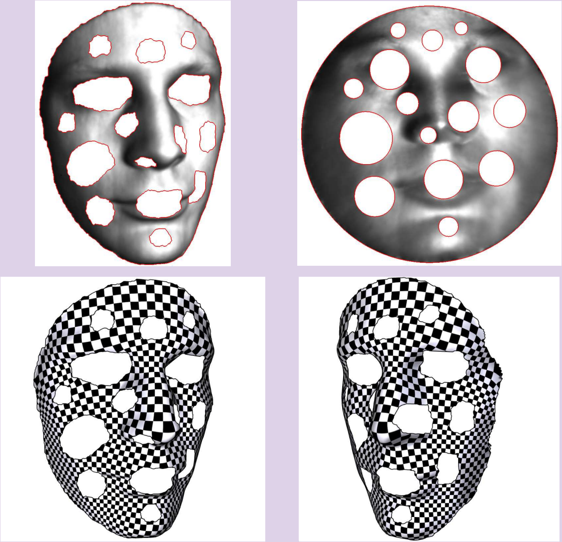
Conformal Canonical Forms

Multiply Connected Domains



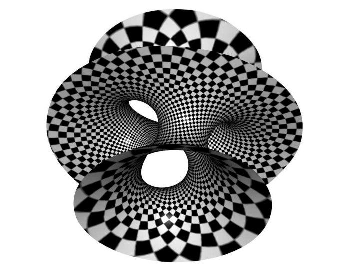
Conformal Canonical Forms

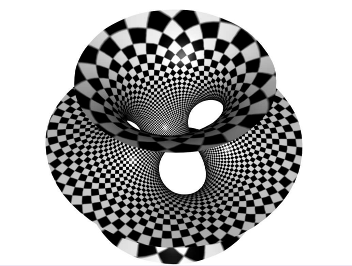
Multiply Connected Domains

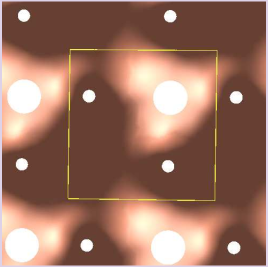


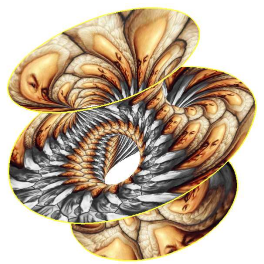
Conformal Canonical Form

Tori with holes



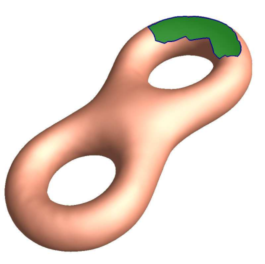


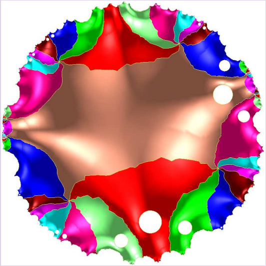




Conformal Canonical Form

High Genus Surface with holes





Teichmüller Space

Shape Space - Intuition

1. All metric surfaces are classified by conformal equivalence relation.
2. All conformal equivalent classes form a finite dimensional manifold.
3. The manifold has a natural Riemannian metric, which measures the distortions between two conformal structures.

Definition (Conformal Mapping)

Suppose $(S_1, \mathbf{g}_1)$ and $(S_2, \mathbf{g}_2)$ are two metric surfaces, $\phi : S_1 \to S_2$ is conformal, if on S_1

$$\mathbf{g}_1 = e^{2\lambda} \phi^* \mathbf{g}_2,$$

where $\phi^* \mathbf{g}_2$ is the pull-back metric induced by ϕ.

Definition (Conformal Equivalence)

Suppose two surfaces S_1, S_2 with marked homotopy group generators, $\{a_i, b_i\}$ and $\{\alpha_i, \beta_i\}$. If there exists a conformal map $\phi : S_1 \to S_2$, such that

$$\phi_*[a_i] = [\alpha_i], \phi_*[b_i] = [\beta_i],$$

then we say two marked surfaces are conformal equivalent.

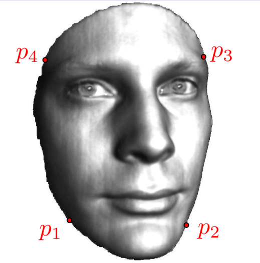
Definition (Teichmüller Space)

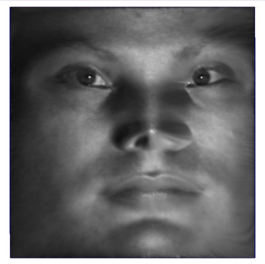
Fix the topology of a marked surface S, all conformal equivalence classes sharing the same topology of S, form a manifold, which is called the Teichmüller space of S. Denoted as T_S.

- Each point represents a class of surfaces.
- A path represents a deformation process from one shape to the other.
- The Riemannian metric of Teichmüller space is well defined.

Teichmüller Space

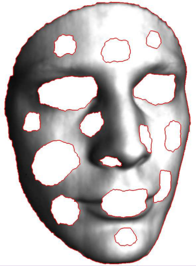
Topological Quadrilateral

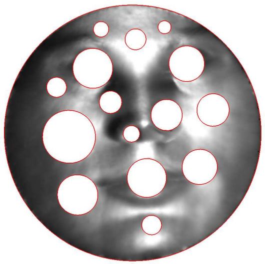




Conformal module: $\frac{h}{w}$. The Teichmüller space is 1 dimensional.

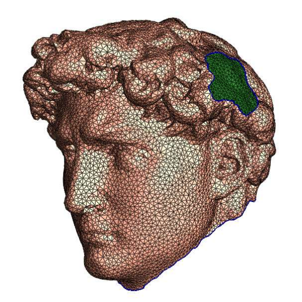
Multiply Connected Domains

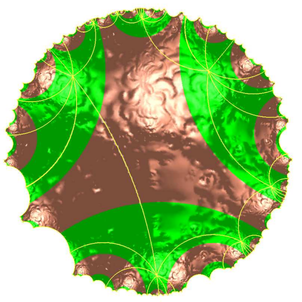




Conformal Module : centers and radii, with Möbius ambiguity. The Teichmüller space is $3n-3$ dimensional, n is the number of holes.

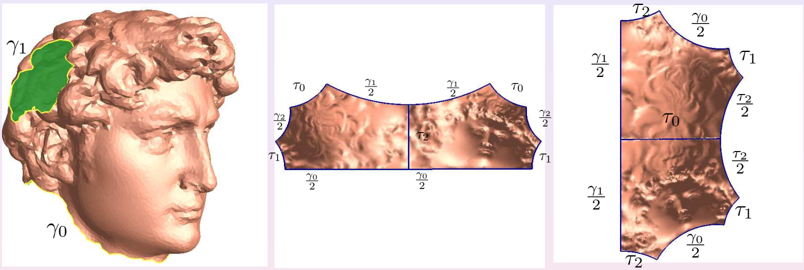
Topological Pants





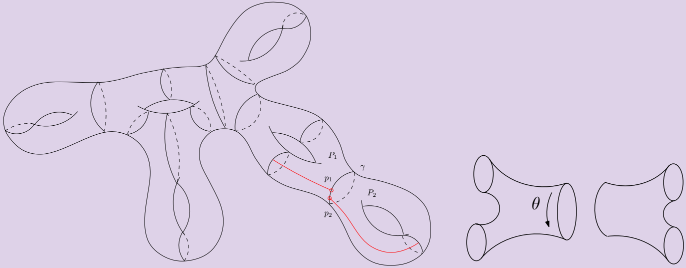
Genus 0 surface with 3 boundaries is conformally mapped to the hyperbolic plane, such that all boundaries become geodesics.

Teichmüller Space - Topological Pants



The lengths of 3 boundaries are the conformal module. The Teichmüller space is 3 dimensional.

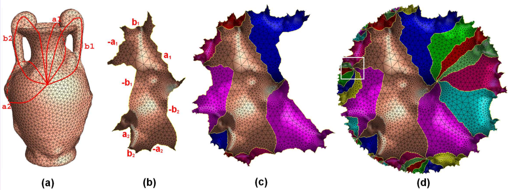
Topological Pants Decomposition - $2g-2$ pairs of Pants



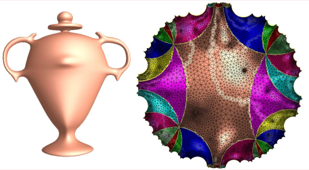
The Teichmüller space is of $6g-6$ dimension, for genus $g>1$.

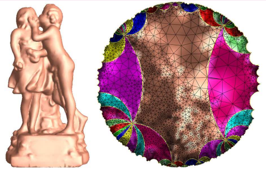
Compute Teichmüller coordinates

Step 1. Compute the hyperbolic uniformization metric.



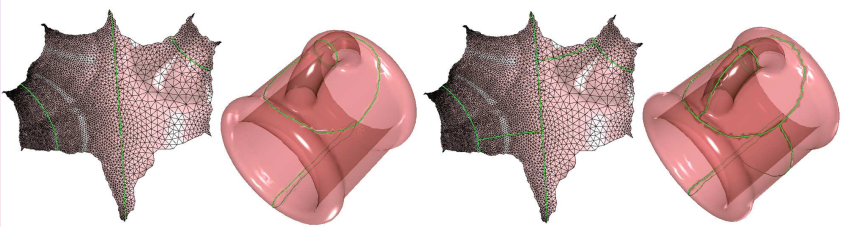
Step 2. Compute the Fuchsian group generators.





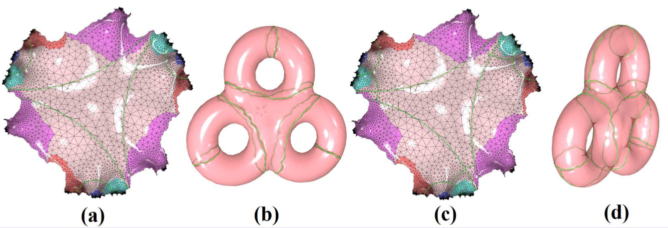
Compute Teichmüller Coordinates

Step 3. Pants decomposition using geodesics and compute the twisting angle.



Compute Teichmüller coordinates

Compute the pants decomposition using geodesics and compute the twisting angle.



General Shape Space

Klein's Program

Klein's Erlangen Program

Different geometries study the invariants under different transformation groups.

Geometries

- Topology - homeomorphisms
- Conformal Geometry - Conformal Transformations
- Riemannian Geometry - Isometries
- Differential Geometry - Rigid Motion

Suppose a mapping $\varphi : (S_1, \mathbf{g}_1) \to (S_2, \mathbf{g}_2)$ is given,

1. Homeomorphism: φ is continuous, bijective, φ^{-1} is also continuous.
2. Conformal: the pull back metric is proportional to the original metric
$$\varphi^* \mathbf{g}_2 = e^{2\lambda} \mathbf{g}_1,$$
where $\lambda : S_1 \to \mathbb{R}$ is the conformal factor function.
3. Isometry: the pull back metric equals to the original metric
$$\varphi^* \mathbf{g}_2 = \mathbf{g}_1.$$
4. Rigid motion: rotation and translation in $\mathbb{R}^3$.

Hierarchy

The transformation groups have the relation:

$$\{\textit{rigid motion}\} \lhd \{\textit{isometry}\} \lhd \{\textit{conformal}\} \lhd \{\textit{homeomorphism}\}$$

The corresponding shape spaces

$$\mathscr{S}/\{\textit{rigid motion}\} \rhd \mathscr{S}/\{\textit{isometry}\} \rhd \mathscr{S}/\{\textit{conformal}\} \rhd \mathscr{S}/\{\textit{homeomo}$$

where
$\mathscr{S} = \{$ *compact orientatable metric surfaces embedded in* $\mathbb{E}^3\}$.

Topology

Definition (Topologically Equivalence)

Two surfaces are topologically equivalent, if there exists a homeomorphism between them.

Definition (Topological Invariants)

Orientability, genus, number of boundaries. Fundamental group, homology group, cohomology group.

Conformal Geometry

Definition (Conformal Equivalence)

Two surfaces are conformal equivalent, if there exists a conformal mapping between them.

Definition (Conformal Invariants)

Conformal module, uniformization domain:

$$S/\Gamma - \cup_{i=1}^{n} C(c_i, r_i),$$

1. S is a constant curvature space, the unit sphere $\mathbb{S}^2$, the Euclidean plane $\mathbb{E}^2$ and the hyperbolic plane $\mathbb{H}^2$.
2. Γ is a fixed point free subgroup of the rigid motion group of S.
3. $C(c_i, r_i)$ is a geodesic circle on S/Γ with center c_i and radius r_i.

Definition (Isometric Equivalence)

Two surfaces are isometric equivalent, if there exists an isometric mapping between them.

Definition (Isometric Invariants)

Suppose the surface $(M,\mathbf{g})$ has the canonical conformal representation $S/\Gamma - \cup_{i=1}^{n} C(c_i, r_i)$, the Riemannnian metric of M is given by

$$\mathbf{g} = e^{2\lambda}\mathbf{g}_S,$$

where λ is the conformal factor, $\mathbf{g}_S$ is the spherical, Euclidean, or hyperbolic metric.

Therefore, a compact, orienatable metric surface has the representation

$$(S/\Gamma - \cup_{i=1}^{n} C(c_i, r_i), \lambda)$$

Differential Geometry in $\mathbb{E}^3$

Suppose two compact surfaces embedded in $\mathbb{E}^3$, $(S_1, \mathbf{g}_1)$ and $(S_2, \mathbf{g}_2)$ differ by a rigid motion, if and only if they share the same

1. conformal representation $S/\Gamma - \cup_{i=1}^n C(c_i, r_i)$,
2. conformal factor λ,
3. mean curvature H.
4. conformal factor and mean curvature satisfies Gauss-Codazzi equations

$$(\log\lambda)_{z\bar{z}} = \frac{\mu\bar{\mu}}{\lambda^2} - \frac{\lambda^2}{4}H^2,$$

$$\mu_{\bar{z}} = \frac{\lambda^2}{2}H_z,$$

$$\mu_{z\bar{z}} = \frac{1}{2}\lambda(2\lambda_z H_z + \lambda H_{zz}).$$

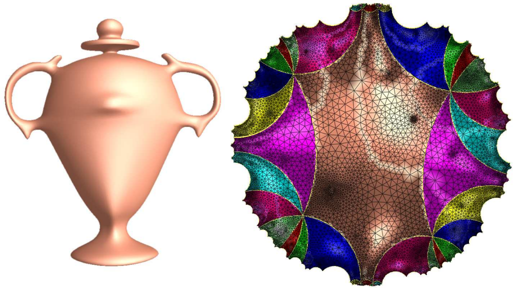
Canonical Representation

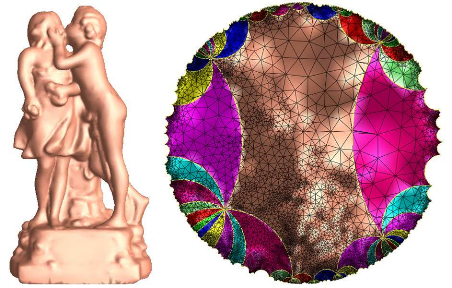
Suppose $(M, \mathbf{g})$ is a compact, orientable, metric surface embedded in $\mathbb{E}^3$, then its representation is a triple

$$(S/\Gamma - \cup_{i=1}^{n} C(c_i, r_i), \lambda, H).$$

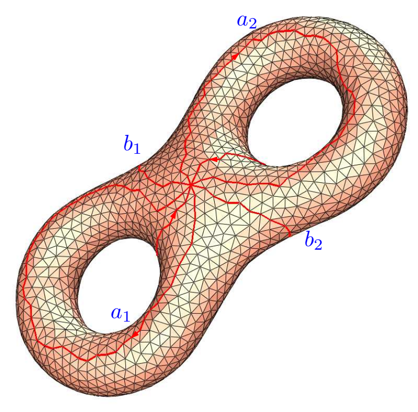
where λ and H satisfy Gauss-Codazzi equations.

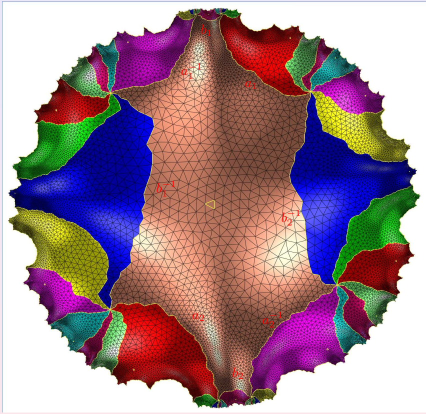
Hyperbolic Ricci Flow





Hyperbolic Yamabe Flow





Example

Suppose $(M, \mathbf{g})$ is a genus two closed surface, then its representation is

$$(\mathbb{H}^2/\Gamma, \lambda, H),$$

where Γ is the Fuchsian group of the surface

$$\Gamma = \langle \alpha_1, \beta_1, \alpha_2, \beta_2 | \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \alpha_2 \beta_2 \alpha_2^{-1} \beta_2^{-1} \rangle$$

each generator α_k or β_k is a Möbius transformation, and can be represented as a complex linear fraction map

$$z \to \frac{az+b}{cz+d}, ad - bc = 1, a, b, c, d \in \mathbb{R}$$

Quasi-Conformal Maps

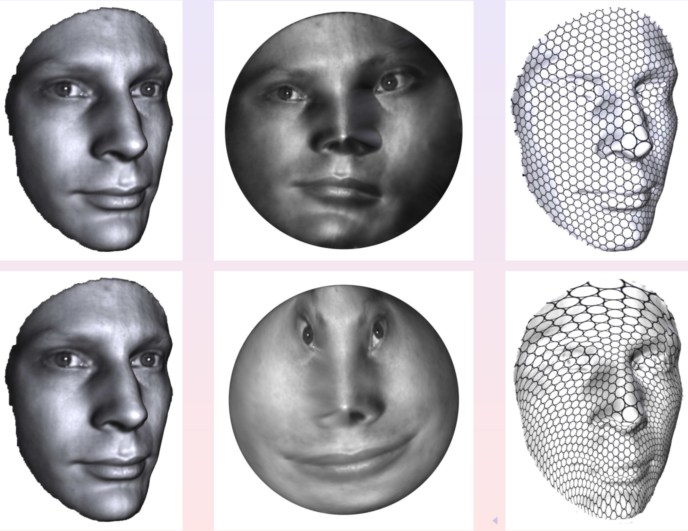
Quasi-Conformal Mappings

Intuition

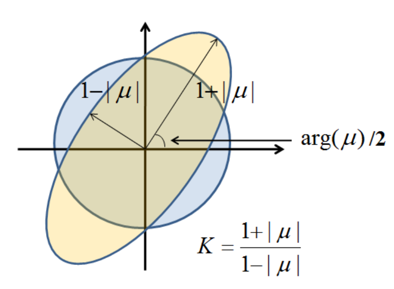
1. All diffeomorphisms between two compact Riemann surfaces are quasi-conformal.
2. Each quasi-conformal mapping corresponds to a unique Beltrami differential.
3. The space of diffeomorphisms equals to the space of all Beltrami differentials.
4. Variational calculus can be carried out on the space of diffeomorphisms.

Quasi-Conformal Map

Most homeomorphisms are quasi-conformal, which maps infinitesimal circles to ellipses.



Beltrami-Equation



Beltrami Coefficient

Let $\phi : S_1 \to S_2$ be the map, z, w are isothermal coordinates of S_1, S_2, Beltrami equation is defined as $\|\mu\|_\infty < 1$

$$\frac{\partial \phi}{\partial \bar{z}} = \mu(z) \frac{\partial \phi}{\partial z}$$

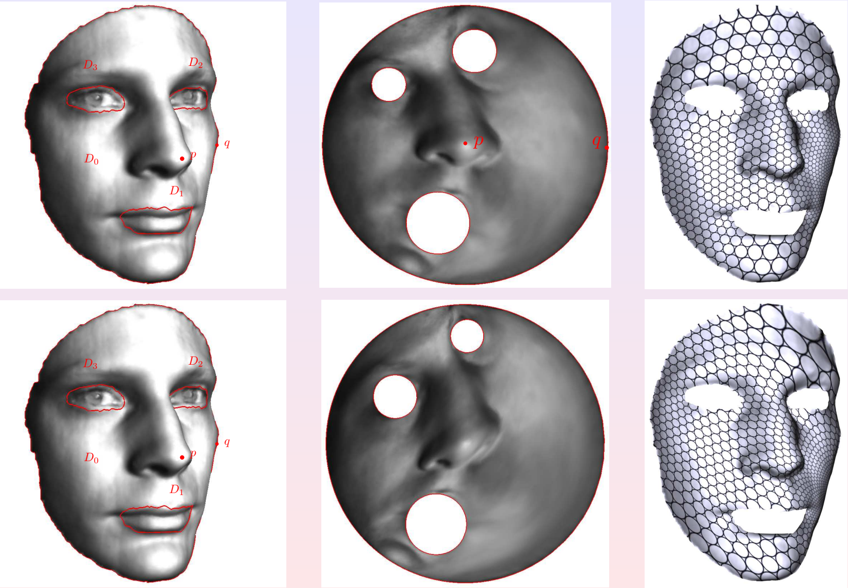
Diffeomorphism Space

Theorem

Given two genus zero metric surface with a single boundary,

$$\{Diffeomorphisms\} \cong \frac{\{Beltrami\ Coefficient\}}{\{Mobius\}}.$$

Quasi-Conformal Map Examples



Let $\{\mu(t)\}$ be a family of Beltrami coefficients depends on a parameter t,

$$\mu(t)(z) = \mu(z) + t\nu(z) + t\varepsilon(t)(z)$$

for $z \in \mathbb{C}$, with suitable μ, $\nu, \varepsilon(t) \in L^{\infty}(\mathbb{C})$,

$$f^{\mu(t)}(w) = f^{\mu}(w) + tV(f^{\mu}, \nu)(w) + o(|t|),$$

where as $t \to 0$,

$$V(f^{\mu}, \nu)(w) = -\frac{f^{\mu}(w)(f^{\mu}(w)-1)}{\pi}\int_{\mathbb{C}}\frac{\nu(z)((f^{\mu})_z(z))^2}{f^{\mu}(z)(f^{\mu}(z)-1)(f^{\mu}(z)-f^{\mu}(w))}dx$$

Solving Beltrami Equation

Landmark-based Registration

Problem

$$\nu^* = argmin_\nu \alpha \int |\nu|^p + \beta \int |\nabla \nu|^2$$

subject to:

1. $\nu^* = \mu(f)$
2. $\|\nu^*\|_\infty < 1$
3. $f(p_i) = q_i$ (landmark constraints).

key ideas

- Beltrami coefficient ν controls conformality distortion $\int |\nu|^p$, smoothness $\int |\nabla \nu|^2$ and bijectivity $\|\nu\|_\infty < 1$.
- Existence of minimizer ν^* has theoretic guaranteed.

Landmark-based Registration

Strategy: Transform the problem to

$$(v^*, f^*) = argmin_{v,f} \alpha \int |v|^p + \beta \int |\nabla v|^2 + \gamma \int |v - \mu(f)|^2$$

subject to:

1. $\|v^*\|_\infty < 1$
2. $f(p_i) = q_i$ (landmark constraints).

key ideas

- Introduce auxiliary variable f
- Alternate optimization over f and v to decouple the minimization

Landmark-based Registration

Algorithm

$$argmin_{v,f} \alpha \int |v|^p + \beta \int |\nabla v|^2 + \gamma \int |v - \mu(f)|^2$$

(A): Fix f_n, minimize

$$\begin{aligned} v_{n+1} &= argmin_v \int \alpha |v|^2 \\ &+ \int \beta |\nabla v|^2 + \int |v - \mu(f_n)|^2 \end{aligned}$$

(B): Fix v_{n+1}, minimize

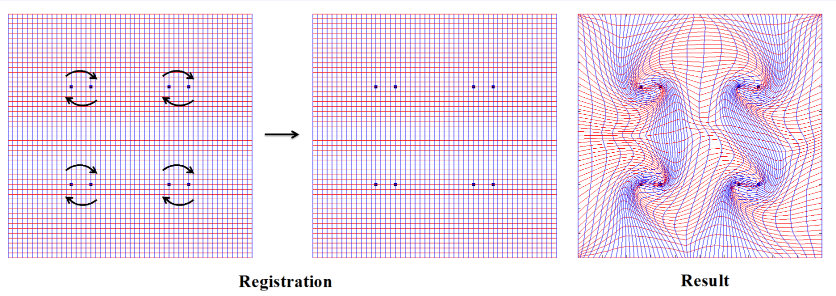
$$f_{n+1} = argmin_f \int |v_{n+1} - \mu(f)|^2$$

Alternatively update v_n and f_n using (A) and (B).

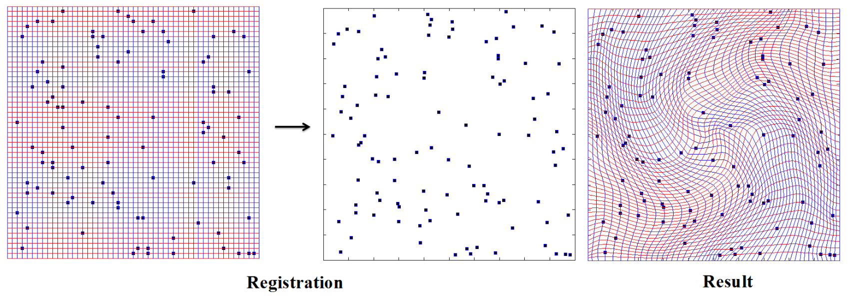
Landmark-based Registration

Algorithm

1. Start with initial Beltrami coefficient ν_0 (set to be 0)
2. Obtain $\nu_{n+1} = argmin_\nu \alpha \int |\nu|^2 + \beta \int |\nabla \nu|^2 + \gamma \int |\nu - \nu_n|^2$
3. Reconstruct f_{n+1} from ν_{n+1} satisfying landmark and boundary constraints.
4. Iterate step 2 and 3 until $|\nu_n - \nu_{n+1}| \leq \varepsilon$ to get the result map $f = f_n$.


Registration
Result

Landmark-based Registration



Applications

Medical Imaging Application

Medical Imaging

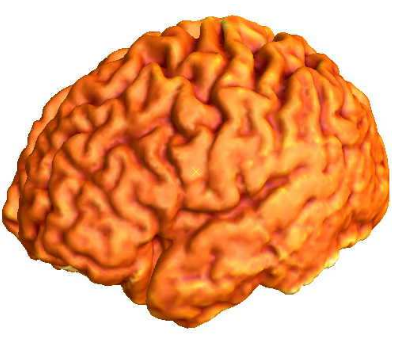
Quantitatively measure and analyze the surface shapes, to detect potential abnormality and illness.

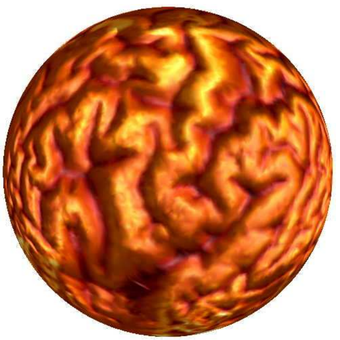
- Shape reconstruction from medical images.
- Compute the geometric features and analyze shapes.
- Shape registration, matching, comparison.
- Shape retrieval.

Conformal Brain Mapping

Brain Cortex Surface

Conformal Brain Mapping for registration, matching, comparison.

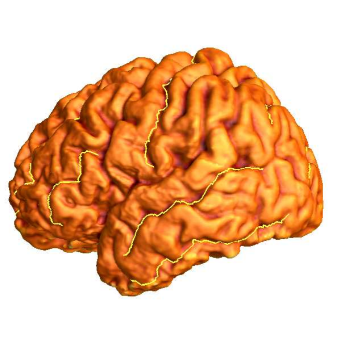


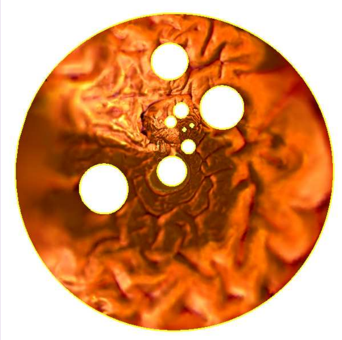


Conformal Brain Mapping

Using conformal module to analyze shape abnormalities.

Brain Cortex Surface

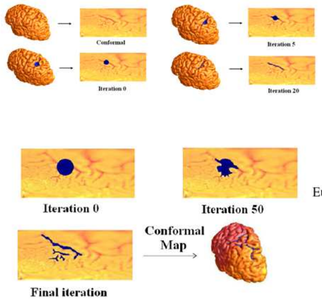




Automatic sulcal landmark Tracking

- With the conformal structure, PDE on Riemann surfaces can be easily solved.
- Chan-Vese segmentation model is generalized to Riemann surfaces to detect sulcal landmarks on the cortical surfaces automatically

Extraction of high mean curvature region by Chan-Vese segmentation



$$F(c_1, c_2, \psi) = \int_S (I_f - c_1)^2 H(\psi) dS + \int_S (I_f - c_2)^2 (1 - H(\psi)) dS + \nu \int_S |\nabla_S H(\psi)| dS$$

$$c_1 = \frac{\int_D I_f \circ \phi(x,y) H(\psi \circ \phi(x,y)) \lambda(x,y) dx dy}{\int_D H(\psi \circ \phi(x,y)) \lambda(x,y) dx dy}$$

$$c_2 = \frac{\int_D I_f \circ \phi(x,y)(1 - H(\psi \circ \phi(x,y))) \lambda(x,y) dx dy}{\int_D (1 - H(\psi \circ \phi(x,y))) \lambda(x,y) dx dy}$$

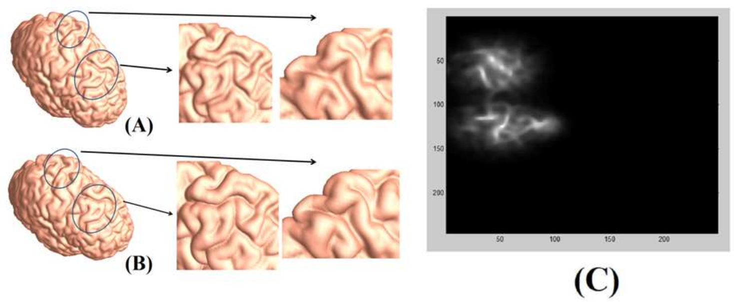
Euler Lagrange equation is:

$$\frac{\partial \psi}{\partial t} = \lambda \delta(\psi)[\ \nu div_S(\frac{\nabla_S \psi}{||\nabla_S \psi||_S}) - (I_f - c_1)^2 - (I_f - c_2)^2] \text{ or}$$

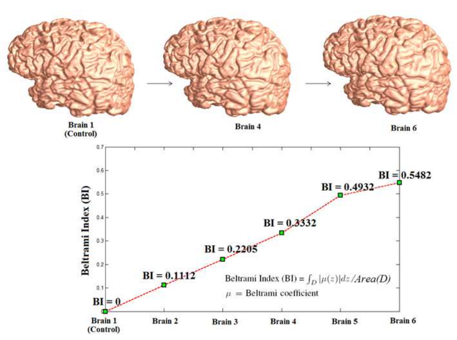
$$\frac{\partial \psi \circ \phi}{\partial t} = \lambda \delta(\psi \circ \phi)[\nu \frac{1}{\lambda} div(\sqrt{\lambda} \frac{\nabla \psi \circ \phi}{|\nabla \psi \circ \phi|}) - (I_f \circ \phi - c_1)^2 - (I_f \circ \phi - c_2)^2]$$

Abnormality detection on brain surfaces

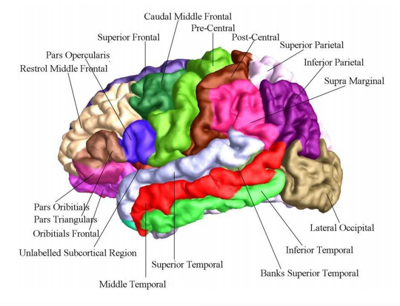
The Beltrami coefficient of the deformation map detects the abnormal deformation on the brain.

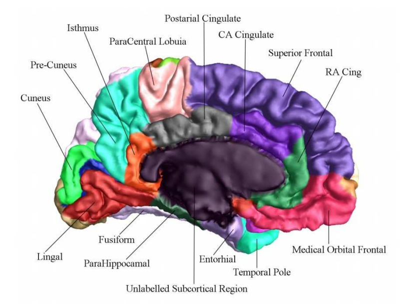


The brain is undergoing gyri thickening (commonly observed in Williams Syndrome) The Beltrami index can effectively measure the gyrification pattern of the brain surface for disease analysis.

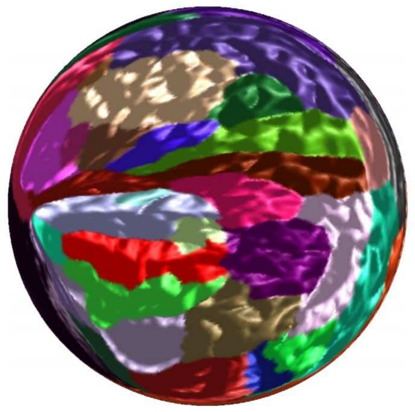


Alzheimer Study



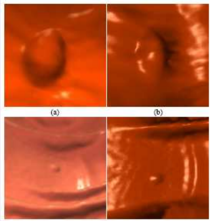


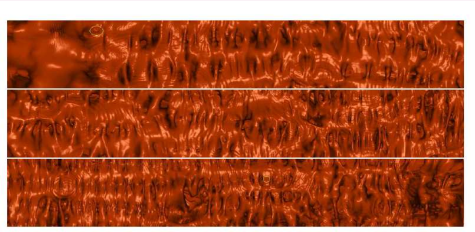
Alzheimer Study



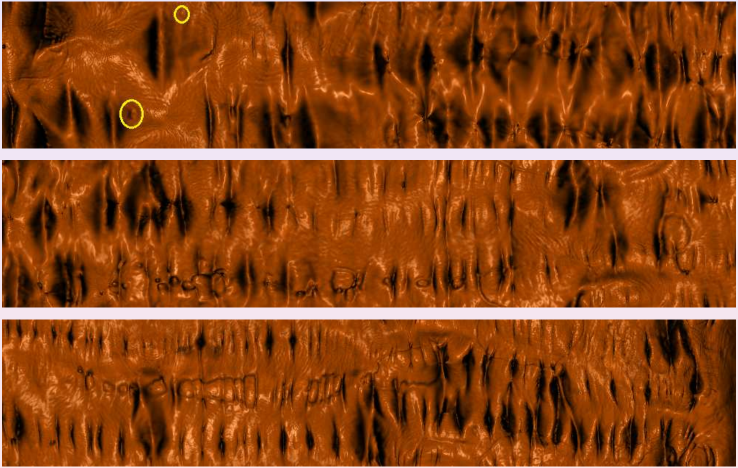
Virtual Colonoscopy

Colon cancer is the 4th killer for American males. Virtual colonosocpy aims at finding polyps, the precursor of cancers. Conformal flattening will unfold the whole surface.





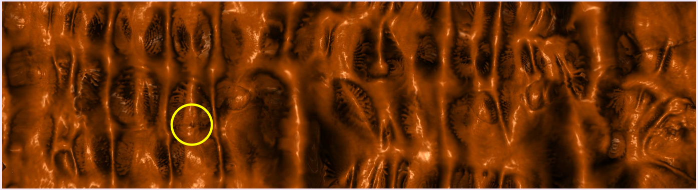
Colon Flattening

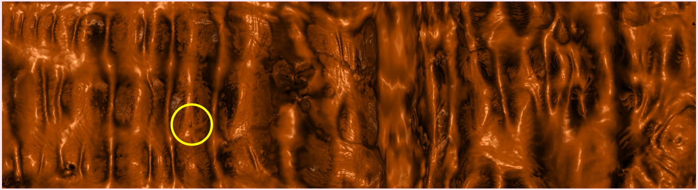


Virtual Colonoscopy

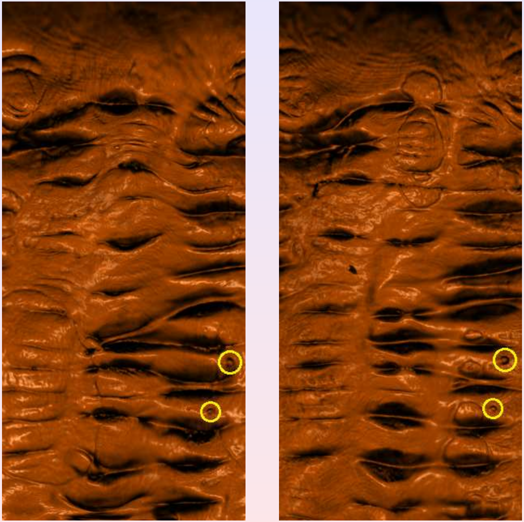
Supine and prone registration. The colon surfaces are scanned twice with different postures, the deformation is not conformal.

Supine and prone registration. The colon surfaces are scanned twice with different postures, the deformation is not conformal.





Colon Registration



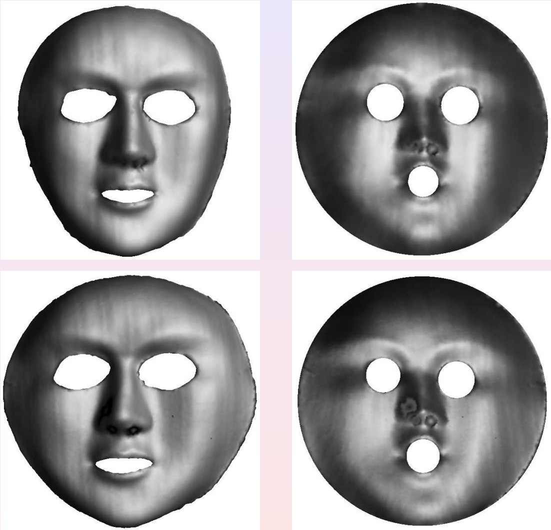
Computer Vision

Vision

- Compute the geometric features and analyze shapes.
- Shape registration, matching, comparison.
- Tracking.

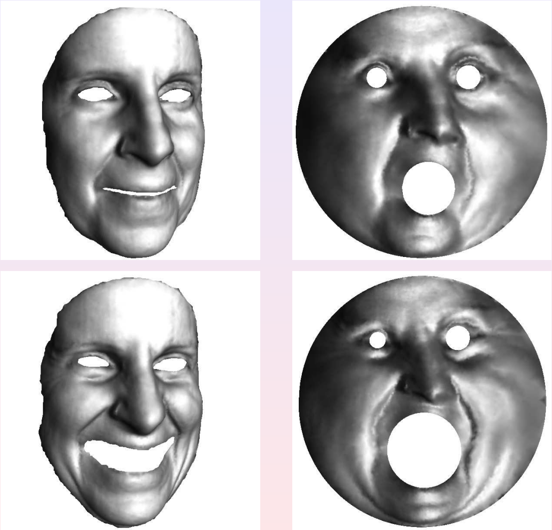
Surface Matching

Isometric deformation is conformal. The mask is bent without stretching.



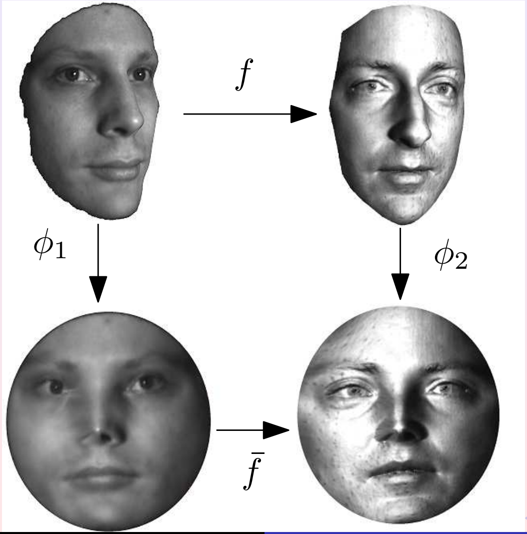
Surface Matching

Facial expression change is not-conformal.

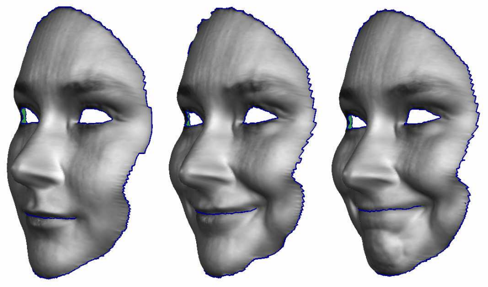


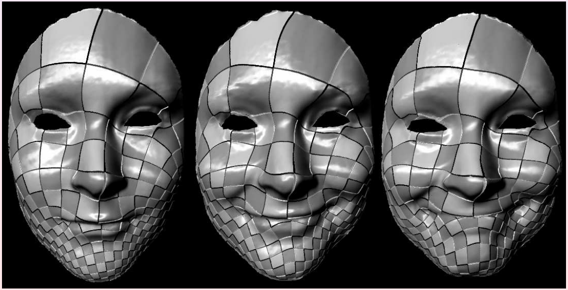
Surface Matching

3D surface matching is converted to image matching by using conformal mappings.

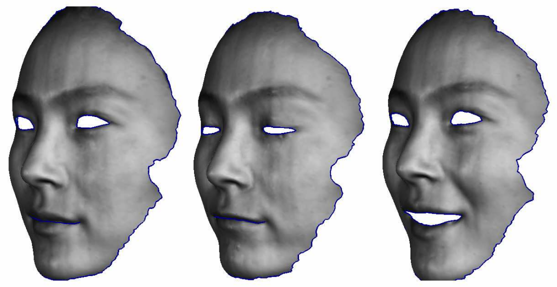


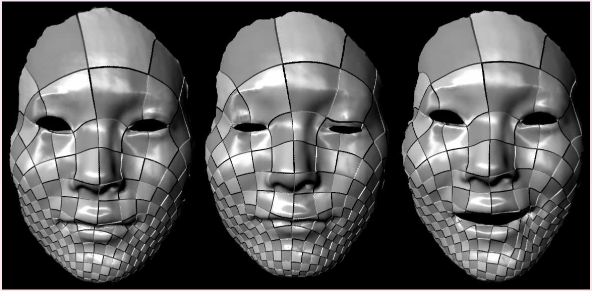
Face Surfaces with Different Expressions are Matched



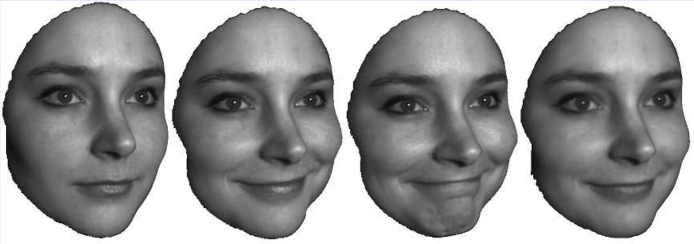


Face Surfaces with Different Expressions are Matched

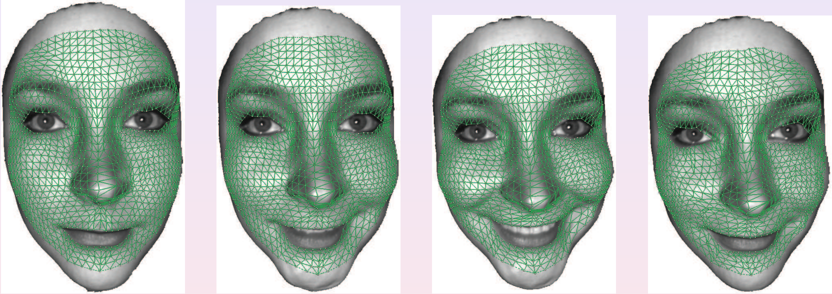




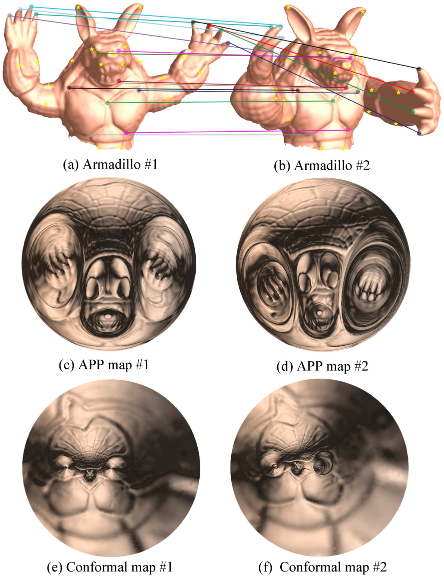
Face Expression Tracking



Face Expression Tracking



Surface Registration

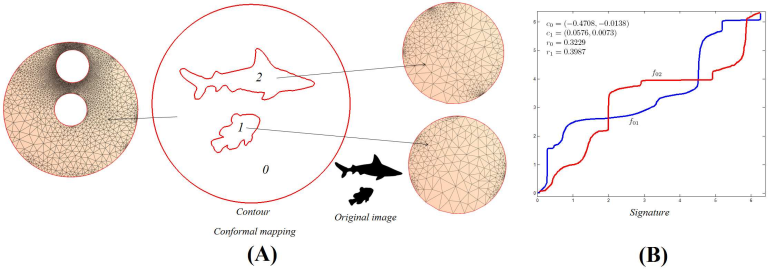


(a) Armadillo #1 (b) Armadillo #2

(c) APP map #1 (d) APP map #2

(e) Conformal map #1 (f) Conformal map #2

$$\{2D\ Contours\} \cong \frac{\{Diffeomorphism\ on\ S^1\} \cup \{Conformal\ Module\}}{\{Mobius\ Transformation\}}$$

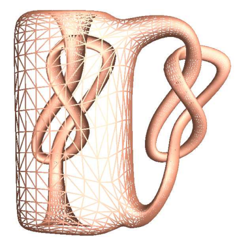


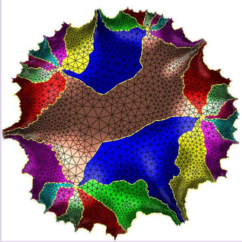
Summary

- Basic concepts in surface differential geometry
- Basic concepts and theories in conformal geometry
- Shape Space - Teichmüller space
- Mapping Space - Quasi-conformal mapping
- Applications

Thanks

For more information, please email to gu@cs.sunysb.edu.





Thank you!