The Stochastic Box-Ball System

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IPAM: Workshop IV

Outline

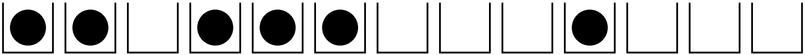
1. Box-Ball System
   - Carrier dynamics
   - Relationship with integrable vertex models
   - Soliton decomposition
2. Stochastic Box-Ball System
   - Definition and examples
   - Result on pair formation
   - Work in progress: scaling limit

Part 1: The Box-Ball System

The box-ball system

The box-ball system is a discrete-time, deterministic dynamical system introduce by Takahashi-Satsuma in 1990:

- Suppose we have a box at every positive integer.
- A finite number d of the boxes contain a single ball, the rest are empty (d-ball system). Record this as a binary sequence $X : \mathbb{N} \to \{0, 1\}$.
- At each step of the dynamics, a “carrier” sweeps across the integers and transports the balls according to certain rules.

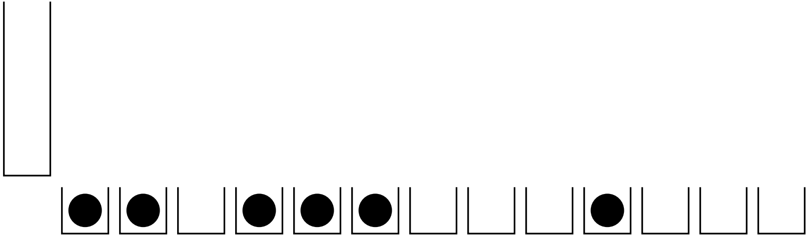


$$X = (1, 1, 0, 1, 1, 1, 0, 0, 0, 1, 0, 0, 0, \ldots)$$

The box-ball system

The carrier

- Sweeps from left to right
- Whenever it reaches a box with a ball, it picks up the ball.
- If it reaches an empty box while holding at least one ball, it drops a ball in the box.



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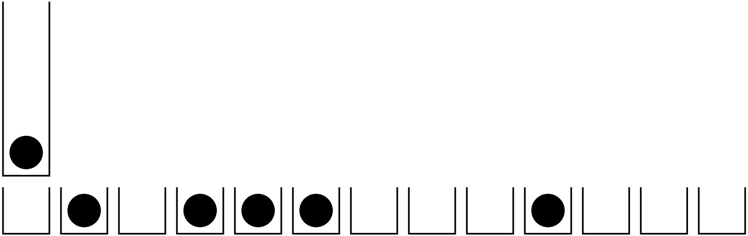

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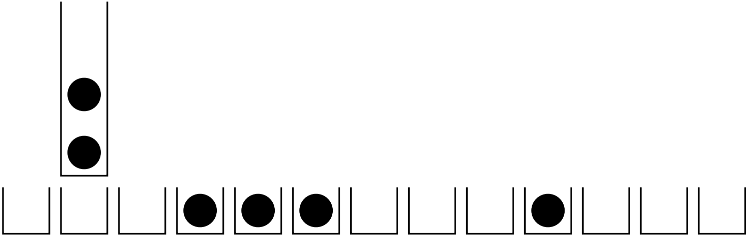

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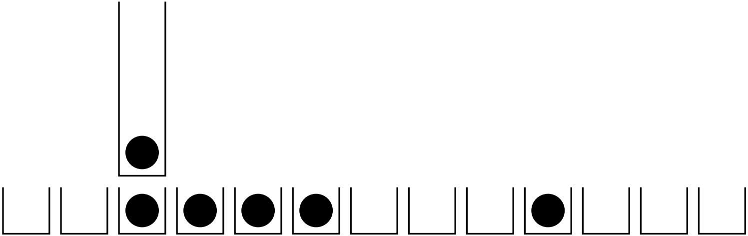

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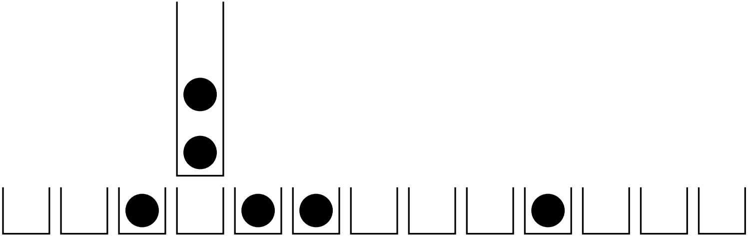

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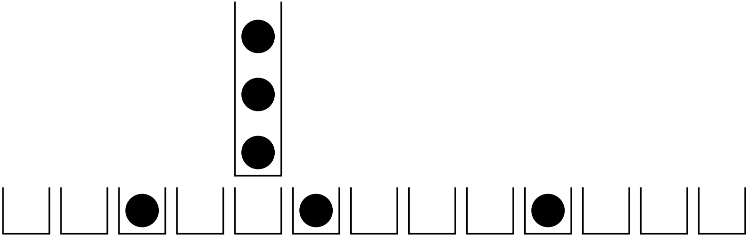

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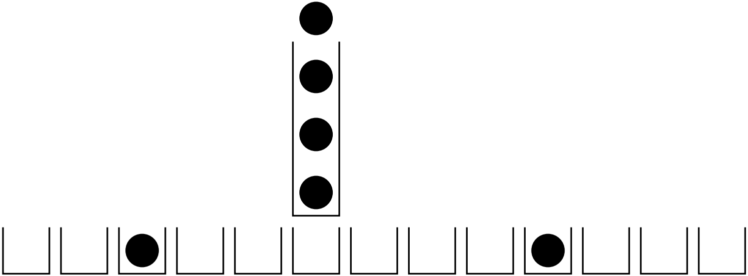

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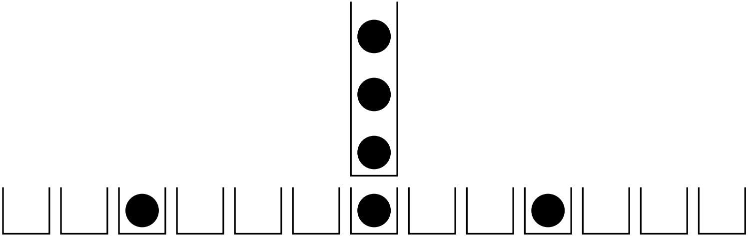

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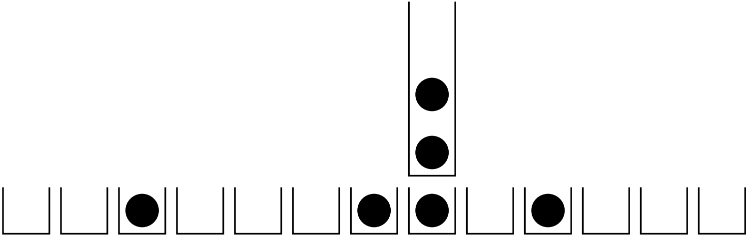

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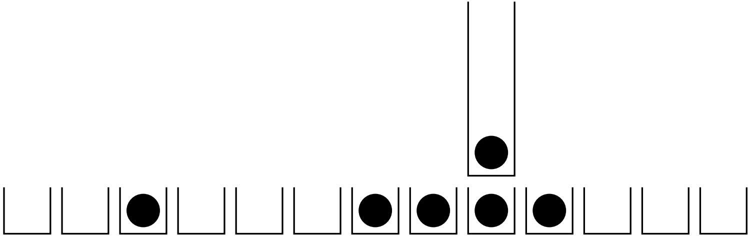

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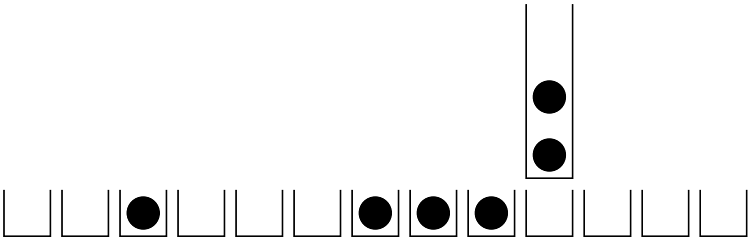

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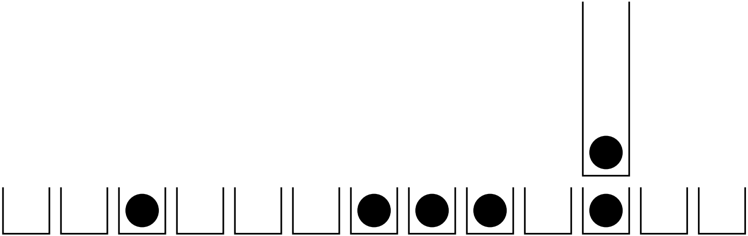

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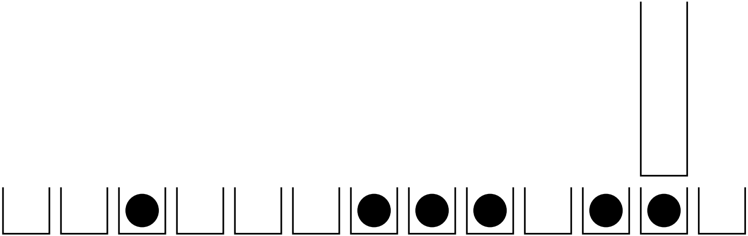

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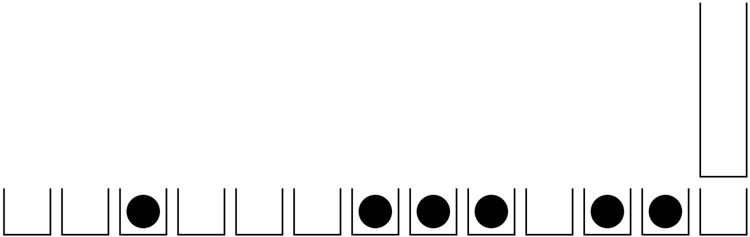
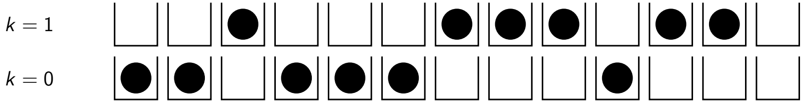


The box-ball system

The carrier

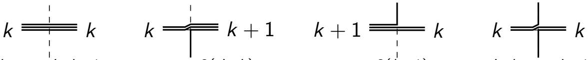
- Sweeps from left to right
- Whenever it reaches a box with a ball, it picks up the ball.
- If it reaches an empty box while holding at least one ball, it drops a ball in the box.
- The full carrier sweep is one step of the dynamics.



The box-ball system

Equivalently, we can express this via a vertex model:
Suppose we have a d-ball system. Consider a vertex model such that

- spin-$\frac{1}{2}$ on the vertical edges
- spin-$\frac{d}{2}$ on the horizontal edges
- Weights given by:

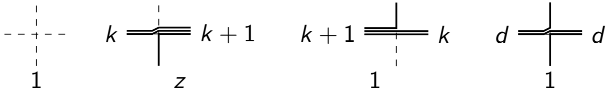


$$q^k - q^{d-k+1}z \qquad (1-q^{2(d-k)})z \qquad 1-q^{2(k+1)} \qquad q^{d-k} - q^{k+1}z$$

- The vertex model is Yang-Baxter integrable.

The box-ball system

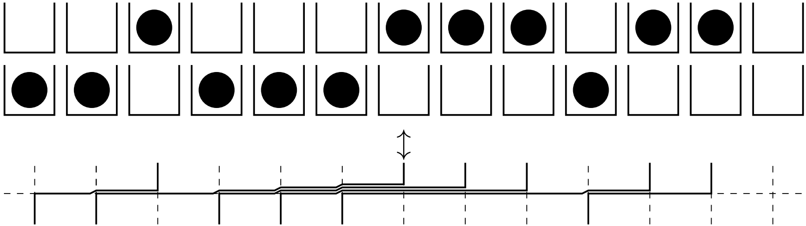
Taking the limit $q \to 0$ the only non-zero weights belong to vertices of the form:



"Combinatorial R matrix"

The box-ball system

- A single step in the dynamics corresponds to a row of the vertex model.
- The number of horizontal paths on an edge corresponds to the number of balls in the carrier.



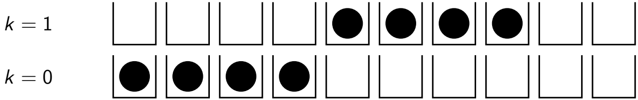
Rich world of connections between the box-ball system and integrable vertex models (Bethe ansatz, KKR bijection, etc.)

- A nice review article: R. Inoue, A. Kuniba, T. Takagi 2012

The box-ball system

The box-ball system exhibits solitons:

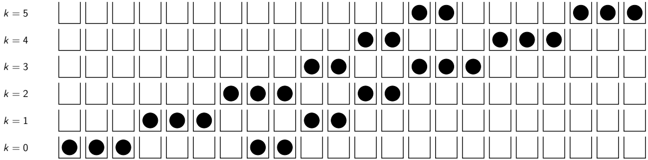
- A soliton of length ℓ is a contiguous interval of ℓ balls.
- It will move at speed ℓ under the dynamics.



The box-ball system

The box-ball system exhibits solitons:

- The shape and speed is maintained after collisions with other solitons.



Collision of a 3-soliton and a 2-soliton.

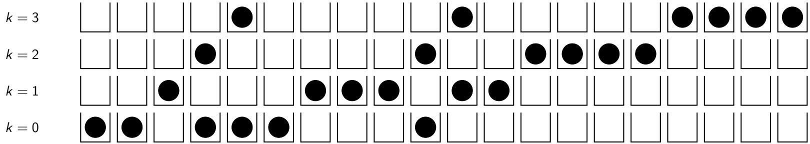
- Discrete limit of the KdV equation

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

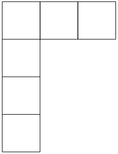
for which solitons take the form of solitary wave solutions.

The box-ball system

After running the dynamics for a sufficiently long time, the ball configuration will decompose into solitons of weakly increasing length (looking left-to-right).



- Decomposes into solitons of length 4, 1, and 1. Conserved quantity of the dynamics.

Young diagram: 

The box-ball system

One way to add randomness to this system is by choosing a random initial distribution of balls. Many works in this direction:

Soliton decomposition:

- A. Kuniba, H. Lyu, M. Okado 2018
- L. Levine, H. Lyu, J. Pike 2020
- J. Lewis, H. Lyu, P. Pylyavskyy, A. Sen 2023

Stationary measures:

- P. Ferrari, C. Nguyen, L. Rolla, M. Wang 2021
- P. Ferrari, D. Gabrielli 2020
- D.A. Croydon, T. Kato, M. Sasada, S. Tsujimoto 2023

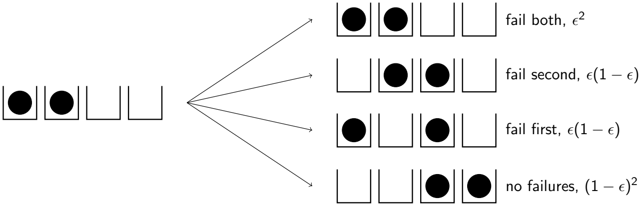
We'll study a different means of adding randomness.

Part 2: The Stochastic Box-Ball System

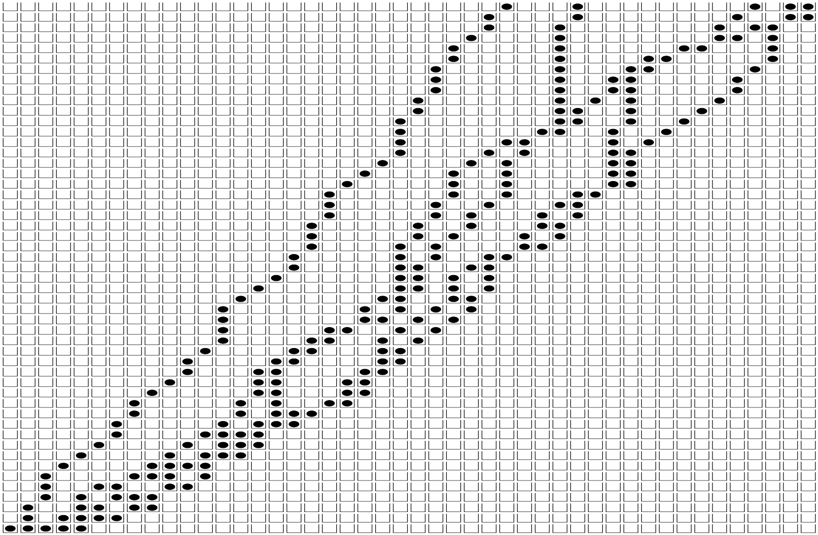
The stochastic box-ball system

Another way to add randomness:

- Let the number of balls to be d and fix $\epsilon \in (0, 1)$.
- Suppose that the carrier is now faulty: with probability ϵ it fails to pick up a ball. If it fails, the ball is left in place and the sweep continues.
- This failure chance is applied independently to each ball.



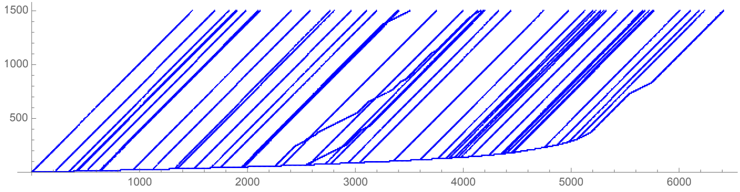
The stochastic box-ball system



5 balls, 50 steps, $\epsilon = 0.5$

The stochastic box-ball system

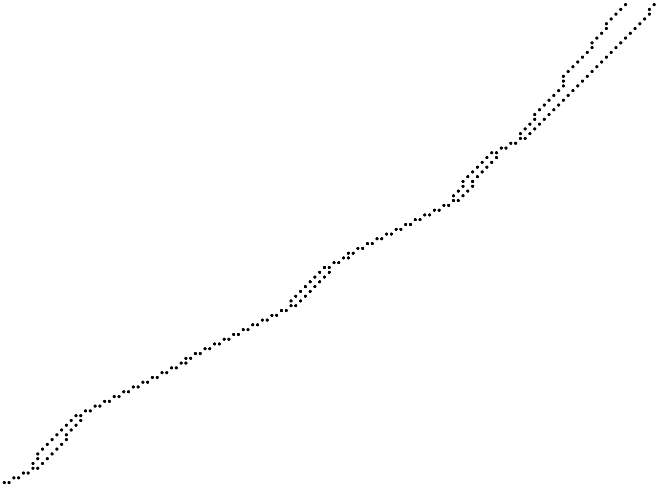
50 balls, 1250 steps, $\epsilon = 0.01$:



- Solitons tend to disintegrate into single particles the travel with average speed $1 - \epsilon$
- Occasionally form pairs that live for geometric amount of time, give a speed boost to the particles

The stochastic box-ball system

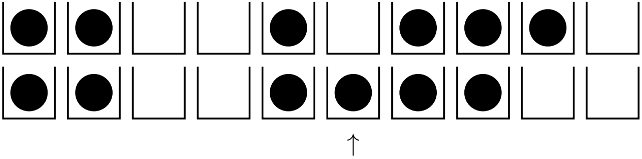
Pair forming and breaking, $\epsilon = 0.1$:



avg. lifespan of a pair: $\frac{1}{\epsilon(1-\epsilon)}$

Limiting values of ϵ

- Clearly, when $\epsilon = 0$, the carrier never fails and we return to the usual box-ball system.
- Consider the limit $\epsilon \to 1$. The balls tend to stay in place. On rare occasions a single ball will be picked up (an independent Geom$(1-\epsilon)$ clock for each ball):



  Rather than the ball being carried to the end of a stack, we can view it as the whole stack in front of it being pushed.
- After rescaling time, in the limit $\epsilon \to 1$, this becomes pushTASEP, with independent Exp(1) clocks for each particle.

The stochastic box-ball system interpolates between these two integrable particle system.

The stochastic box-ball system

- Can we find some integrable structures that can be used to help study this model?
- Are there other ways to add randomness to the dynamics that preserves some integrability?

The inter-distance process

Rather than study the ball positions themselves, we will instead focus on the inter-distance process.

- Suppose at step k the balls are located at positions

$$\zeta_k^{(1)} < \zeta_k^{(2)} < \ldots < \zeta_k^{(d)}.$$

- Define the inter-distance vector

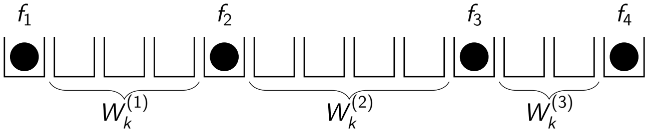
$$W_k = (W_k^{(1)}, \ldots, W_k^{(d-1)}) \in \mathbb{Z}_{\geq 0}^{d-1}$$

at step k by

$$W_k^{(i)} = \zeta_k^{(i+1)} - \zeta_k^{(i)} - 1$$

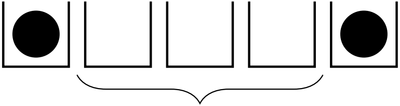
that is, the number of empty boxes between balls i and $i+1$.

The inter-distance process

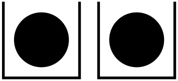


- Stochastic box-ball dynamics are translationally invariant, so we don't need to care about the overall positions of the balls.
- At each step of the dynamics, we flip a sequence of independent biased coins $f_1 f_2 \dots f_d$, one for each ball to determine whether it gets picked up.
- Once we know which balls are picked up, where they are dropped is completely determined.
- Don't keep track of which ball is which.
- Markov chain on $\mathbb{Z}_{\geq 0}^{d-1}$.

Two-ball system



$W_k^{(1)} > 0$



$W_k^{(1)} = 0$

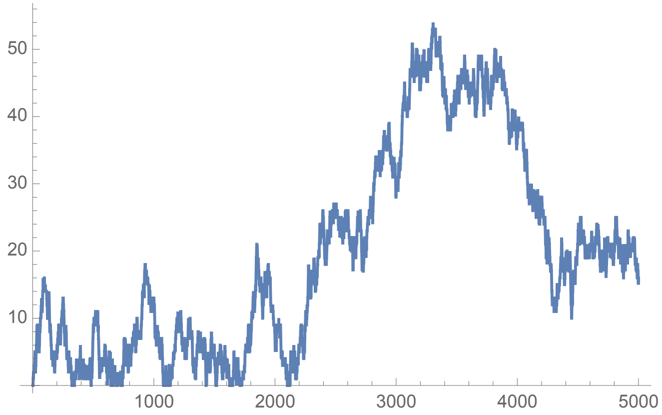
$$\Delta W_k^{(1)} = \begin{cases} +1, & \text{w/ probability } \epsilon(1-\epsilon) \\ -1, & \text{w/ probability } \epsilon(1-\epsilon) \\ 0, & \text{w/ probability } \epsilon^2 + (1-\epsilon)^2 \end{cases}$$

$$\Delta W_k^{(1)} = \begin{cases} +1, & \text{w/ probability } \epsilon(1-\epsilon) \\ 0, & \text{w/ probability } 1-\epsilon(1-\epsilon) \end{cases}$$

- Lazy symmetric random walk in the bulk (get's lazier as ϵ moves away from $1/2$).
- Boundary behavior when $W_t^{(1)} = 0$.
- Gambler's ruin: Pairs reform after some random time T with $\mathbb{E}[T] = \infty$.
- Pair breaks up after a geometric amount of time with average $\frac{1}{\epsilon(1-\epsilon)}$
- Symmetric wrt $\epsilon \leftrightarrow 1-\epsilon$

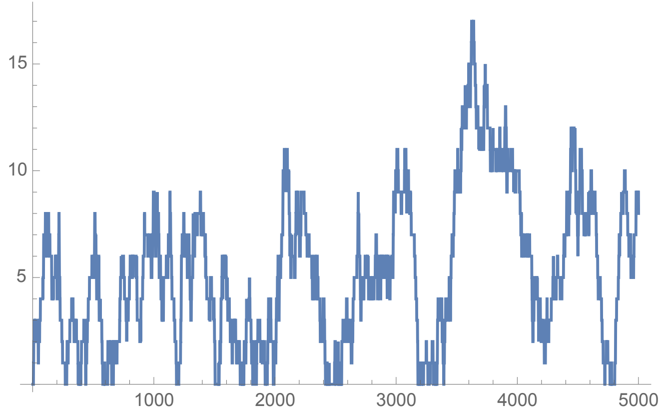
Two-ball system

Simulation of inter-distance in 2-ball system with $\epsilon = 0.5$ over 5000 steps:



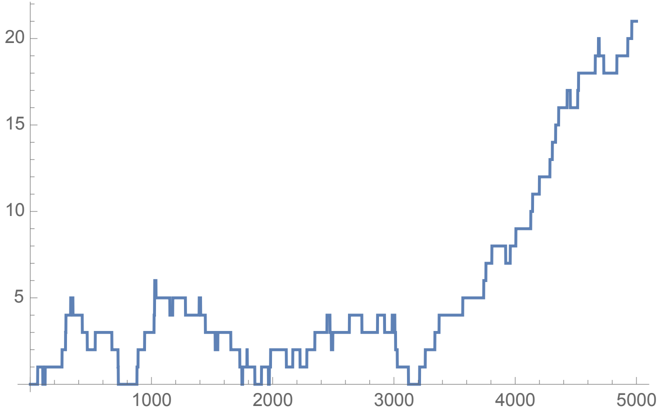
Two-ball system

5000 steps, $\epsilon = 0.1$:

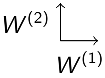


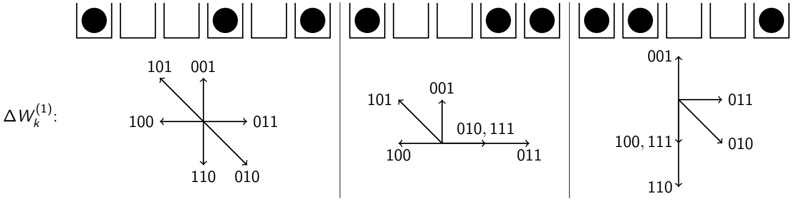
Two-ball system

5000 steps, $\epsilon = 0.01$:



Three-ball system



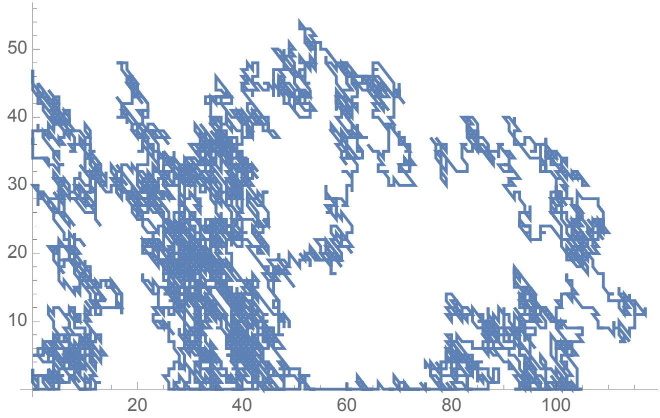


- In the interior behaves as a lazy symmetric random walk.
- On the boundaries we see remnants of the soliton behavior.
- $W^{(2)} = 0$: then 111 results in $\Delta W^{(1)} = +1$.

This breakdown continues to hold for d-ball systems.

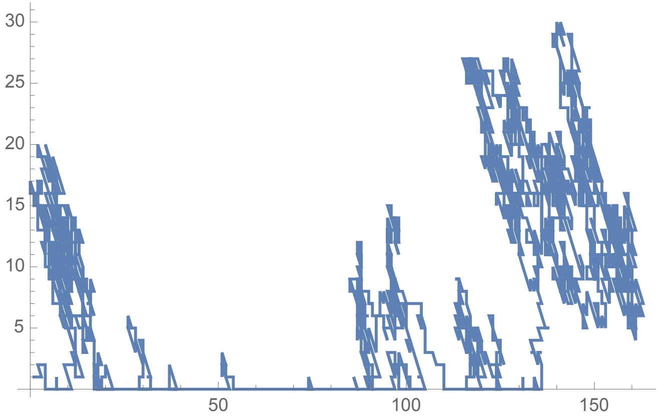
Three-ball system

10000 steps, $\epsilon = 0.5$:



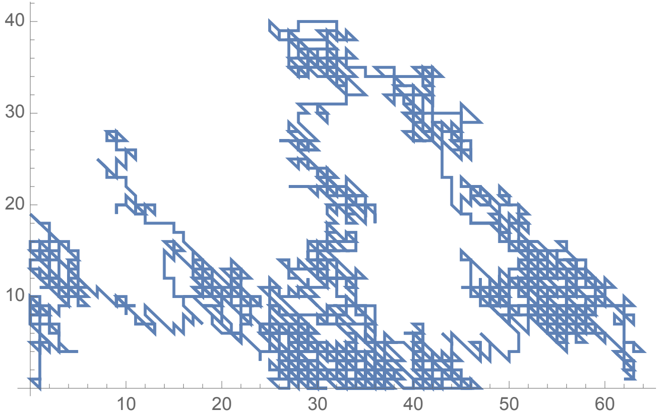
Three-ball system

10000 steps, $\epsilon = 0.1$:



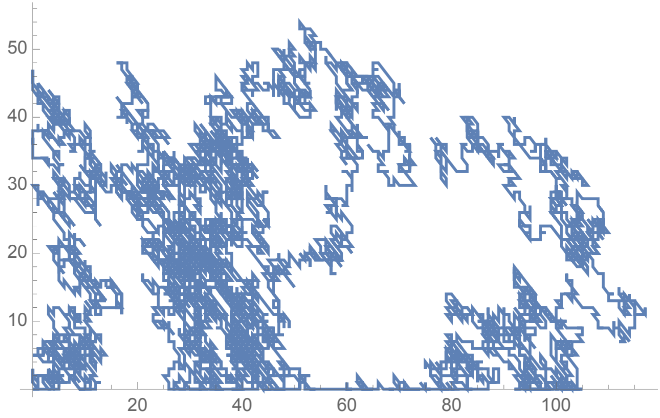
Three-ball system

10000 steps, $\epsilon = 0.9$



Three-ball system

10000 steps, $\epsilon = 0.5$:



Scaling limit for the inter-distance process

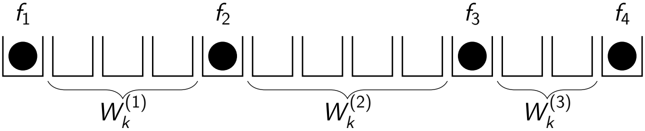
Work in progress:

Consider the stochastic box-ball system with d balls. Let $\bar{W}_t =$ linear interpolation of W_k. Fix $\epsilon \in (0,1)$. Then we conjecture

$$n^{-1/2}(\bar{W}_{nt})_{0\leq t\leq 1} \to \text{semimartingale RBM}$$

on $\mathbb{R}^{d-1}_{\geq 0}$.

The local time at the boundary



Proposition (K.-Lyu 2024)

For each $k = 1, \ldots, d-1$, the expected number of times $W^{(k)} = 0$ in n steps is $O(n^{1/2})$.

- For a simple random walk on $\mathbb{Z}^{d-1}$, this is true.

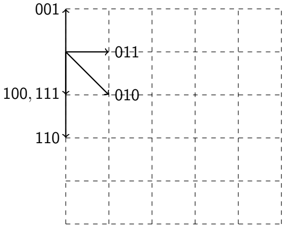
The local time at the boundary

Proposition (K.-Lyu 2024)

For each $k = 1, \ldots, d-1$, the expected number of times $W^{(k)} = 0$ in n steps is $O(n^{1/2})$.

- On our case, the remnant soliton behavior makes this non-trivial.

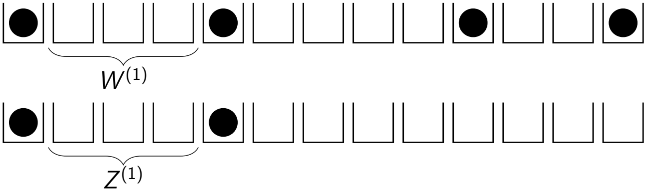
$W^{(1)} = 0$:



Hitting the vertical axis biases us towards the horizontal axis. Need to control how many extra hits to the horizontal axis this causes.

The local time at the boundary

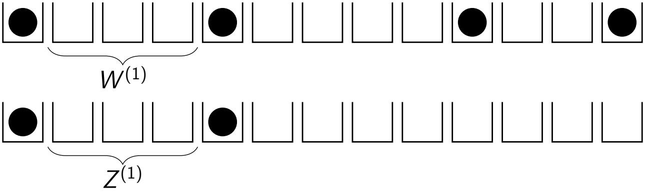
For the first pair of balls things are easy...



- Couple the first two balls with a 2-ball system $Z^{(1)}$ using the same coin flips f_1 and f_2.

The local time at the boundary

For the first pair of balls things are easy...



- Easy to check that the presence of balls to the right will only serve to make $W^{(1)}$ increase faster than $Z^{(1)}$.

$$W_k^{(1)} \geq Z_k^{(1)} \text{ for all } k.$$

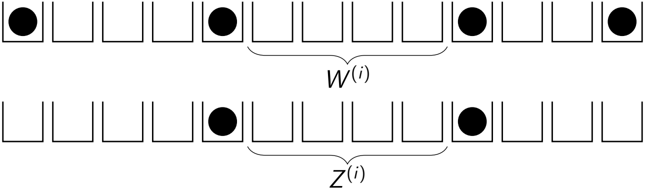
Immediately get

$$\#\{\text{time } W^{(1)} \text{ spends at } 0\} \leq \#\{\text{time } Z^{(1)} \text{ spends at } 0\}$$

LHS is $O(n^{1/2})$ on average since $Z^{(1)}$ is a lazy symmetric random walk.

The local time at the boundary

Try the same thing with a different pair,



Couple $W^{(i)}$ and $Z^{(i)}$ using the same coin flips, f_i and f_{i+1}. Unfortunately, presence of balls to the left can cause $W^{(i)}$ to shrink faster than $Z^{(i)}$. So it is no longer true that

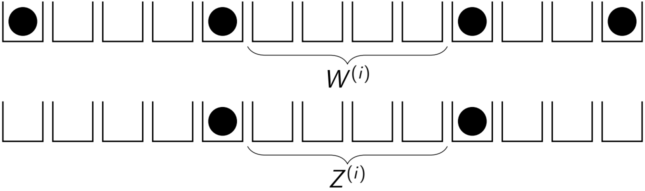
$$W_k^{(i)} \geq Z_k^{(i)} \text{ for all } k.$$

However, we will see that we have

$$\#\{\text{time } W^{(i)} \text{ spends at } 0\} \leq \#\{\text{time } Z^{(i)} \text{ spends at } 0\} + \#\{\text{extra hits}\}.$$

The local time at the boundary

Try the same thing with a different pair,



- Let T_1 be the first time a pair forms within the balls to the left.
- During the interval $[0, T_1]$ we have $W^{(i)} \geq Z^{(i)}$.
- After T_1 there is some time γ_1 until the left balls are all singletons again.
- From T_1 onward we can no longer guarantee $W^{(i)} \geq Z^{(i)}$.

The local time at the boundary

Let T_2 be the first time after $T_1 + \gamma_1$ that a pair forms in the balls to the left.

- During the interval $(T_1 + \gamma_1, T_2]$ the increment is well-behaved. That is,

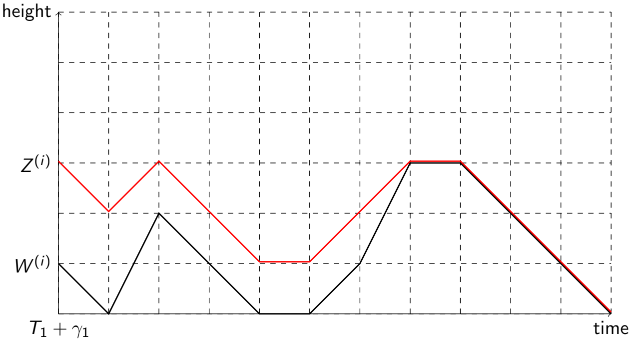
$$\Delta W^{(i)} \geq \Delta Z^{(i)}$$

  during this time interval.
- If at any point $W^{(i)} \geq Z^{(i)}$ then this will be true for the remainder of the interval. In particular, this will be true when $Z^{(i)} = 0$.
- Need to count the extra hits before this point.

The local time at the boundary

The increment is well-behaved: $\Delta W^{(i)} \geq \Delta Z^{(i)}$

- $W^{(i)} = 0$ only if $Z^{(i)}$ is at a running minimum.



- Will only hit each level a geometric number of times.
- Difference in starting height is at most $(i-1) \cdot \gamma_1$.

The local time at the boundary

We can control how much things deviates from the coupled 2-ball system on the interval $[T_1 + \gamma_1, T_2)$:

$$\begin{aligned}\#\{\text{steps for which } W^{(i)} = 0\} \leq \#&\{\text{steps for which } Z^{(i)} = 0\} \\ &+ \#\{\text{extra hits when } Z^{(i)} > W^{(i)}\}\end{aligned}$$

with

$\#\{\text{extra hits when } Z^{(i)} > W^{(i)}\}$ bounded by
a sum of geometric random variables.

The local time at the boundary

- Break our steps into excursions:

$$[0, T_1] \cup (T_1, T_1 + \gamma_1], \ldots, (T_{M-1} + \gamma_{M-1}, T_M] \cup (T_M, T_M + \gamma_M]$$

- Each excursion behaves as before.

$$\begin{aligned}\mathbb{E}[\#\{\text{steps for which } W^{(i)} = 0\}] &\leq \underbrace{\mathbb{E}[\#\{\text{steps for which } Z^{(i)} = 0\}]}_{O(n^{1/2})} \\ &\quad + \mathbb{E}[\sum_{i=1}^{M} \underbrace{\#\{\text{extra hits in interval } i\}}_{\leq C \text{ on average}}] \\ &\leq \mathbb{E}[\#\{\text{steps for which } Z^{(i)} = 0\}] + C\mathbb{E}[M]\end{aligned}$$

by induction $\mathbb{E}[M] = O(n^{1/2})$.

The stochastic box-ball system

Work in progress:

Consider the stochastic box-ball system with 3-balls. Let $\bar{W}_t =$ linear interpolation of W_k. Fix $\epsilon \in (0,1)$. Then we conjecture

$$n^{-1/2}(\bar{W}_{nt})_{0\leq t\leq 1} \to \text{semimartingale RBM}$$

on $\mathbb{R}^2_{\geq 0}$.

SRBM for the inter-distance process

Preliminary Skorokhod decomposition:

$$W = \underbrace{X}_{\substack{\text{interior}\\ \text{process}}} + \overbrace{R}^{\substack{\text{reflection}\\ \text{matrix}}} \underbrace{Y}_{\substack{\text{pushing}\\ \text{process}}}$$

- In the limit the interior the process behaves like a Brownian motion X.
- At the boundary there is an instantaneous reflection into the interior governed by the 2×2 reflection matrix.
- The pushing process Y is 2-dimensional process, non-decreasing, $Y^{(i)}$ can only increase when $W^{(i)} = 0$.

SRBM

Theorems of Dai-Williams (1995), Williams (1998), and Kang-Williams (2007) give sufficient conditions for when a sequence of processes (W^n, X^n, Y^n), $W^n = X^n + RY^n$, converge to an SRBM as $n \to \infty$.

- Main difficulty for us: Reflection matrix
- For these theorems the reflection matrix is deterministic and the reflections on each axis must point into the interior, not along the boundary.
- In the inter-distance process, the reflection is random (depends on the coin flips), some reflections push along the boundary (remnant of the soliton behavior).

SRBM for the inter-distance process

Let $\Delta Y^{(i)} = 1(W^{(i)} = 0)$. Reflection matrix for the inter-distance process is determined by:

$$R\Delta Y = \Delta W - \Delta X$$

Suppose that $W^{(2)} = 0$. Compare ΔW and ΔX:

$$\text{second column of } R\text{: } R_2 = \begin{cases} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, & \text{for flips 010 and 110} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \text{for flips 011 and 111} \end{cases}$$

The reflection depends on the sequence of coin flips.

SRBM for the inter-distance process

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For 010 and 110 the reflection point along the boundary.

SRBM for the inter-distance process

Let $\bar{R}$ be the average reflection matrix.

$$\bar{R}_2 = (1-\epsilon)\begin{pmatrix} 1 & -\epsilon \\ & \epsilon \end{pmatrix}$$

- Idea: Replace R by $\bar{R}$, show that the error goes to zero after the scaling.
- Let $\delta_k = (R - \bar{R})\Delta Y_k$, the error from this replacement.
- Our new Skorohod decomposition:

$$W = X + \bar{R}Y + \underbrace{\sum_k \delta_k}_{\text{error term}}$$

SRBM for the inter-distance process

Let $\bar{R}$ be the average reflection matrix.

$$\bar{R}_2 = (1-\epsilon)\begin{pmatrix} 1 & -\epsilon \\ & \epsilon \end{pmatrix}$$

- Idea: Replace R by $\bar{R}$, show that the error goes to zero after the scaling.
- Note along each boundary the choice of reflection direction is an iid random variable.
- By central limit theorem:

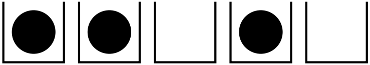
$$||\sum_k \delta_k^{(2)}|| = O((\underbrace{\#\{\text{times } W^{(2)} = 0\}}_{O(n^{1/2})})^{1/2}) = O(n^{1/4})$$

- Since we will eventually scale everything by $n^{-1/2}$, this will vanish.

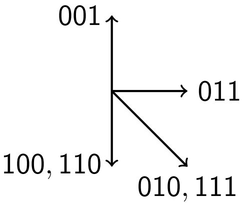
SRBM for the inter-distance process

Remaining issue:

- Close to the origin the increment changes

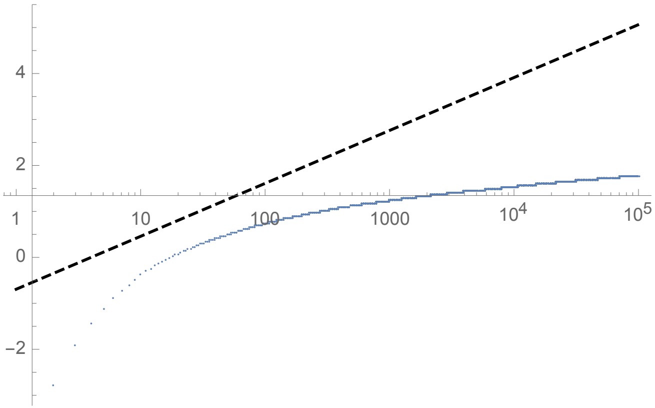


ΔW :



- Additional error whenever we hit this spot on the boundary.
- Need to show that the number of hits is $o(n^{1/2})$ so it vanishes after scaling.
- For a 2d SRW: $\#\{\text{hits to origin}\} \sim \log n$.

SRBM for the inter-distance process



Plot of $\log \#\{\text{hits to origin in 3-ball system by time } n\}$ (blue) and $\log n^{1/2}$ (dashed).

Thank You!