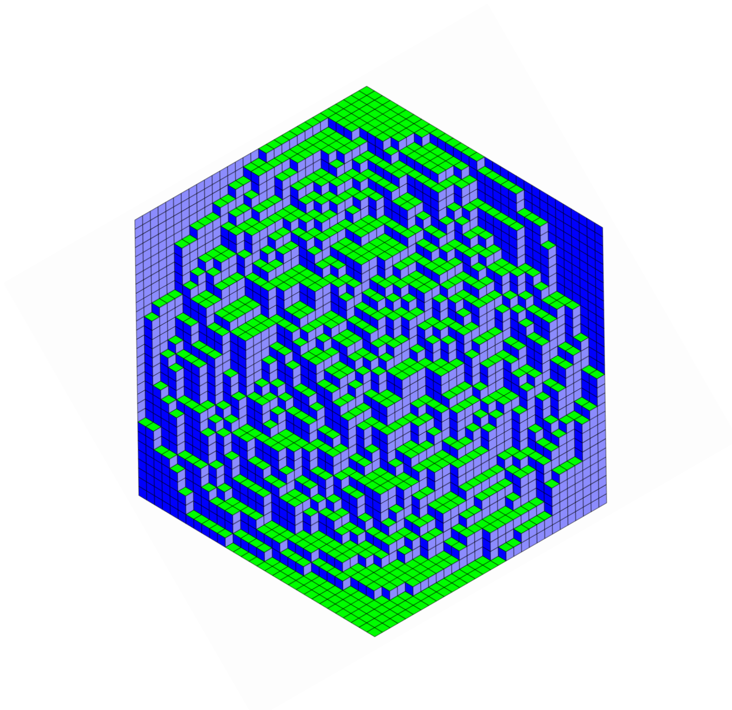
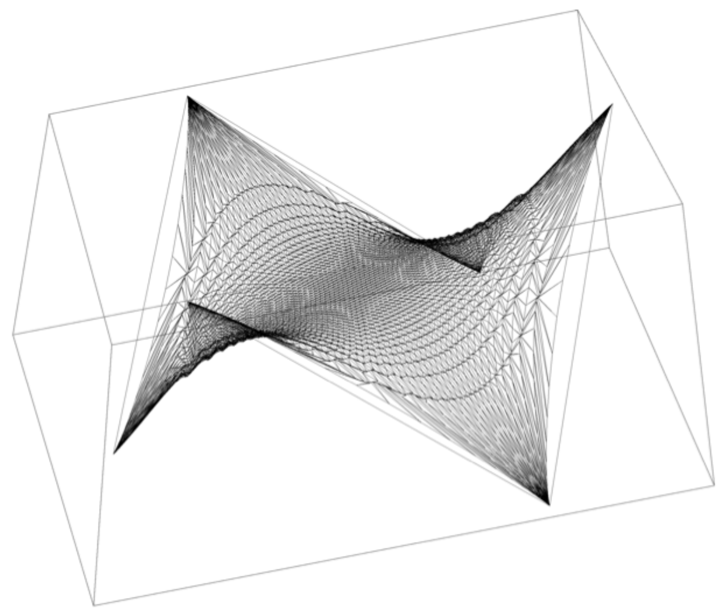
Perfect t-embeddings of Hexagon

Marianna Russkikh

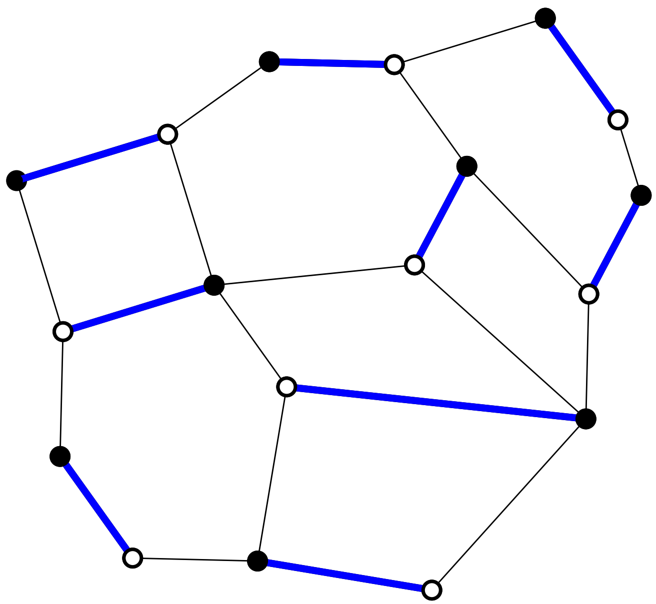
University of Notre Dame

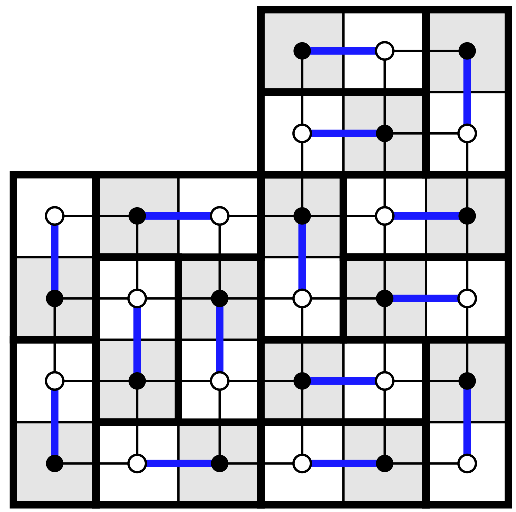


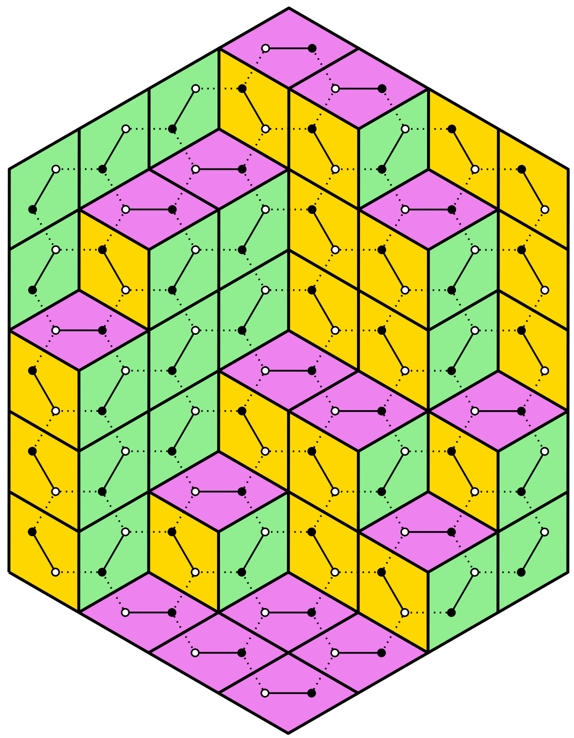


joint with T. Berggren and M. Nicoletti.

Dimer model







A dimer cover of a planar bipartite graph is a set of edges with the property: every vertex is contained in exactly one edge of the set.

(On the square lattice / honeycomb lattice it can be viewed as a tiling of a domain on the dual lattice by dominos / lozenges.)

Weighted dimers

Weight function on edges:

$$\nu : E \to \mathbb{R}_{>0}$$

Associated weight of a dimer cover:

$$\nu(m) = \prod_{e \in m} \nu(e)$$

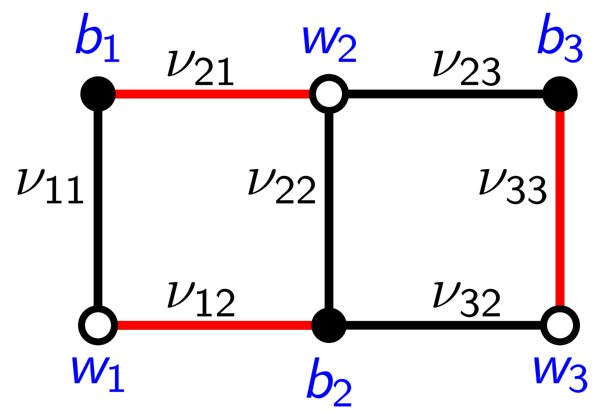
Partition function:

$$Z = \sum_{m \in M} \nu(m)$$

Probability measure on dimer coverings:

$$\mu(m) = \frac{1}{Z} \nu(m)$$

An example for 2×3 graph:



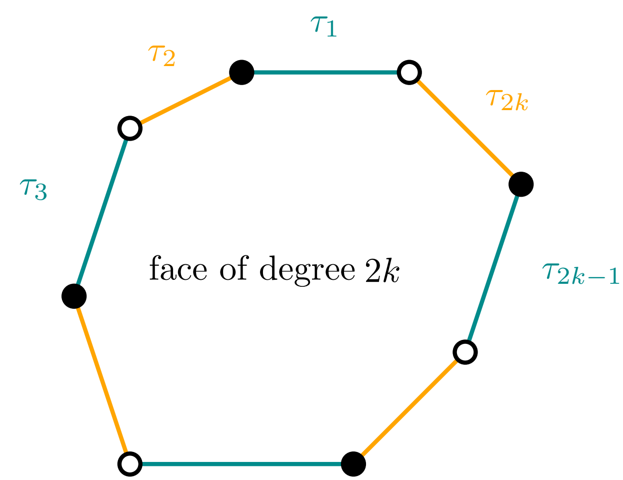
$$\nu(m) = \nu_{12} \cdot \nu_{21} \cdot \nu_{33}$$

$$\begin{aligned} Z = &\nu_{12} \cdot \nu_{21} \cdot \nu_{33} \\ &+\nu_{23} \cdot \nu_{32} \cdot \nu_{11} \\ &+\nu_{11} \cdot \nu_{22} \cdot \nu_{33} \end{aligned}$$

Kasteleyn matrix

Complex Kasteleyn signs:
$\tau_i \in \mathbb{C}$, $|\tau_i| = 1$,

$$\boxed{\frac{\tau_1}{\tau_2} \cdot \frac{\tau_3}{\tau_4} \cdot \ldots \cdot \frac{\tau_{2k-1}}{\tau_{2k}} = (-1)^{(k+1)}}$$



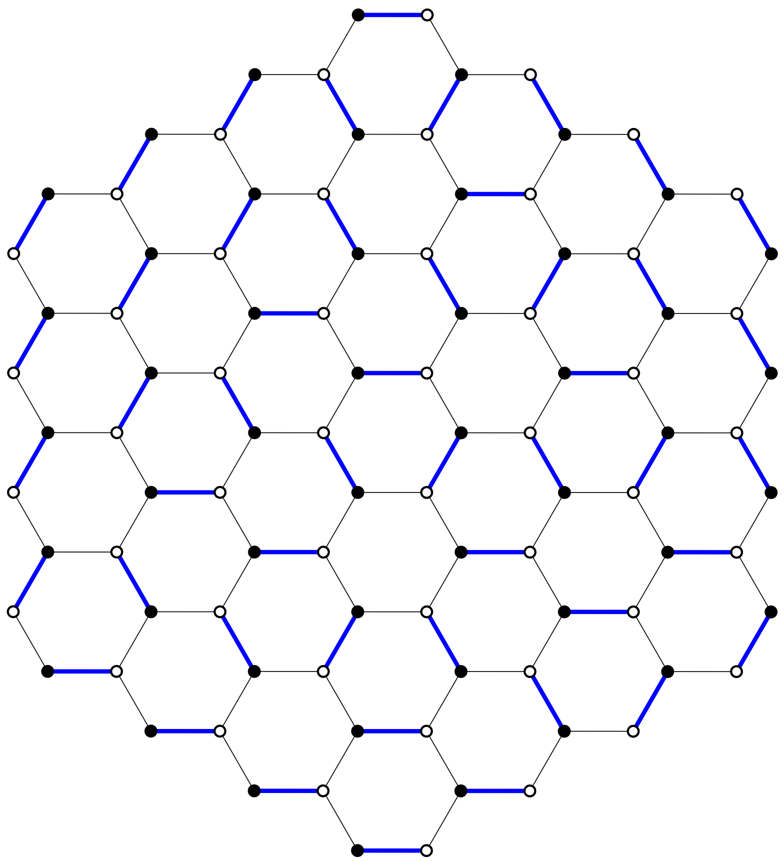
A **(Percus–)Kasteleyn matrix** K is a weighted, **signed** adjacency matrix whose rows index the white vertices and columns index the black vertices: $K(w, b) = \tau_{wb} \cdot \nu(wb)$.

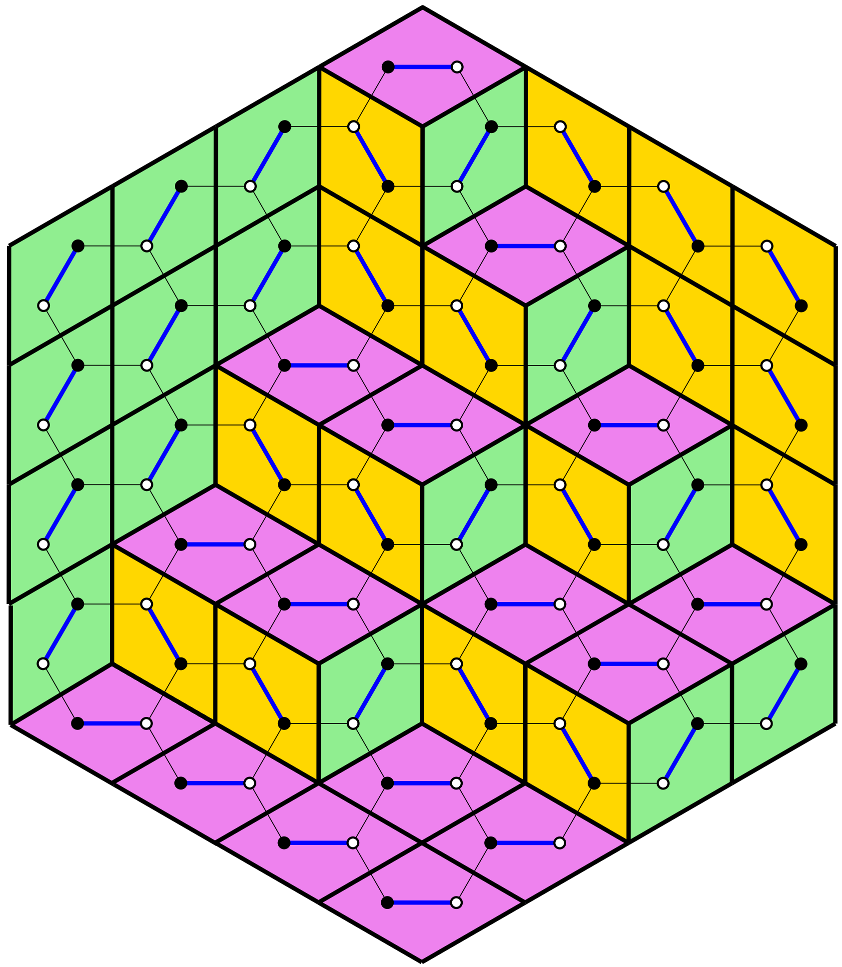
- [Percus'69, Kasteleyn'61]: $Z = |\det K| = \sum_{m \in M} \nu(m)$
- The local statistics for the measure μ on dimer configurations can be computed using **the inverse Kasteleyn matrix**.

Hexagon

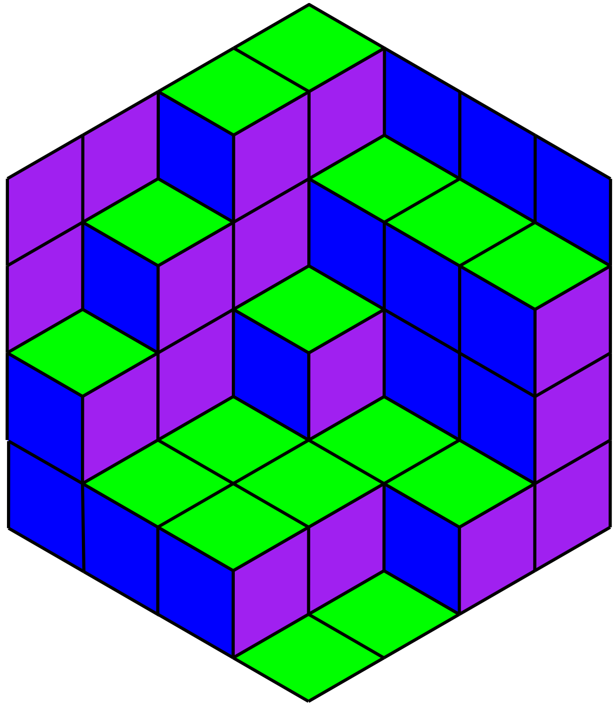
(uniformly weighted)

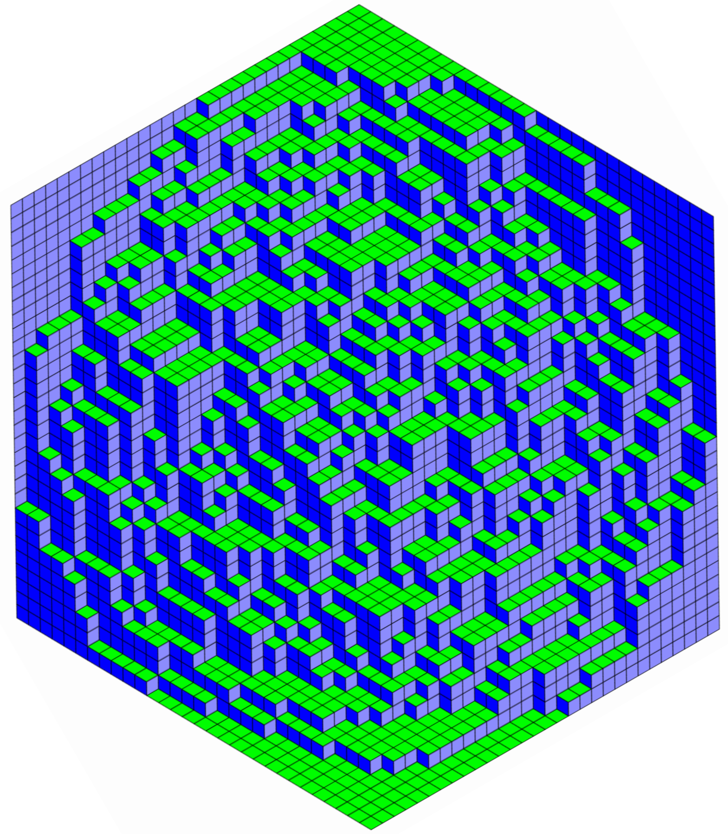
Lozenge tilings of hexagon



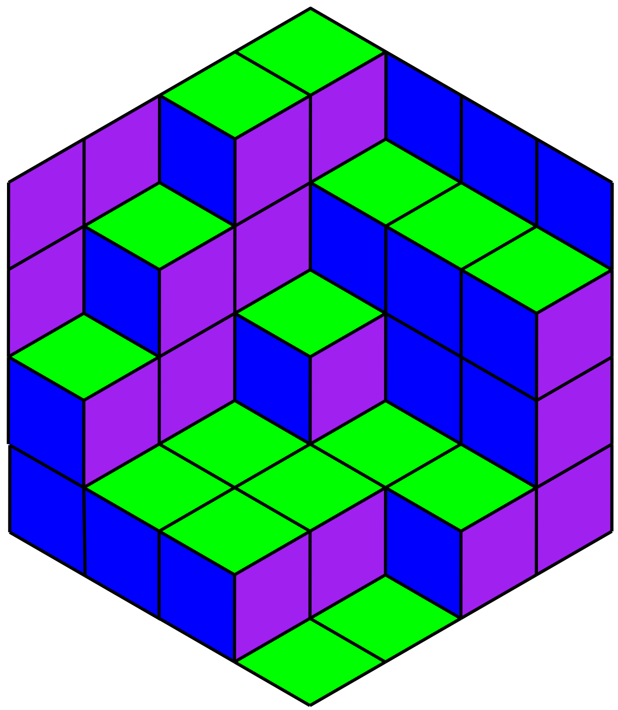


Uniformly distributed lozenge tilings of the hexagon





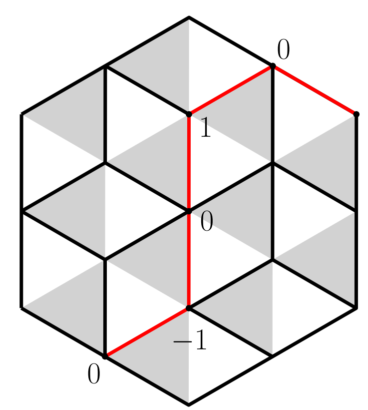
Height function



Lozenge tilings can be viewed as orthogonal projection onto the plane $\{x+y+z=0\}$ of stepped surfaces (polygonal surfaces in $\mathbb{R}^3$ whose faces are squares in the 2-skeleton of $\mathbb{Z}^3$)

Dimer height function on vertices: The height function is equal to $\sqrt{3}$ times the distance from the surface to the plane $\{x+y+z=0\}$

Height function: along each edge not covered by a lozenge the height changes by ± 1, increases by 1 if this edge has a black face on its left, and decreases by 1 otherwise.



Height function

The key questions: the large-scale behavior of

(a) the limit shape of the height function,

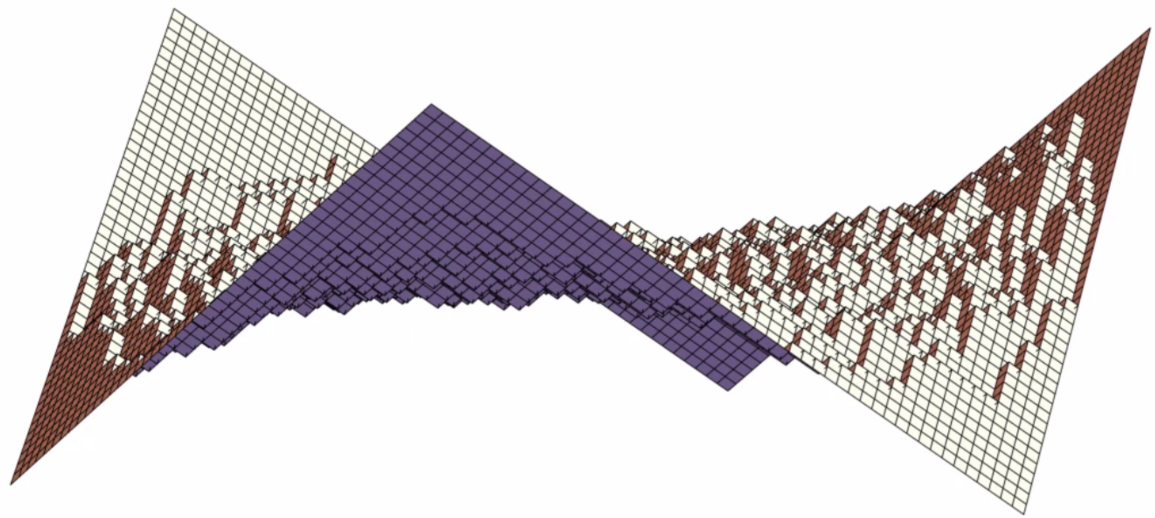
(b) fluctuations of the height function.

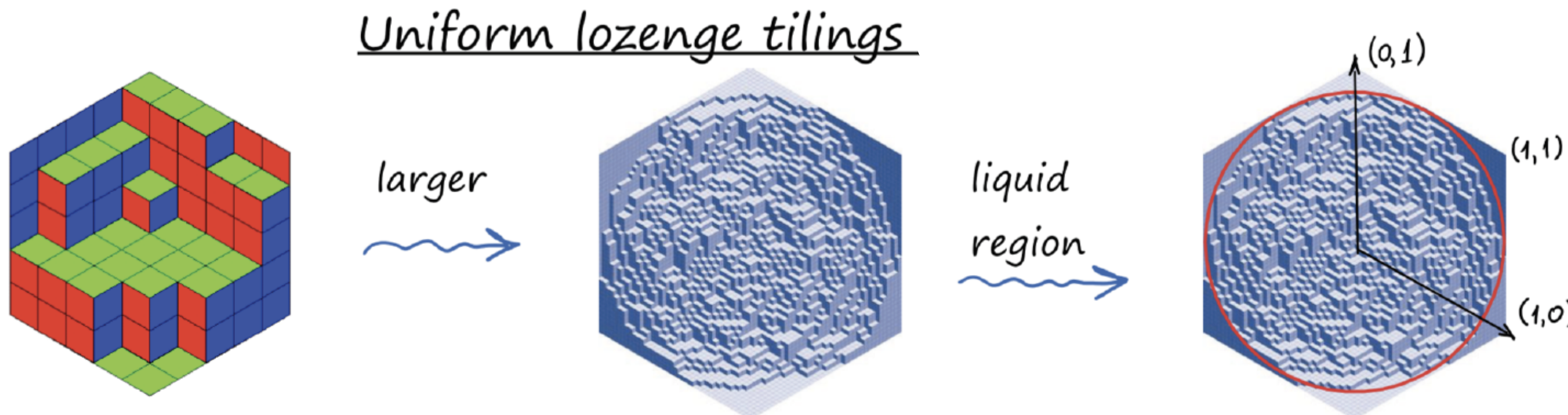
Intuition:

(a) Law of Large Numbers

(b) Central Limit Theorem

Limit shape

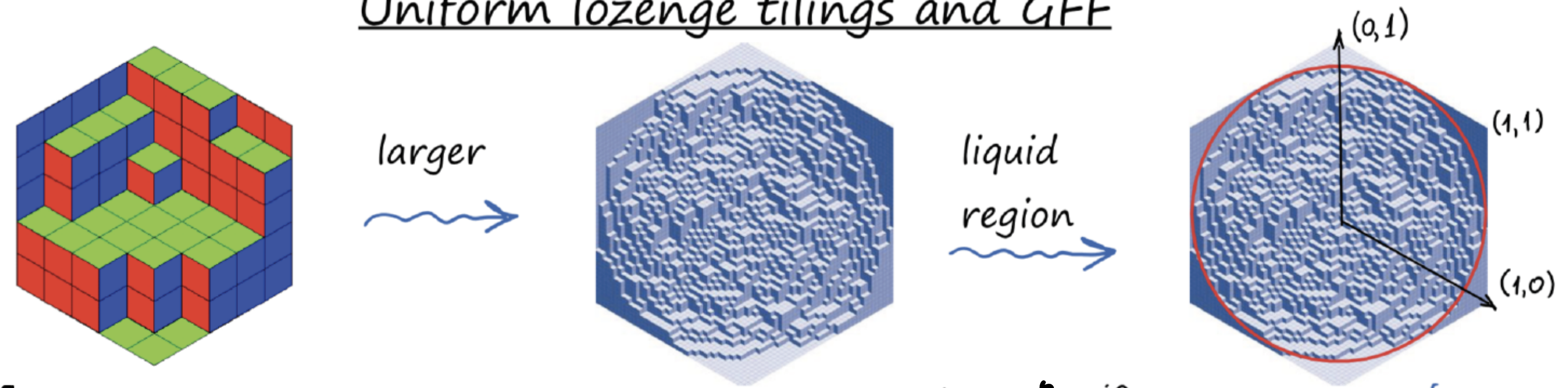




- [Cohn–Kenyon–Propp, 2000]:
  The local density of each type of edge converges to a deterministic limit. **Equivalently:** random profiles δh^{δ} concentrate near a surface (with given boundary) that maximizes a certain entropy functional.

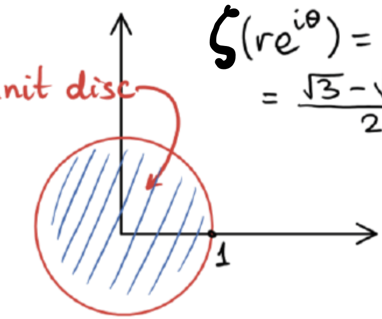
Fluctuations

Uniform lozenge tilings and GFF



[Petrov]
Theorem As mesh goes to zero,
Fluctuations of height $\Rightarrow$
Gaussian Free Field on $\mathbb{D}$ with
zero boundary conditions.

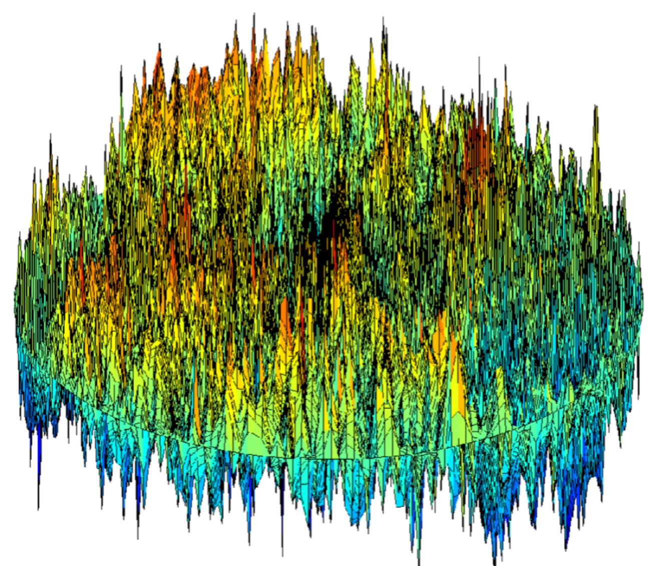
$\mathbb{D}$ = unit disc



$$\zeta(re^{i\theta}) = \frac{\sqrt{3}-\sqrt{3-4r^2}}{2r}e^{i\theta}$$

new complex structure!

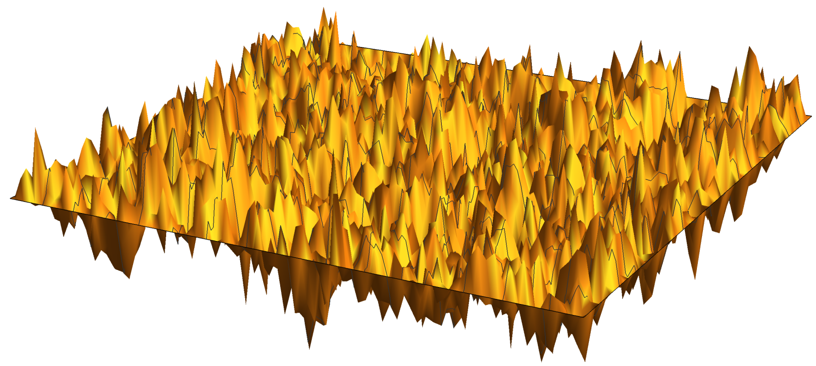
- The GFF is a random generalized function (distribution) on a domain $D \subset \mathbb{C}$, a 2-dimensional analogue of a Brownian bridge.
- [Kenyon-Okounkov '05] conjectured it to appear universally in tiling models.



Gaussian Free Field

The Gaussian Free Field is not a random function, but a random distribution.

[1d analog: Brownian Bridge]



The Gaussian free field Φ on $\mathcal{D}$ is the random distribution such that pairings with test functions $\int_{\mathcal{D}} f\Phi$ are jointly Gaussian with covariance

$$\mathrm{Cov}\left(\int_{\mathcal{D}} f_1\Phi, \int_{\mathcal{D}} f_2\Phi\right) = \int_{\mathcal{D}\times\mathcal{D}} f_1(z)G(z,w)f_2(w).$$

Results

Theorem (Berggren, Nicoletti, R. '24)

Construction and exact (double integral) formula of a perfect t-embedding of the uniformly weighted hexagon.

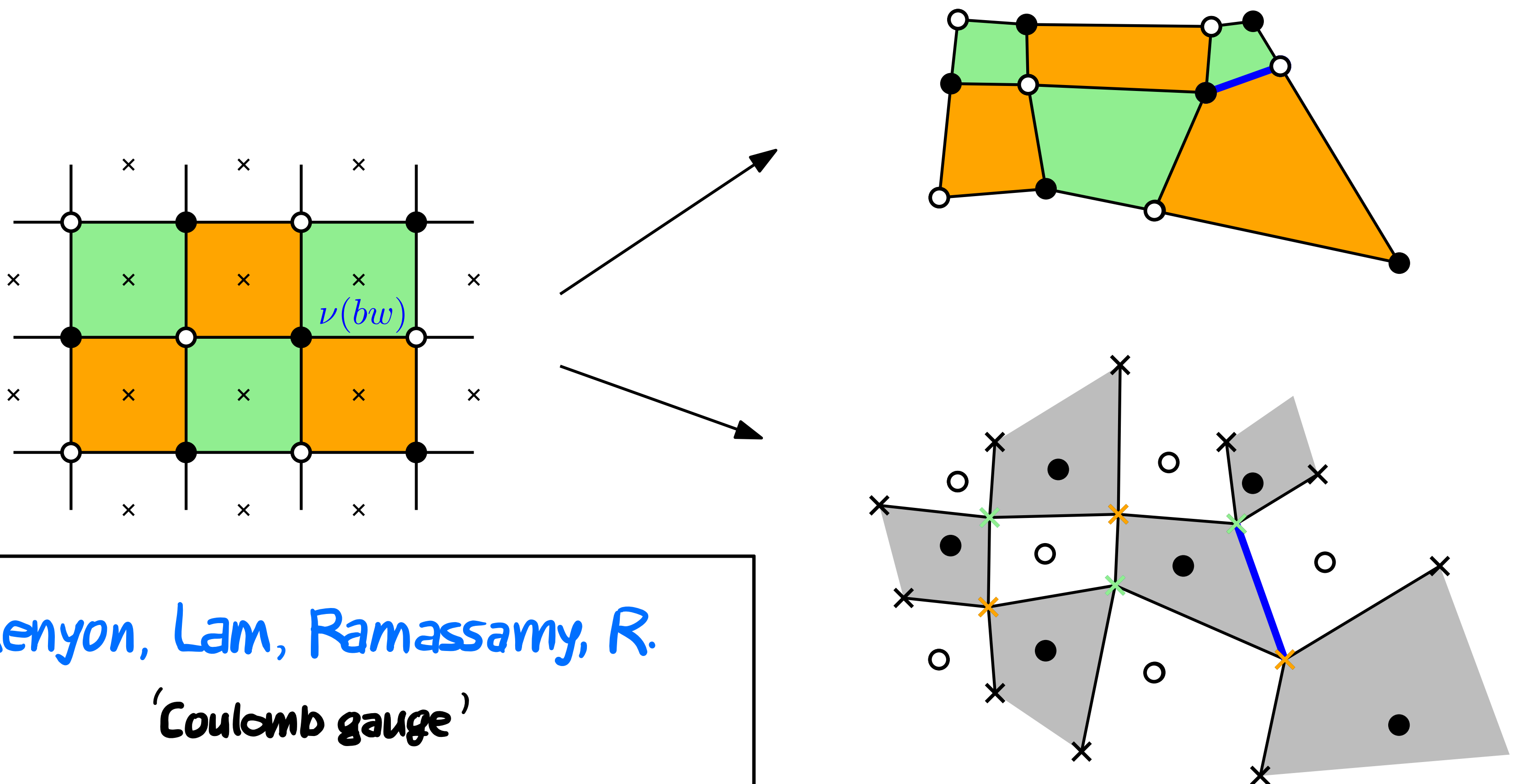
Theorem (Berggren, Nicoletti, R. '24)

Perfect t-embeddings of the uniformly weighted hexagon converge to a maximal surface in the Minkowski space $\mathbb{R}^{2,1}$.

$\Rightarrow$ *convergence of height fluctuations to the Gaussian free field in the conformal parametrization of this surface.*

Perfect t-embeddings

Embeddings of a dimer graph



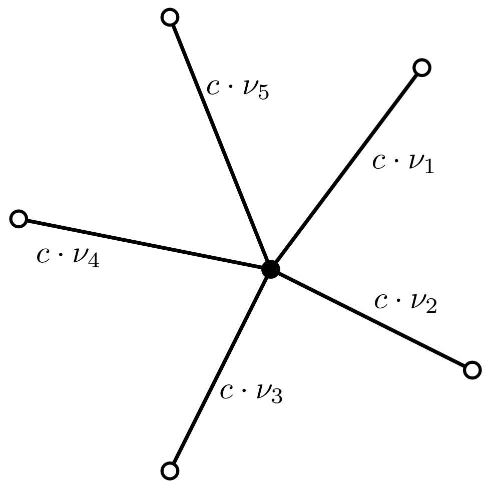
- Kenyon, Lam, Ramassamy, R.

  'Coulomb gauge'

- Chelkak, Laslier, R.

  't-embedding'

Weighted dimers and gauge equivalence



Weight function $\nu : E(\mathcal{G}) \to \mathbb{R}_{>0}$

Probability measure on dimer covers:

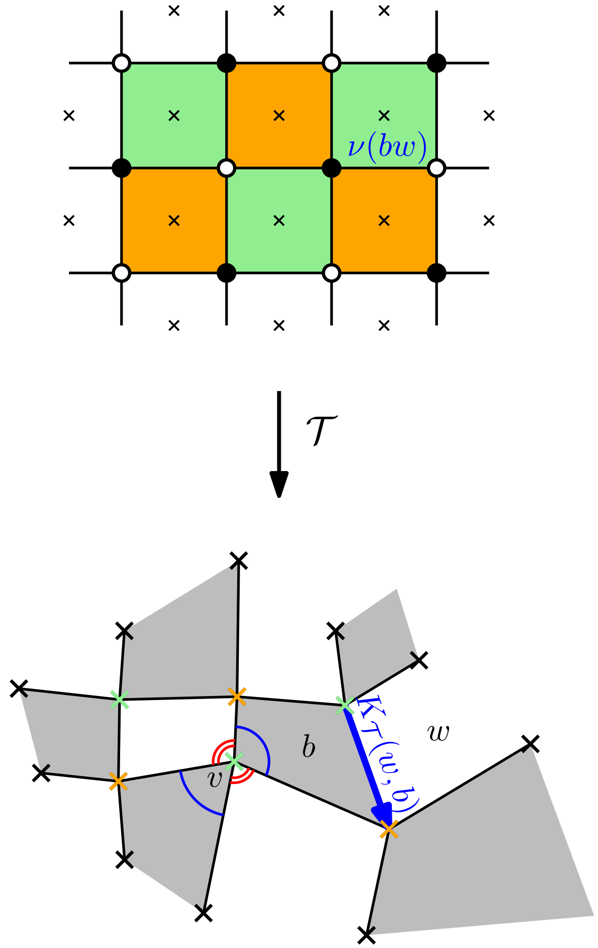
$$\mu(m) = \frac{1}{Z} \prod_{e \in m} \nu(e)$$

Definition

Two weight functions ν_1, ν_2 are said to be *gauge equivalent* if there are two functions $F : B \to \mathbb{R}$ and $G : W \to \mathbb{R}$ such that for any edge bw, $\nu_1(bw) = F(b)G(w)\nu_2(bw)$.

Gauge equivalent weights define the same probability measure μ.

Definition: t-embedding



$K_{\mathcal{T}}$ is a Kasteleyn matrix.

[Chelkak, Laslier, R.]

$\mathcal{T}$ is embedding of $\mathcal{G}^*$ such that

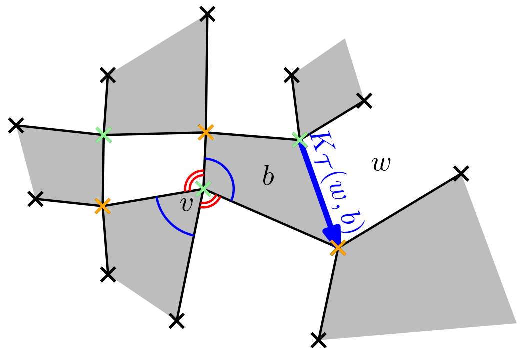
1) **lengths** are gauge equivalent to (given) dimer weights
2) **angles** at (inner) vertices are balanced:

$$\sum_{f \text{ white}} \theta(f,v) = \sum_{f \text{ black}} \theta(f,v) = \pi.$$

Rmk: (2) $\implies$ Kasteleyn sign condition.

Origami map

t-embedding $\mathcal{T}(\mathcal{G}^*)$:



1) **lengths** are gauge equivalent to (given) dimer weights

2) **angles** at vertices are balanced:

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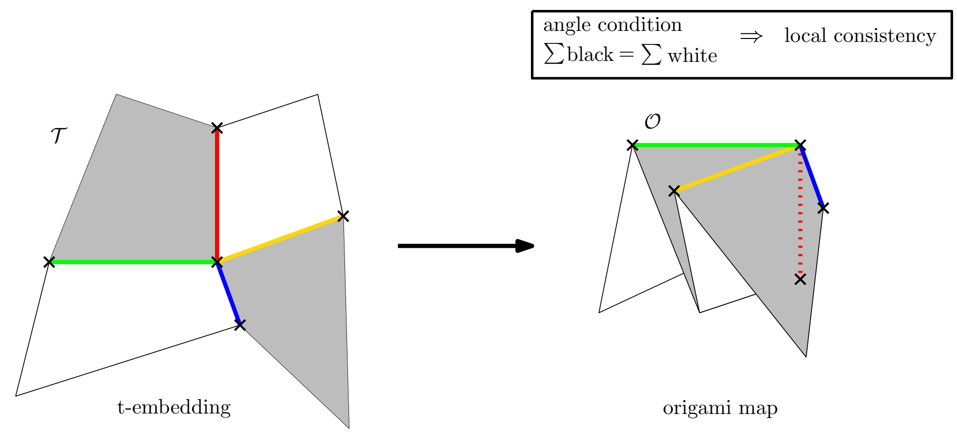
[Chelkak, Laslier, R.]

To get an origami map $\mathcal{O}(\mathcal{G}^*)$ from $\mathcal{T}(\mathcal{G}^*)$ one can fold the plane along every edge of the embedding.



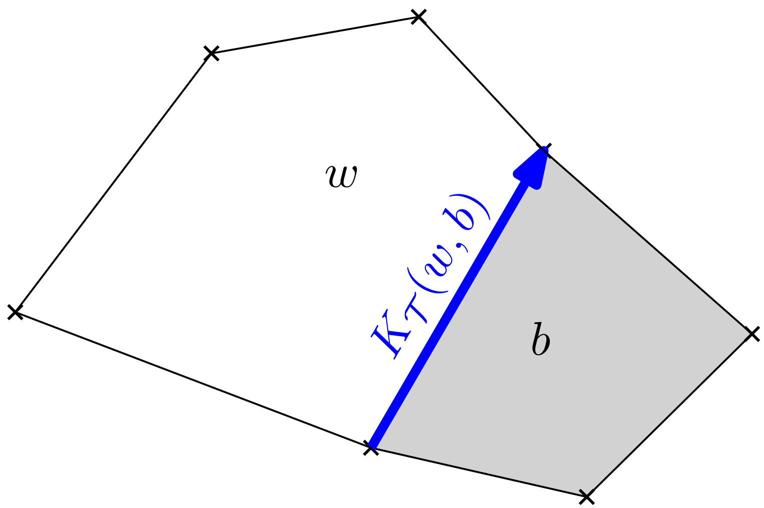
t-embeddings: $(\mathcal{T}, \mathcal{O}) \subset \mathbb{R}^{2+2}$

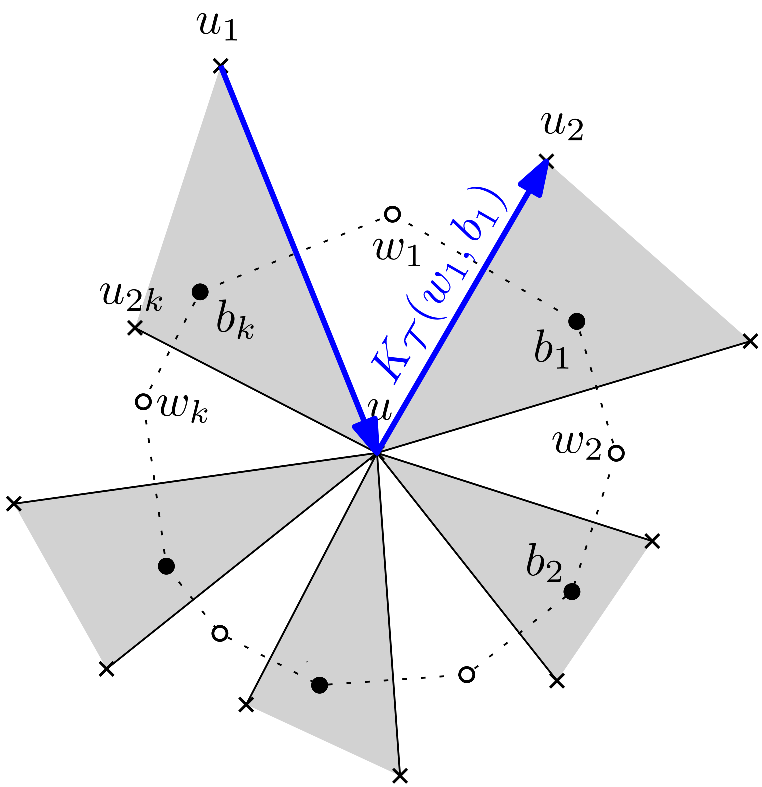
$|\mathcal{O}(z) - \mathcal{O}(z')| \leq |\mathcal{T}(z) - \mathcal{T}(z')|$

discrete space-like surfaces in Minkowski space $\mathbb{R}^{2+2}$

Kasteleyn weights

$$\mathcal{T} \to (\mathcal{G}, K_{\mathcal{T}}), \quad \text{where} \quad \sum_b K_{\mathcal{T}}(w,b) = \sum_w K_{\mathcal{T}}(w,b) = 0$$





Then $K_{\mathcal{T}}$ is a Kasteleyn matrix.

$\boxed{\text{Kasteleyn sign condition}} \longleftrightarrow \boxed{\text{angle condition}}$

$\prod \frac{K_{\mathcal{T}}(w_i,b_i)}{K_{\mathcal{T}}(w_{i+1},b_i)} \in (-1)^{k+1}\mathbb{R}_+$ $\quad\quad$ $\sum \text{white} = \pi \mod 2\pi$

Kasteleyn weights

$G(w)\left(\sum_b F(b)\, K_{\mathbb{R}}(w,b)\right)$

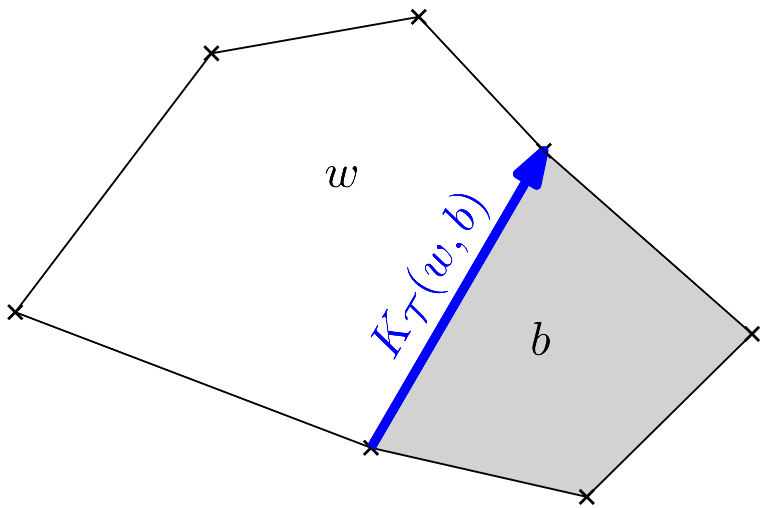
$\parallel$

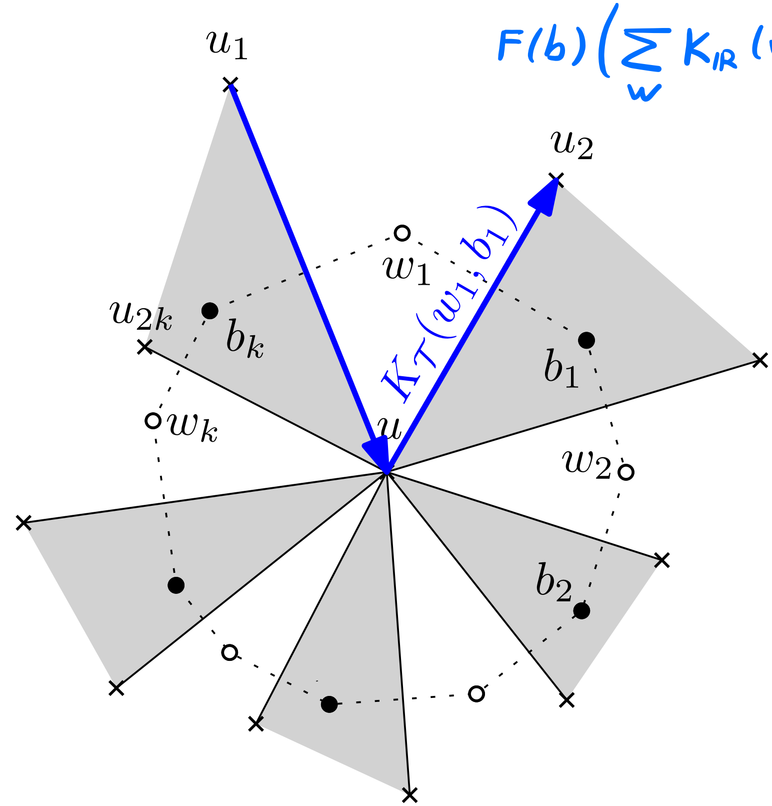
$$\mathcal{T} \to (\mathcal{G}, K_{\mathcal{T}}), \quad \text{where} \quad \sum_b K_{\mathcal{T}}(w,b) = \sum_w K_{\mathcal{T}}(w,b) = 0$$

$(\mathcal{G}, K_{\mathbb{R}})$

$\parallel$

$F(b)\left(\sum_w K_{\mathbb{R}}(w,b)\, G(w)\right)$





Then $K_{\mathcal{T}}$ is a Kasteleyn matrix.

$K_{\mathcal{T}}(w,b) = F(b)\, K_{\mathbb{R}}(w,b)\, G(w)$

$$\boxed{\text{Kasteleyn sign condition}} \longleftrightarrow \boxed{\text{angle condition}}$$

$$\prod \frac{K_{\mathcal{T}}(w_i,b_i)}{K_{\mathcal{T}}(w_{i+1},b_i)} \in (-1)^{k+1}\mathbb{R}_+ \qquad \sum \text{white} = \pi \mod 2\pi$$

Definition: Coulomb gauge

[Kenyon, Lam, Ramassamy, R.]

A pair of functions $F : B \to \mathbb{C}$ and $G : W \to \mathbb{C}$ are called Coulomb gauge functions if

$$
\begin{aligned}
[K_{\mathbb{R}}F](b) = 0 \quad & \text{for all } b \in B \smallsetminus \partial B, \\
[GK_{\mathbb{R}}](w) = 0 \quad & \text{for all } w \in W \smallsetminus \partial W,
\end{aligned}
$$

where ∂B and ∂W are boundary black and white vertices of $\mathcal{G}$.

One can define a t-realisation $\mathcal{T} = \mathcal{T}_{(F,\,G)}$ together with the associated origami map $\mathcal{O} = \mathcal{O}_{(F,\,G)}$ by setting

$$
\begin{aligned}
d\mathcal{T}(bw^*) \; &:= \; F(b)K_{\mathbb{R}}(b,w)G(w), \\
d\mathcal{O}(bw^*) \; &:= \; F(b)K_{\mathbb{R}}(b,w)\overline{G(w)}\,.
\end{aligned}
$$

Rmk: Note that Kasteleyn sign condition implies the angle condition only modulo 2π.

General setup

Theorem (Kenyon, Lam, Ramassamy, R. '19)

t-embeddings exist at least in the following cases:

- *If $\mathcal{G}^\delta$ is a bipartite finite graph with outer face of degree 4.*
- *If $\mathcal{G}^\delta$ is a biperiodic bipartite graph.*

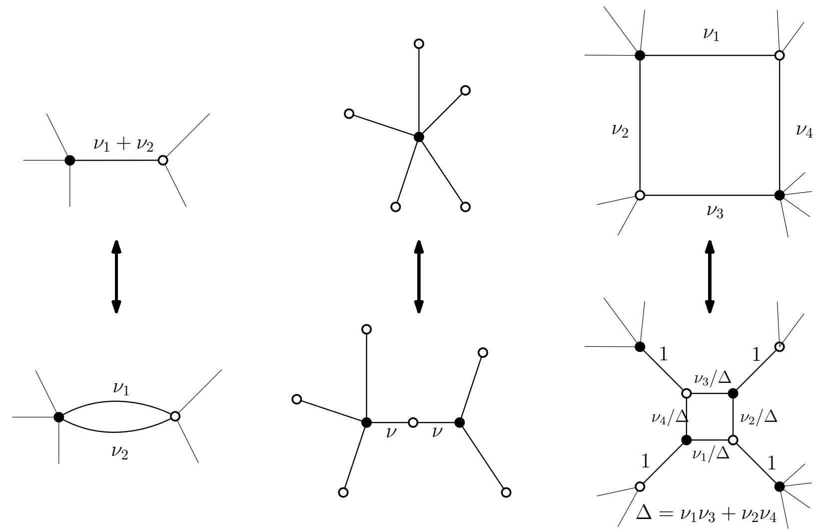
Scaling limit results: [Chelkak, Laslier, R. '20-21]

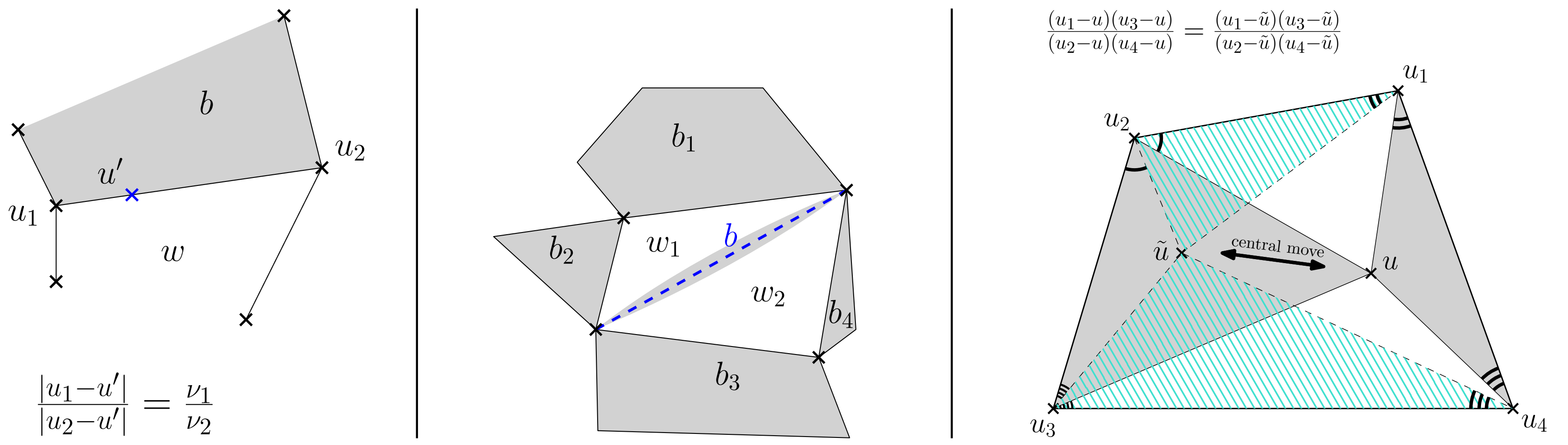
- New discrete complex analysis techniques on t-embeddings developed
- Perfect t-embeddings reveal the relevant conformal structure of the Dimer model

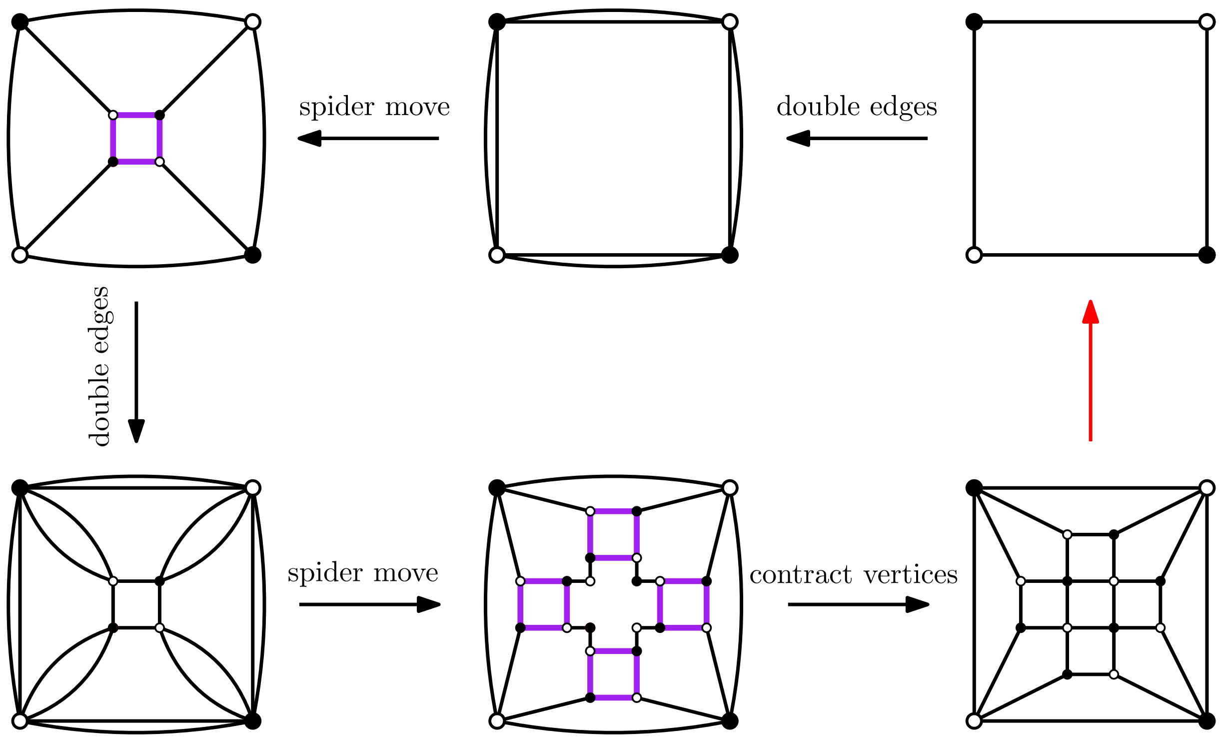
[Kenyon, Lam, Ramassamy, R. '19]:

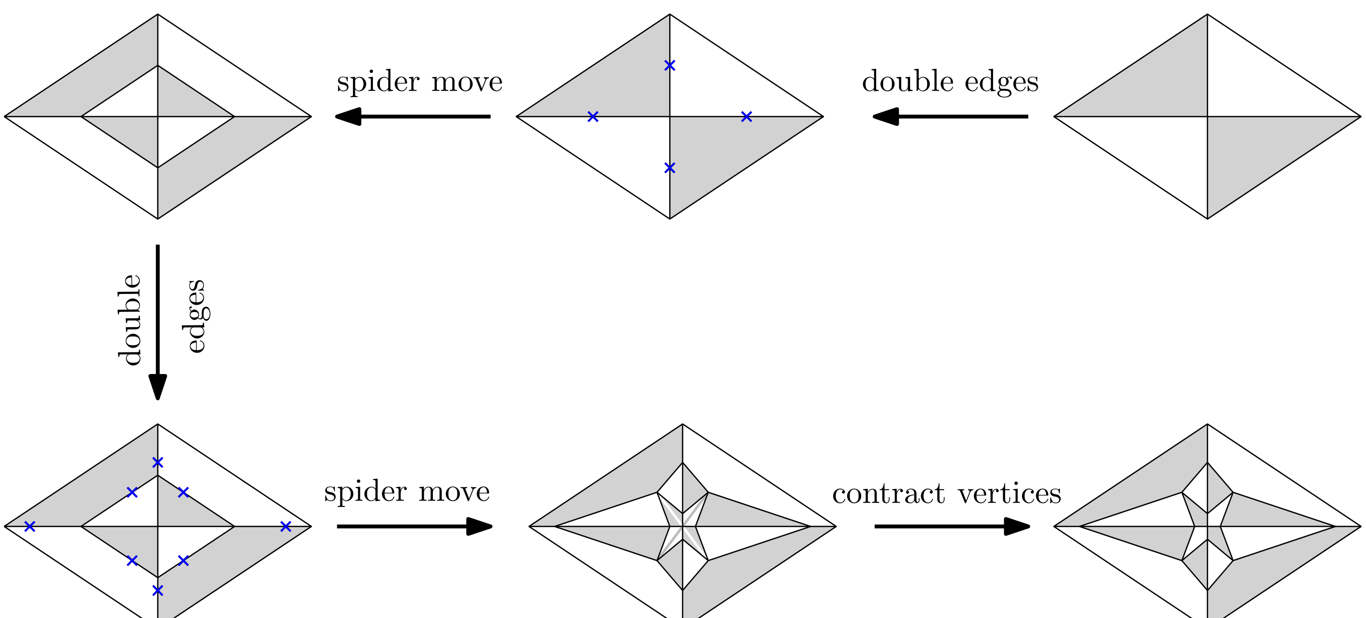
T-embeddings of $\mathcal{G}^*$ are preserved under elementary transformations of $\mathcal{G}$.

Elementary transformations preserving the dimer measure






spider move
double edges
double edges
spider move
contract vertices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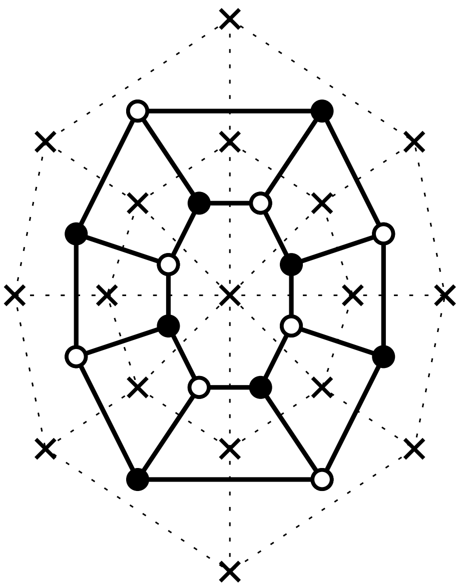
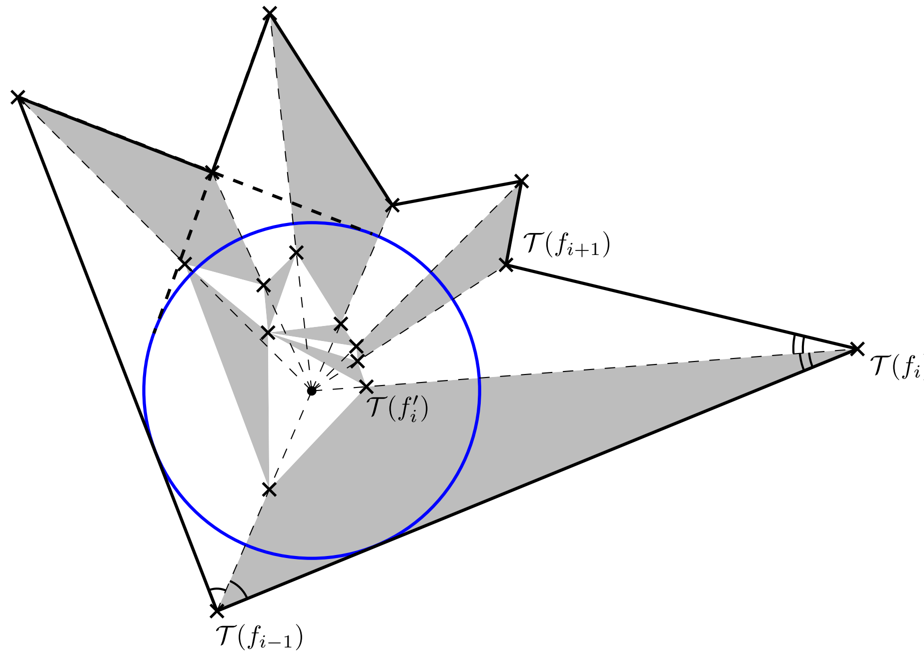
Perfect t-embeddings

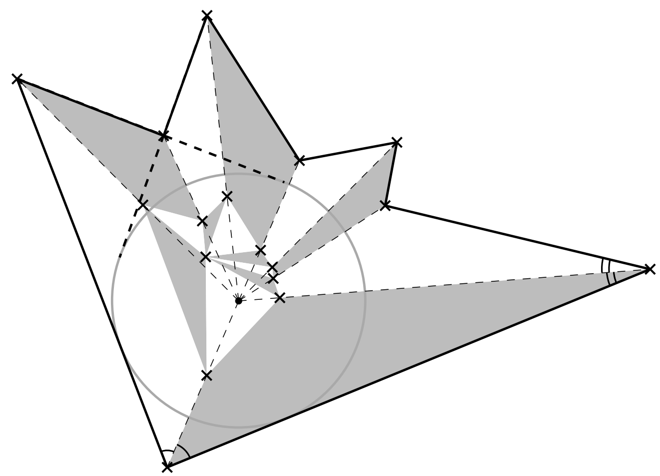


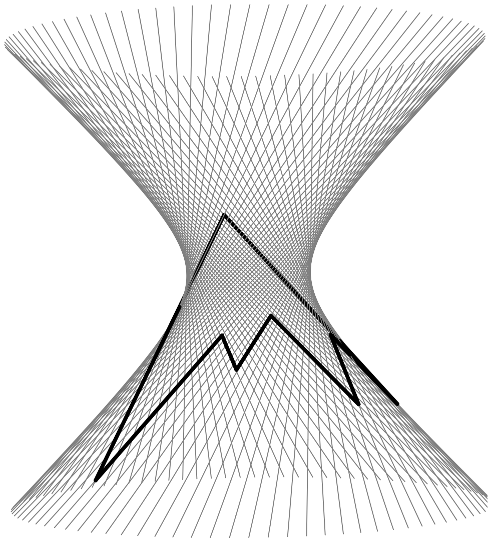


Definition [Chelkak, Laslier, R.] Perfect t-embeddings:

- outer face is tangental (not necessary convex)
- outgoing edges $=$ bisectors

General setup





Theorem (Chelkak, Laslier, R. '21)

Assume $\mathcal{G}^\delta$ are perfectly t-embedded.

a) *Technical assumptions on faces*

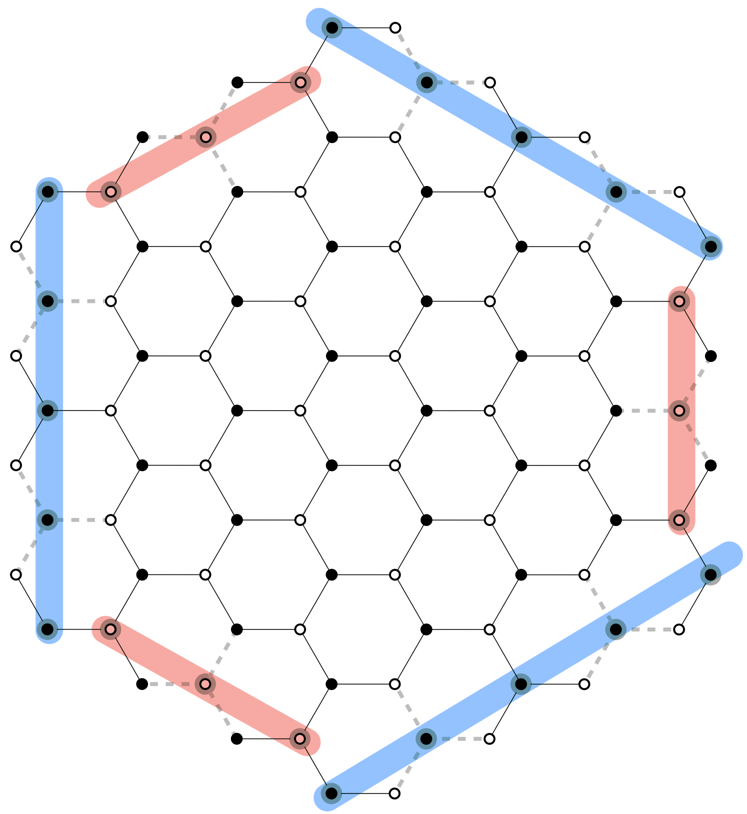
b) *The origami maps converge to a maximal surface in the Minkowski space $\mathbb{R}^{2,1}$*

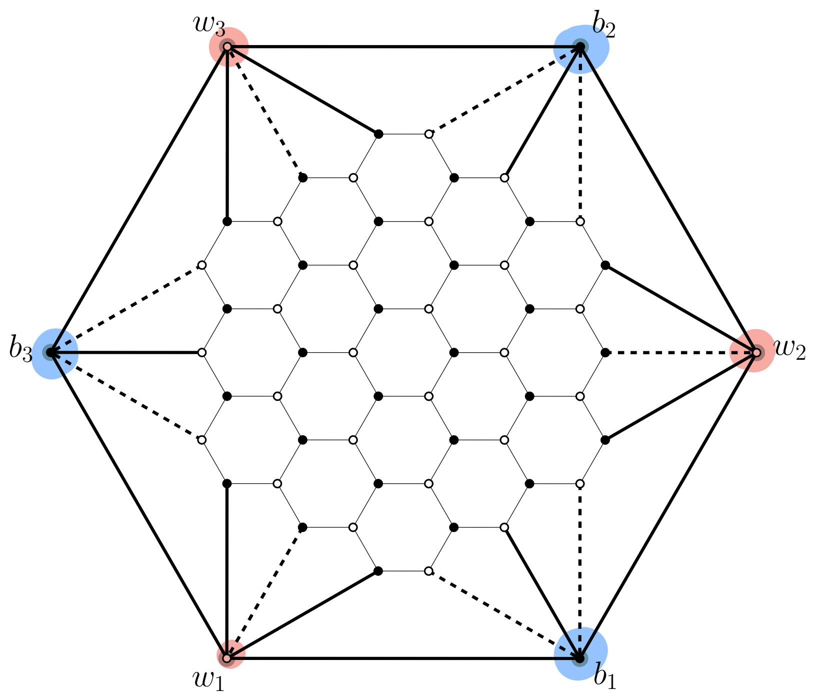
$\Rightarrow$ *convergence to the Gaussian free field in the conformal parametrization of this surface.*

Rmk: Existence of perfect t-embeddings remains an open question.

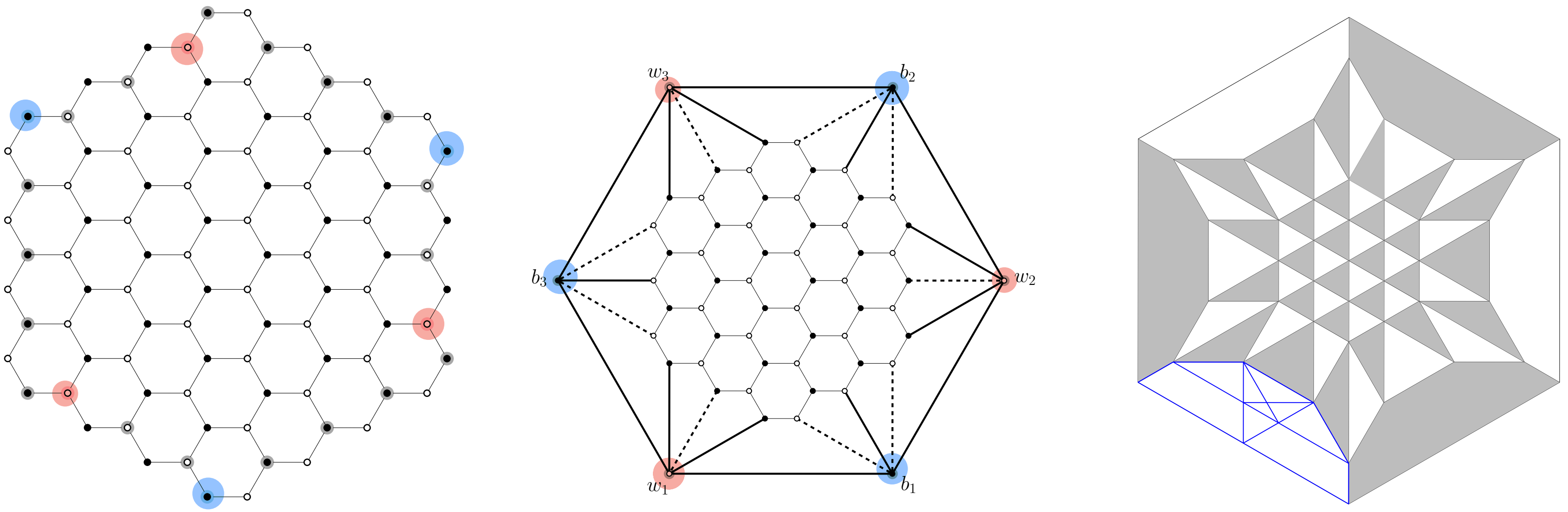
Perfect t-embedding of Hexagon

Reduced hexagon





Perfect t-embedding of the hexagon



Theorem (Berggren, Nicoletti, R. '24)

Define Coulomb gauge functions $F : B \to \mathbb{C}$ and $G : W \to \mathbb{C}$ by

$$F(b) := e^{i2\pi/3} K_{\mathbb{R}}^{-1}(w_2, b) + K_{\mathbb{R}}^{-1}(w_3, b) + e^{-i2\pi/3} K_{\mathbb{R}}^{-1}(w_1, b),$$
$$G(w) := -K_{\mathbb{R}}^{-1}(w, b_1) + e^{i\pi/3} K_{\mathbb{R}}^{-1}(w, b_2) + e^{-i\pi/3} K_{\mathbb{R}}^{-1}(w, b_3).$$

Then $\mathcal{T} = \mathcal{T}_{(F,G)}$ is a perfect t-embedding of $(H'_n)^*$.

Perfect t-embedding of the hexagon

$$F(b) = e^{i2\pi/3} K_{\mathbb{R}}^{-1}(w_2, b) + K_{\mathbb{R}}^{-1}(w_3, b) + e^{-i2\pi/3} K_{\mathbb{R}}^{-1}(w_1, b),$$
$$G(w) = -K_{\mathbb{R}}^{-1}(w, b_1) + e^{i\pi/3} K_{\mathbb{R}}^{-1}(w, b_2) + e^{-i\pi/3} K_{\mathbb{R}}^{-1}(w, b_3).$$

$$\begin{aligned} d\mathcal{T}(bw^*) &= F(b) K_{\mathbb{R}}(b, w) G(w), \\ d\mathcal{O}(bw^*) &= F(b) K_{\mathbb{R}}(b, w) \overline{G(w)}. \end{aligned}$$

- **[Petrov '12]:** Inverse Kasteleyn matrix of the uniformly weighted hexagon admits a double integral formula.
- This provides us with expressions of $\mathcal{T}_n$ and $\mathcal{O}_n$ in terms of double integrals.
- The integral expression allows for asymptotic analysis using a classical stepest descent analysis

Rigidity condition

Theorem (Berggren, Nicoletti, R. '24)

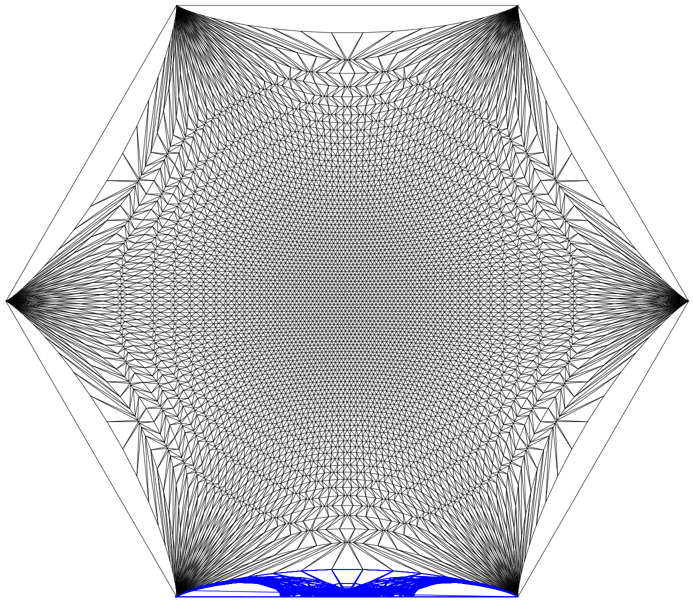
Given a compact set $\mathcal{K} \subset \mathrm{Hex}$, there exist positive $N_{\mathcal{K}}, C_{\mathcal{K}}$ and $\varepsilon_{\mathcal{K}}$ which only depend on $\mathcal{K}$, such that for all pairs of vertices $v \sim v'$ of the dual graph $(H_n')^*$ such that both $\mathcal{T}_n(v), \mathcal{T}_n(v') \in \mathcal{K}$ we have

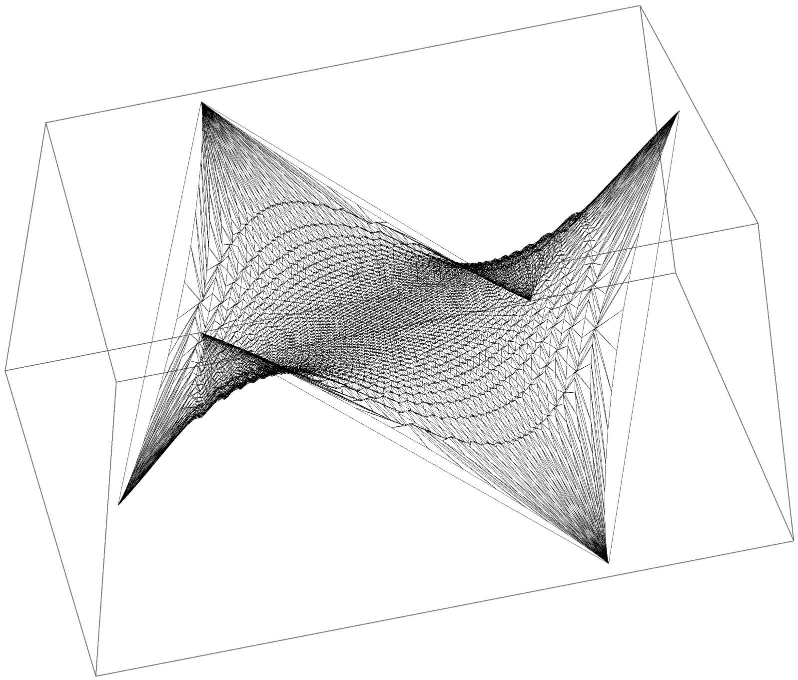
$$\frac{1}{nC_{\mathcal{K}}} \leq |\mathcal{T}_n(v') - \mathcal{T}_n(v)| \leq \frac{C_{\mathcal{K}}}{n}$$

for all $n > N_{\mathcal{K}}$.

In addition the angles of the faces of the perfect t-embedding inside $\mathcal{K}$ are contained in $(\varepsilon_{\mathcal{K}}, \pi - \varepsilon_{\mathcal{K}})$ for all $n > N_{\mathcal{K}}$.

Perfect t-embeddings of the hexagon

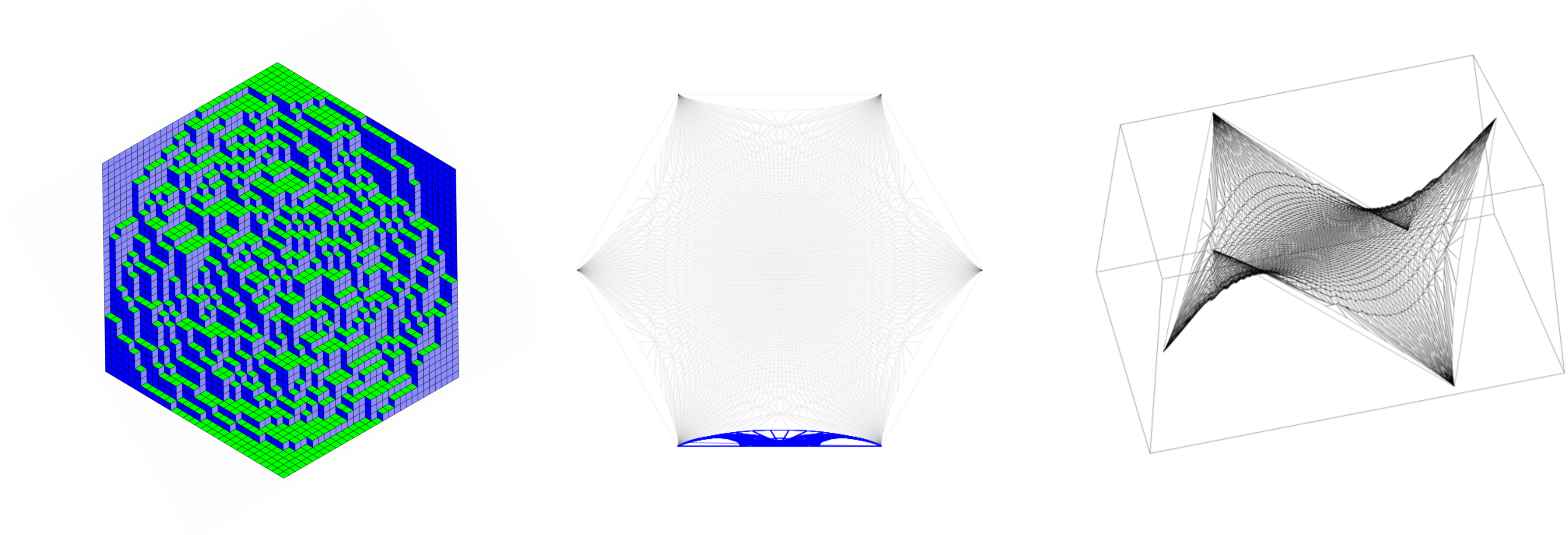




Theorem (Berggren, Nicoletti, R. '24)

The pair $(\mathcal{T}_n, \mathcal{O}_n) \to (z, \vartheta(z))$*, as* $n \to \infty$*, where* $(z, \vartheta(z)) \in \mathbb{R}^2 \times \mathbb{R}$ *is the graph of a maximal surface in* $\mathbb{R}^{2,1}$.

The scaling limit of dimer fluctuations in homogeneous hexagon via the intrinsic conformal structure of a Lorentz-minimal surface.



Theorem (Berggren, Nicoletti, R. '24)

Let $\mathcal{T}_n$ be the sequence of perfect t-embeddings of the reduced uniformly weighted hexagon H'_n, with corresponding origami maps $\mathcal{O}_n$. All assumptions of the main theorem of [CLR'21] hold for the sequence $\mathcal{T}_n$.