Church and
The probabilistic language of thought

Noah D. Goodman
Stanford University

IPAM graduate summer school
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Motivation

The mind is an information processing system.
Motivation

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But what *kind* of information processor?
Statistics and composition
Statistics and composition

Thought is useful in an uncertain world
Statistics and composition

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Statistics and composition

Thought is useful in an uncertain world

Why did he yell at me?
Statistics and composition

Thought is useful in an uncertain world

Why did he yell at me?

He wanted to hurt me.
He thought I was a telemarketer.
Statistics and composition

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Belief
Desire
Action

Probabilistic inference
Statistics and composition

Probabilistic inference

Thought is useful in an uncertain world
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Thought is productive: “the infinite use of finite means”

Probabilistic inference
Statistics and composition

Thought is useful in an uncertain world

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Probabilistic inference

..a big green bear who loves chocolate..
Thought is useful in an uncertain world

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Statistics and composition

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Probabilistic inference

$p=mv$
Statistics and composition

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Probabilistic inference

Compositional representations
Thought is useful in an uncertain world

Thought is productive: “the infinite use of finite means”

Probabilistic inference

Compositional representations

∀x King(x) ⇒ Man(x)
∀y Man(y) ⇐⇒ ¬Woman(y)
Statistics and composition

Thought is useful in an uncertain world

Probabilistic inference

Thought is productive: “the infinite use of finite means”

Generative models

Rules:

\[ \forall x \text{King}(x) \implies \text{Man}(x) \]
\[ \forall y \text{Man}(y) \iff \neg \text{Woman}(y) \]

Compositional representations

The Language of Thought

Jerry A. Fodor
Statistics and composition

Probabilistic language of thought hypothesis

Thought is useful in an uncertain world

Thought is productive: “the infinite use of finite means”

∀x King(x) ⇒ Man(x)
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Generative models

Compositional representations

Probabilistic inference
Probabilistic LoT

• The **probabilistic** language of thought hypothesis:
  • Mental representations are compositional,
  • Their meaning is probabilistic,
  • They encode generative knowledge,
  • Hence, they support thinking and learning by probabilistic inference.
Probabilistic LoT

- The **probabilistic** language of thought hypothesis:
  - Mental representations are compositional,
  - Their meaning is probabilistic,
  - They encode generative knowledge,
  - Hence, they support thinking and learning by probabilistic inference.

Can this hypothesis be formalized?
Probabilistic generative models

• Mental model of the causal process that gives rise to observations.
Probabilistic generative models

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- E.g. Bayes nets.

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</thead>
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<tr>
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<td>0.7</td>
</tr>
<tr>
<td>no flu</td>
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Probabilistic generative models

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- E.g. Bayes nets.
- But what’s “hidden inside” the nodes and arrows?

![Graph showing relationships between flu, TB, cough, and their probabilities]
Probabilistic generative models

- Mental model of the causal process that gives rise to observations.
- E.g. Bayes nets.
- But what’s “hidden inside” the nodes and arrows?

<table>
<thead>
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<td>TB</td>
</tr>
<tr>
<td>flu</td>
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</tr>
<tr>
<td>no flu</td>
<td>0.7</td>
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<table>
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<tr>
<th>Action</th>
<th>Desire</th>
<th>Belief</th>
</tr>
</thead>
<tbody>
<tr>
<td>going to IPAM</td>
<td>0.7</td>
<td>0.05</td>
</tr>
<tr>
<td>it’s raining</td>
<td>0.002</td>
<td>0.004</td>
</tr>
</tbody>
</table>

Flu:cough:TB

Belief:Desire:Action
\( \lambda \) calculus

- **Notation:**
  - Function have parentheses on the wrong side: \((\sin x)\)
  - Operators always go at the beginning: \((+ x y)\)

- \( \lambda \) makes functions, define binds values to symbols:

\[
\begin{align*}
\text{(define } \textbf{double} & \text{ (} \lambda (x) (+ x x) \text{))} \\
\text{(define } \textbf{repeat} & \text{ (} \lambda (f) (\lambda (x) (f (f x))) \text{))} \\
\text{(define } \textbf{2nd-derivative} & \text{ (} \textbf{repeat} \textbf{ derivative} \text{))}
\end{align*}
\]
Random primitives:

(define a (flip 0.3))  =>  1 0 0
(define b (flip 0.3))  =>  0 0 0
(define c (flip 0.3))  =>  1 0 1
(+ a b c)  =>  2 0 1...

Conditioning (inference):

(query
  (define a (flip 0.3))
  (define b (flip 0.3))
  (define c (flip 0.3))
  (+ a b c)
  (= (+ a b) 1))

Query
Condition, must be true

Goodman, Mansinghka, Roy, Bonawitz, Tenenabum (2008)
PLoT

• Formalizing the PLoT:
  Mental representations (concepts) are functions in a *stochastic* lambda calculus (e.g. Church).

http://projects.csail.mit.edu/church
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...or pieces of probabilistic programs, anyhow.

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PLoT

• Formalizing the PLoT: Mental representations (concepts) are functions in a **stochastic** lambda calculus (e.g. Church).

• ...or pieces of probabilistic programs, anyhow.

• Separate the process of inference from representations and the inferences they license (Cf. Marr’s levels).

http://projects.csail.mit.edu/church
Outline (next 3 lectures)

- Church: language design considerations.
- Church: inference techniques.
- Social cognition.
- Natural language pragmatics (etc).
- Concept learning as program induction.
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- Church: language design considerations.
- Church: inference techniques.
- Social cognition.
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Church language design

- Start with (pure subset of) LISP.
- Add random primitives.
- Add `mem`.
- Add `query`.
Why functions?

• Composition of probabilistic functions represents *directed* generative models.

• Captures the intuition that much human knowledge is about causal process.

  • For example, explaining away: conditioning an undirected model can introduce new independence but not new dependence.

• Hierarchical models, etc.
Undirected modeling?

- Complex conditions in query represent (any) undirected constraints.
- For example an Ising model:

```scheme
(query
  (define a (flip))
  (define b (flip))
  (define c (flip))

  (list a b c)

  (and (flip (if (equal? a b)) 1.0 0.3)
       (flip (if (equal? b c)) 1.0 0.3)))
```
Undirected modeling?

- Complex conditions in query represent (any) undirected constraints.
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```

\[ e^{\ln(0.3)\delta_{a=b}} \]
Undirected modeling?

• Complex conditions in query represent (any) undirected constraints.

• For example Markov Logic:

```lisp
(query
  (define world (repeat N flip))
  world
  (and
    (log-flip (if (clause1 world) 0.0 weight1)))
  (log-flip (if (clause2 world) 0.0 weight2)))
...))
```
Undirected modeling?

• Complex conditions in query represent (any) undirected constraints.

• So, Church can represent directed, undirected, and hybrid models.

• Complex conditions can be built out of directed pieces.
  (This seems important for natural language. Cf. Montegue.)
Why functional?

• First class abstraction.

• The pieces needed for model specification are representable in the modeling language.

• Abstraction permits domain specific languages -- changes the effective language of thought.

• Higher-order functions are useful. (E.g. the higher-order distribution DPmem.)

• Recursion.

• Models are parsimonious.
Why LISP?

- Small core language.
- Simple semantics.
- Very little syntax.
- Simple type system.
- Code as data (meta-circular, homoiconic).
- Well-worked out concepts for computation (see SICP).
Random primitives

• Just the fair-coin $\text{flip}$ primitive is enough ($\psi\lambda$-calculus),

• But it is convenient (and well-formed) to add other random primitives.

• An ERP is a primitive (with arguments) that can sample values and score.

• For each set of arguments, multiple samples from an ERP must be independent identically distributed (iid)

• Examples: $\text{flip}, \text{sample-discrete}, \text{gaussian}$. 
Exchangeability

• If all the ERPs return iid samples, then is any Church thunk (procedure without arguments) an iid distribution?

• No. Consider:

• But they are exchangeable.

• Theorem (Freer & Roy, 2010):
  Any church thunk is exchangeable and any exchangeable distribution may be represented by a Church thunk.

```
(define weight (beta 1 1))
(define (mycoin) (flip weight))
(repeat 10 mycoin) => (t t t t t t t t t t)
```
Random primitives

• Relax the interface for random primitives: an XRP can sample and score. Multiple calls exchangeable.

• Examples:
  • Uniform draw without replacement.
  • Gensym.
  • Beta-binomial.
  • CRP.
Memoization

- The `mem` primitive memoizes a procedure.
- This changes the semantics (unlike in deterministic languages).
- Interacts with symbols and `gensym`, to enable “BLOG style” style world models.

```scheme
(define mem-flip (mem flip))
(= (mem-flip) (mem-flip))

True with probability 0.5
True with probability 1.0
```

```scheme
(define people (repeat (poisson 1.0) gensym))
(define eye-color
  (mem (λ (person) (uniform-draw '(blue brown))))))
(define blue-eyed-people
  (filter (λ (person) (equal? 'blue (eye-color person)))
          people))
```
Stochastic memoization

- **DPmem**: stochastically reuse return values from functions.
- A *higher-order* distribution: it takes a stochastic function (of any type) and samples a new function (of same type) that concentrates the original.
- Adapts a generative process to balance reuse with re-computation (see Johnson, O’Donnell).
- Implemented in Church via stick-breaking or using an XRP.

```scheme
(define mem-flip (DPmem 1.0 flip))
(= (mem-flip) (mem-flip))
```
(define draw-class
  (DPmem 1.0 gensym))
(define class
  (mem (lambda (obj) (draw-class)))
(define class-weight
  (mem (lambda (obj-class feature)
    (beta 1.0 1.0)))))
(define observe-feature
  (lambda (obj feature)
    (flip (class-weight (class obj) feature))))
**DP mixture model**

A pool of cluster symbols with DP prior.

<table>
<thead>
<tr>
<th>Code</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>(define draw-class (DPmem 1.0 gensym))</code></td>
<td>The cluster of an object and cluster parameters are persistent.</td>
</tr>
<tr>
<td><code>(define class (mem (lambda (obj) (draw-class)))))</code></td>
<td></td>
</tr>
<tr>
<td><code>(define class-weight (mem (lambda (obj-class feature) (beta 1.0 1.0)))))</code></td>
<td></td>
</tr>
<tr>
<td><code>(define observe-feature (lambda (obj feature) (flip (class-weight (class obj) feature)))))</code></td>
<td></td>
</tr>
</tbody>
</table>
Infinite relational model

The IRM model from: Kemp, et al, 2006

(define draw-class
  (DPmem 1.0 (lambda (obj-domain) (gensym)))))

(define class
  (mem (lambda (obj) (draw-class (domain obj)))))

(define class-weight
  (mem (lambda (obj-class1 obj-class2)
    (beta 1.0 1.0)))))

(define observe-relation
  (lambda (obj1 obj2)
    (flip (class-weight (class obj1) (class obj2))))))
Infinite relational model

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(define draw-class
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```
(define category
  (DPmem 1.0
    (λ (parent-category)
      (pair (gensym) parent-category))))

(define (category-hierarchy N)
  (if (= N 0)
    '(top)
    (category (category-hierarchy (- N 1))))

(category-hierarchy 3) x4...
(define category
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(define (category-hierarchy N)
  (if (= N 0)
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      (category (category-hierarchy (- N 1))))))

(category-hierarchy 3) →

=> ((g0 g1 g2 top)
    (g3 g1 g2 top)
    (g4 g1 g2 top)
    (g5 g6 g7 top))
(define draw-state (DPmem 1.0 gensym))
(define transition (DPmem 1.0 (λ (state) (draw-state))))
(define obsfn
  (mem (λ (state) (make-dirichlet-multi terminals))))

(define (hdp-hmm state N)
  (if (= N 0)
      '() (pair ((obsfn state))
         (hdp-hmm (transition state) (- N 1)))))
first-class query

• How should we specify observations in probabilistic programs?
  • Separate model from data / data variables?
  • ‘Observe’ statements within model?
  • Better: make inference an ordinary function.
    • Can define inference within the language (e.g. by rejection).
    • Gives proper scoping, complex conditions, nested query (‘inference about inference’).
Nested query

Alice and Bob arrange to meet at the bar, each later realizes they didn’t fix which bar and must guess where to meet.
(A coordination game, see Schelling, 1960)

Recursive social inference:

(define (sample-location)
  (if (flip .55) 'popular-bar 'unpopular-bar))

(define (bob depth)
  (query
    (define bob-location (sample-location))
    bob-location
    (equal? bob-location (alice (- depth 1))))

(define (alice depth)
  (query
    (define alice-location (sample-location))
    alice-location
    (or (= depth 0)
      (equal? alice-location (bob depth))))
Nested query

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Figure : A simple coordination game. Two agents reason about each other in order to coordinate where to meet. As we increase the depth of recursive reasoning, the probability of meeting at the more popular bar—the Schelling point of this game—converges to 1.

<table>
<thead>
<tr>
<th>Depth of recursion</th>
<th>Prob. Unpopular bar</th>
<th>Prob. Popular bar</th>
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<tbody>
<tr>
<td>0</td>
<td>0.9854</td>
<td>0.0146</td>
</tr>
<tr>
<td>1</td>
<td>0.9783</td>
<td>0.0217</td>
</tr>
<tr>
<td>2</td>
<td>0.9681</td>
<td>0.0319</td>
</tr>
<tr>
<td>3</td>
<td>0.9314</td>
<td>0.0686</td>
</tr>
<tr>
<td>4</td>
<td>0.9009</td>
<td>0.0991</td>
</tr>
<tr>
<td>5</td>
<td>0.8589</td>
<td>0.1411</td>
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<td>6</td>
<td>0.8030</td>
<td>0.1971</td>
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<tr>
<td>7</td>
<td>0.7317</td>
<td>0.2683</td>
</tr>
<tr>
<td>8</td>
<td>0.6461</td>
<td>0.3193</td>
</tr>
<tr>
<td>9</td>
<td>0.55</td>
<td>0.4407</td>
</tr>
<tr>
<td>10</td>
<td>0.45</td>
<td>0.55</td>
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Nested query

Policy-free MDP planning as a recursively optimal action selection.

(define (choose-action state)
  (query
    (define action (action-prior))
    action
    (flip (normalize-reward
      (sample-reward action state)))))

(define (sample-reward action state)
  (let ((next-state (state-transition state action)))
    (+ (reward next-state)
      (if (terminal? next-state)
          0
          (sample-reward
            (choose-action next-state)
            next-state)))))
Church vs. ...

- **Bugs**: Less expressive language for directed models. More stable engine.

- **BLOG**: Nice constructs for unknown objects. No abstraction.

- **IBAL (etc)**: Similar. No continuous variables, mem, xrps. Obs not query.

- **MLN**: Undirected, based on FoL. Good implementations.

- **Csoft, Factorie, Figaro, Hansei, ProbLog**...
Church inference

- Universal representation (PLoT) needs universal inference.
- The mind can’t have a separate algorithm for every task.
- Rejection sampling is sound, but slow.
- Efficient universal inference?
Metropolis-Hastings

To sample from $P(x)$, repeatedly:

- propose $x'$ from $Q(x \to x')$,
- accept with probability: $\min\left(1, \frac{P(x)Q(x \to x')}{P(x')Q(x' \to x)}\right)$
- This converges in distribution to $P(x)$ (even if $P(x)$ isn't normalized).
MH for Church

• What is a state?
  • The set of random choices made in executing (forward) the query expression.

• How can choices be individuated?
  • Not by evaluation order (can change)…
What is a state?

The set of random choices made in executing (forward) the query expression.

How can choices be individuated?

Not by evaluation order (can change)...
MH for Church

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(\text{and}
  (\text{if} (\text{flip})
    (\text{flip})
    \text{true})
  (\text{flip}))
MH for Church

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```lisp
(and
  (if (flip)
    (flip)
    true)
  (flip))
```

1: T
2: F
MH for Church

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(\textbf{and} \\
\textbf{(if (flip)} \\
\textbf{(flip)} \\
\textbf{true)} \\
\textbf{(flip)})

\begin{itemize}
\item 1:T
\item 2:F
\item 3:T
\item 1:F
\end{itemize}
MH for Church

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(\text{and} \\
(\text{if} \ (\text{flip}) \\
(\text{flip}) \\
\text{true}) \\
(\text{flip}))

1:T  2:F  3:T

1:F  2:F
What is a state?

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Named via dynamic calling address:

MH for Church
MH for Church

• What is a state?

• The set of random choices made in executing (forward) the query expression.

• How can choices be individuated?

• Not by evaluation order (can change)...

• Named via dynamic calling address:

$$A^{top}[E] = ((\lambda (addr) \ A[E]) \ 'top)$$

$$A[\lambda (addr \ (I^n_{i=1}) \ E_{body})] = (\lambda (addr \ . I^n_{i=1}) \ A[E_{body}])$$

where $S$ is a globally unique symbol.

$$A[(\)mem \ E] = (\lambda (maddr \ f) \ (\lambda (addr \ . args) \ (apply \ f \ (cons \ args \ maddr) \ args))) \ addr \ A[E]$$

$$A[\begin{array}{l}
\text{begin} \ E_{i=1}^n \\
\text{letrec} \ (\{I_i \ E_i\}_{i=1}^n) \ E_{body}
\end{array}] = \begin{array}{l}
\text{begin} \ A[E_{i=1}^n] \\
\text{letrec} \ (\{I_i \ A[E_i]\}_{i=1}^n) \ A[E_{body}]
\end{array}$$

$$A[\begin{array}{l}
\text{if} \ E_t \ E_c \ E_a \\
\text{define} \ I \ E \\
\text{quote} \ E
\end{array}] = \begin{array}{l}
\end{array}$$

$$A[\\begin{array}{l}
\text{op} \ E_{i=1}^n \\
\text{cons} \ S \ addr
\end{array}] = (\ A[\text{op}] \ (\text{cons} \ S \ addr) \ A[E_{i=1}^n]$$

$$A[E] = E, \text{otherwise.}$$
MH for Church

- What is a state?
- The set of random choices made in executing (forward) the query expression.
- How can choices be individuated?
- Not by evaluation order (can change)...
- Named via dynamic calling address:

\[
\begin{align*}
A^{top}[E] &= ((\lambda \text{addr} \ A[E]) \ '\text{top}') \\
A[(\lambda (I_i) \ E_{body})] &= (\lambda \text{addr}. I_i) \\
&\quad \text{where } S \text{ is a globally unique symbol.}
A[(\text{mem } E)] &= (\lambda \text{maddr } f) \ (\lambda \text{addr} \ A[E])
A[(\text{begin } E_i)] &= (\text{begin } A[E_i])
A[(\text{letrec } (I_i E_i) \ E_{body})] &= (\text{letrec } (I_i E_i)
A[(\text{if } E_t \ E_c \ E_a)] &= (\text{if } A[E_t] A[E_c] A[E_a])
A[(\text{define } I \ E)] &= (\text{define } I \ A[E])
A[(\text{quote } E)] &= (\text{quote } E)
A[(E_{op} E_{i=1})] &= (A[E_{op}] \ (\text{cons } S \text{ addr}) A[E_i])
A[E] &= E, \text{otherwise.}
\end{align*}
\]

(begin
  (define geometric
    (lambda (p)
      (if (flip p)
          1
          (+ 1 (geometric p)))))
(geometric .7))

\[
\begin{align*}
A^{top}[E] &= ((\lambda \text{addr} \ A[E]) \ '\text{top}') \\
A[(\lambda (I_i) \ E_{body})] &= (\lambda \text{addr}. I_i) \\
&\quad \text{where } S \text{ is a globally unique symbol.}
A[(\text{mem } E)] &= (\lambda \text{maddr } f) \ (\lambda \text{addr} \ A[E])
A[(\text{begin } E_i)] &= (\text{begin } A[E_i])
A[(\text{letrec } (I_i E_i) \ E_{body})] &= (\text{letrec } (I_i E_i)
A[(\text{if } E_t \ E_c \ E_a)] &= (\text{if } A[E_t] A[E_c] A[E_a])
A[(\text{define } I \ E)] &= (\text{define } I \ A[E])
A[(\text{quote } E)] &= (\text{quote } E)
A[(E_{op} E_{i=1})] &= (A[E_{op}] \ (\text{cons } S \text{ addr}) A[E_i])
A[E] &= E, \text{otherwise.}
\end{align*}
\]

\[
\begin{align*}
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A[(\text{begin } E_i)] &= (\text{begin } A[E_i])
A[(\text{letrec } (I_i E_i) \ E_{body})] &= (\text{letrec } (I_i E_i)
A[(\text{if } E_t \ E_c \ E_a)] &= (\text{if } A[E_t] A[E_c] A[E_a])
A[(\text{define } I \ E)] &= (\text{define } I \ A[E])
A[(\text{quote } E)] &= (\text{quote } E)
A[(E_{op} E_{i=1})] &= (A[E_{op}] \ (\text{cons } S \text{ addr}) A[E_i])
A[E] &= E, \text{otherwise.}
\end{align*}
\]
MH for Church

- MH over ‘execution traces’:
  - propose: change a single random choice,
  - trace update: re-execute to update the state, reusing existing choices (new choices from conditional prior),
  - collect, the (now) unused choices,
  - check condition and compute score,
  - accept/reject.

- Initial state must satisfy condition (rejection, annealing, constraint propagation).
MH for Church

• Very similar to BLOG algorithm.
  • Evaluation automatically instantiates minimal “partial world” (but cf. lazy evaluation).

• “Lightweight” technique, addressing by program transformation, works for almost any programming language -- enter Stochastic X.

• Current implementation doesn’t scale well (due to software engineering issues).

Goodman, et al. (2008)
Wingate, Stuhlmueller, Goodman (2011)
Hamiltonian Monte Carlo

- Add an auxiliary “momentum” variable for each continuous variable.

- Hamiltonian evolution is reversible and conserves probability (hence satisfies detailed balance).

- Numeric integration introduces conservation errors, so add an MH accept step.

- Need the gradient of the score.

\[
\phi(x) = -\ln(P(x))
\]

\[
H(x, p) = \phi(x) + mp^2
\]

\[
\frac{dx}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}
\]
Automatic differentiation

• Since programs are compositions of functions, can compute derivatives automatically (remember the chain rule?)

• A “non-standard interpretation” -- implemented by operator overloading.

• Exact (to machine precision) gradient of score at each point. (No silly errors!)

• Low overhead (roughly 2x in Bher).

• Generalizes back-prop.
HMC via AD

- We use AD to provide gradients needed in HMC.
- HMC can be much more efficient.
- Sampling an eccentric 3-dim gaussian:

![Graph showing distance to true expectation vs. samples for MCMC and HMC](image)

Wingate, Stuhlmueller, Siskind, Goodman (under review)
HMC via AD

• We use AD to provide gradients needed in HMC.

• Conditioning Perlin noise on symmetry:

~1000 dims

Wingate, Stuhlmueller, Siskind, Goodman (under review)
Other techniques

- Sequential Monte Carlo
  - Particle filter with rejuvenation,
  - Any model, and any sequentialization (specified via free variable in query).

- Dynamic programming (UDP/cosh)
  - Inference as marginalization,
  - Builds compact system of polynomial equations...
Other techniques

Factored coroutine

Original program

(define (game player)
  (if (flip .6)
      (not (game (not player)))
      (if player
          (flip .2)
          (flip .7))))
(game true)

Factored computation graph

Subproblem dependency graph

Subproblem component DAG

Systems of polynomial equations

Marginal probabilities

Marginal probabilities

true
false

Subproblem component DAG
Many directions open..

I WANT YOU
To work on Church inference algorithms
• Structured models do more than specify distributions. See Pearl, next.

• Can build a counterfactual operator in Church from the same “update” operator needed for MH.

(counterfactual
  (define dad-eyes (uniform-draw '(blue brown)))
  (define mom-eyes (uniform-draw '(blue brown)))
  (if (or (eq? dad-eyes 'brown)
           (eq? mom-eyes 'brown)
           (flip 0.1))
    'brown
    'blue)) => (blue . brown)
Conclusion

• The PLoT: mental representations as stochastic functions.

• Church: an expressive probabilistic programming language suited for cognitive modeling.

• Next lectures: applications to social cognition, natural language, concept learning.