Skew RSK dynamics

T. Sasamoto

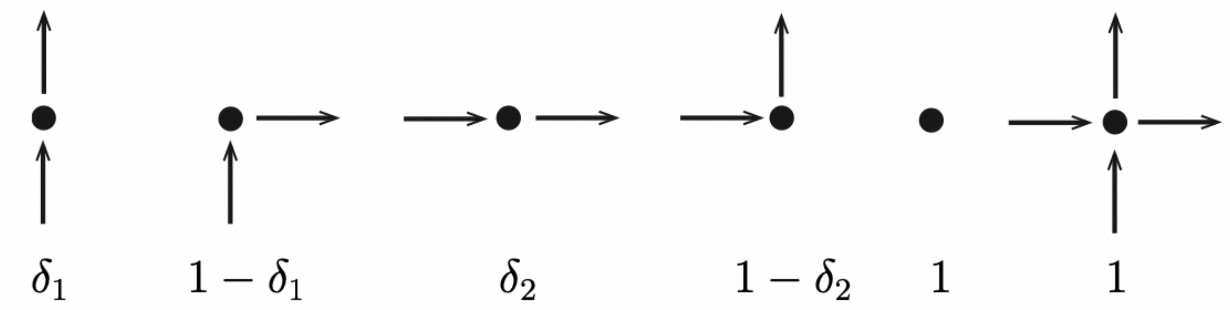
(Joint works with T. Imamura, M. Mucciconi, T. Scrimshaw)

23 Mar 2024 @ IPAM

Refs: Forum of Mathematics Pi (2023) e27 1–101,
arXiv: 2204.08420, arXiv: 2406(?).*****

0. Stochastic 6-vertex model

Stochastic 6-vertex model



- Continuous time limit is ASEP.
- Stochastic 5-vertex model with $\delta_{1,2} = 0, 1$ is a TASEP.

Q: Are 6-vertex models (or some aspects of them) free fermionic?
TASEP? ASEP? $\Delta = \frac{1}{2}$?

Jimbo "There is a huge gap between free fermion models and integrable but non-free fermion models."

Plan

1. TASEP (or stochastic 5-vertex model), RSK and Schur measure, and $T = 0$ free fermion

2. KPZ models (or stochastic higher-spin 6 vertex model) and q-Whittaker measure

   Relation between q-Whittaker and periodic Schur measures

   $\Rightarrow$ Relation between KPZ models and $T > 0$ free fermion!

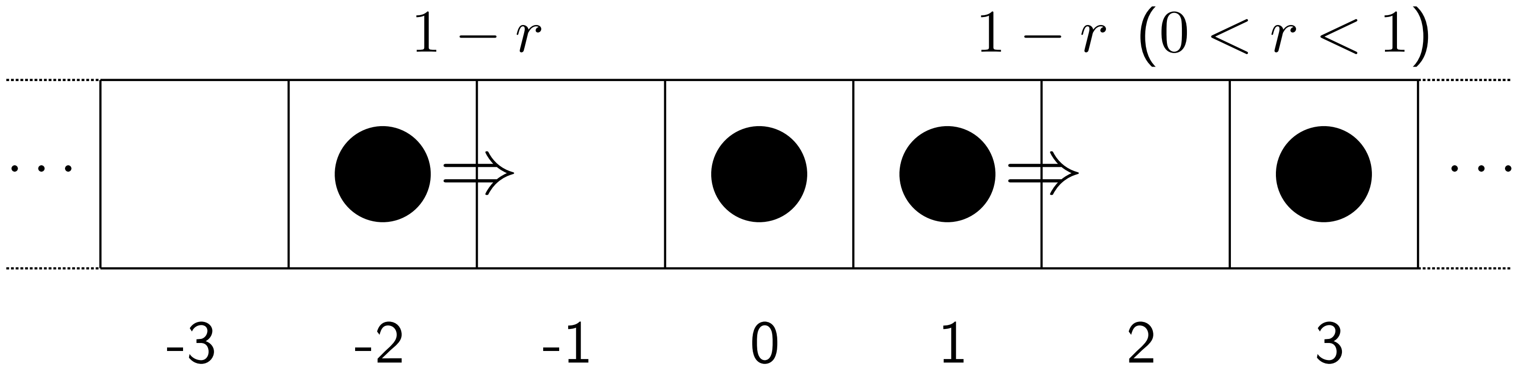
3. Bijection by skew RSK dynamics

4. Ideas of proof

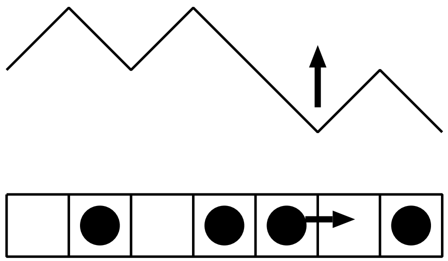
5. Column skew RSK dynamics (connection to BBS)

1. TASEP, RSK and Schur measure, and $T = 0$ free fermion

TASEP

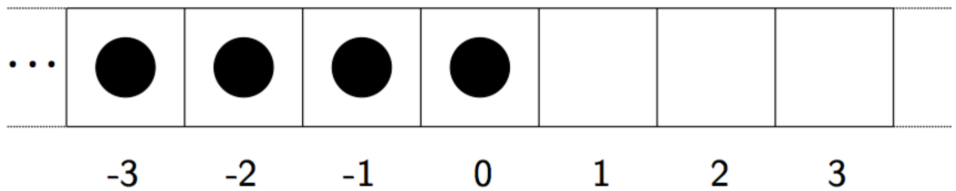


$N(t)$: Integrated current at $(0, 1)$ upto time t from step i.c.



$N(t) \sim h(0, t)$: height

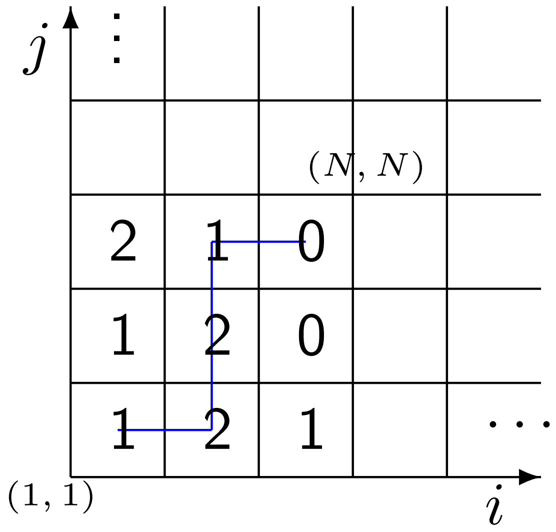
Step i.c.



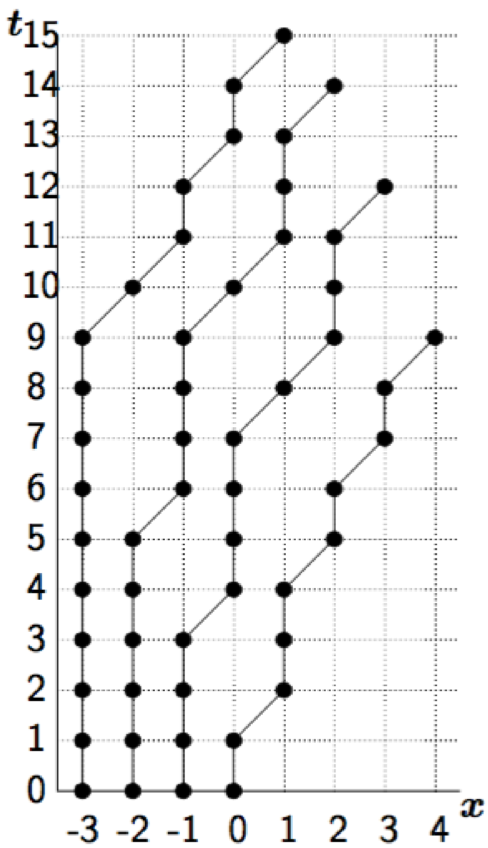
Mapping to combinatorics

2000 Johansson

Waiting times w_{ij}: iid geo(r)



Trajectories



$$G_N = \max_{\substack{\text{up-right paths from} \\ (1,1)\text{to}(N,N)}} \left(\sum_{\substack{(i,j) \\ \text{on a path}}} w_{i,j} \right)$$

$$\mathbb{P}[N(t) \geq N] = \mathbb{P}[G_N \leq t]$$

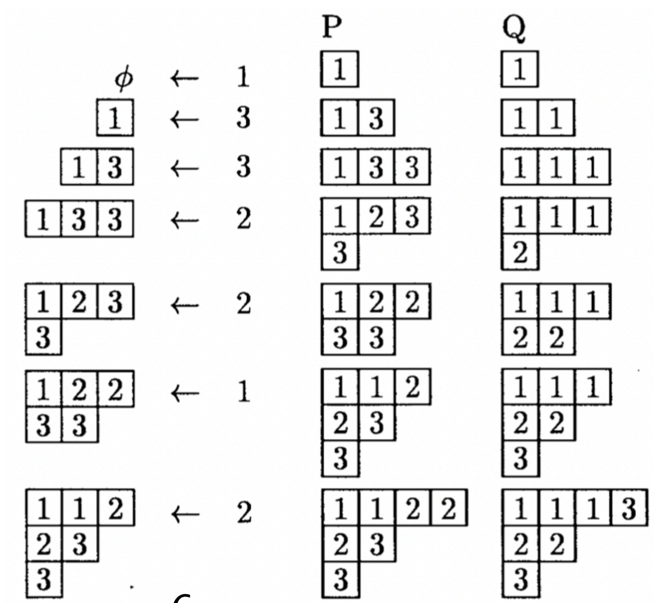
RSK

Robinson-Shensted-Knuth correspondence: Bijection between $N \times M$ $\mathbb{N}$-matrices and pairs of semi-standard tablueaux (SST)

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \Leftrightarrow \begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 3 & 3 & 2 & 2 & 1 & 2 \end{pmatrix} \Leftrightarrow \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}$$

RSK algorithm

Insertion and bumping



Schur function and its Cauchy identity

- Schur function (Combinatorial definition)

$$s_\lambda(a) = \sum_{T \in \mathrm{SST}(\lambda)} a^T, a^T = \prod_i a_i^{\#i \text{ in } T}$$

- By RSK, one can prove its Cauchy identity.

$$\sum_{\lambda \in \mathcal{P}} s_\lambda(a) s_\lambda(b) = \prod_{i=1}^{N} \prod_{j=1}^{N} \frac{1}{1 - a_i b_j} \quad (=: Z)$$

- General a_i, b_j corresponds to w_{ij} with geo($a_i b_j$)

Current distribution

- By restricting the sum and noting $G_N = \lambda_1$, we have

$$\mathbb{P}[G_N \le u] = \frac{1}{Z} \sum_{\lambda, \lambda_1 \le u} s_\lambda(a) s_\lambda(b)$$

- Schur measure

$$\frac{1}{Z} s_\lambda(a) s_\lambda(b)$$

  By Jacobi-Trudi formula $s_\lambda(x) = \det(\phi_n(x_m))$, the Schur measure is a DPP (determinantal point process) associated with $T = 0$ free fermion.

- 2000 Baik Rains

  Symmetrized version: $P = Q$

2. KPZ models and q-Whittaker measure

- KPZ models: such as ASEP, q-TASEP, stochastic HS6VM.
  2011 Borodin-Corwin, 2016 Borodin-Bufetov-Wheeler, 2021 Bufetov-Mucciconi-Petrov

  Related to q-Whittaker (or Hall-Littewood) measure.

- Geometric q-PushTASEP(2015 Matveev-Petrov) is related to the q-Whittaker measure of the form

$$\frac{1}{Z} b_\mu(q) P_\mu(a) P_\mu(b), \quad b_\mu(q) = \prod_{i \geq 1} \frac{1}{(q;q)_{\mu_i - \mu_{i+1}}}$$

  where $a = (a_1, \cdots, a_N), b = (b_1, \cdots, b_M)$.

  The Nth particle position at time M is related to μ_1 as $X_N(M) \stackrel{d}{=} \mu_1 + N$. Note: No single det formula for P_μ.

Periodic Schur measure

- Periodic Schur measure (2007 Borodin, 2018 Betea-Bouttier)

$$\frac{1}{Z}\sum_{\rho\in\mathcal{P},\rho(\subset\lambda)} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b)$$

- Its shift mixed version ($\lambda_i \to \lambda_i + S$) with

$$\mathbb{P}(S=\ell) = \frac{t^\ell q^{\ell^2/2}}{(q;q)_\infty \theta(-tq^{1/2})}, \qquad \ell \in \mathbb{Z}, \text{ for } t>0$$

with $\theta(x) = (x;q)_\infty (q/x;q)_\infty$, is a DPP associated with $T>0$ free fermion and hence

$$\mathbb{P}\left(\lambda_1 + S \le n\right) = \det\left(1-K\right)_{\ell^2(\mathbb{Z})}$$

where K is a free fermion kernel at $T>0$.

Relation between q-Whittaker and periodic Schur

- Theorem: μ_1: q-Whittaker, λ_1: periodic Schur

$$\mathbb{E}\left[1/(-tq^{\frac{1}{2}+n-\mu_1};q)_\infty\right] = \mathbb{P}(\lambda_1 + S \leq n)$$

Connection between q-Whittaker & periodic Schur measures

- This is equivalent to the following identity

$$\sum_{\ell=0}^{N} \frac{q^\ell}{(q;q)_\ell} \sum_{\mu:\mu_1=N-\ell} b_\mu(q) P_\mu(a) P_\mu(b) = \sum_{\lambda,\rho:\lambda_1=N} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b)$$

where $b_\mu(q) = \prod_{i\geq 1} \frac{1}{(q;q)_{\mu_i - \mu_{i+1}}}$.

We found a bijective proof of this!

4. Bijection by skew RSK dynamics

Skew Schur function

$$s_{\lambda/\rho}(x) = \sum_{T \in \mathrm{SST}(\lambda/\rho)} x^T$$

				1
		2	3	4
1	3	5		
2				

where SST is the set of skew semistandard tableaux.

RHS of the identity is related to a pair (P, Q). Try to find a bijection from (P, Q) to something which is related to q-Whittaker function!

Sqeezing: $(P, Q) \to (P_1, Q_1)$

P:

				1
		2	3	4
1	3	5		
2				

Q:

				2
		1	3	3
2	2	5		
3				

$\longrightarrow$

P_1:

			1
	2	3	4
1	3	5	
2			

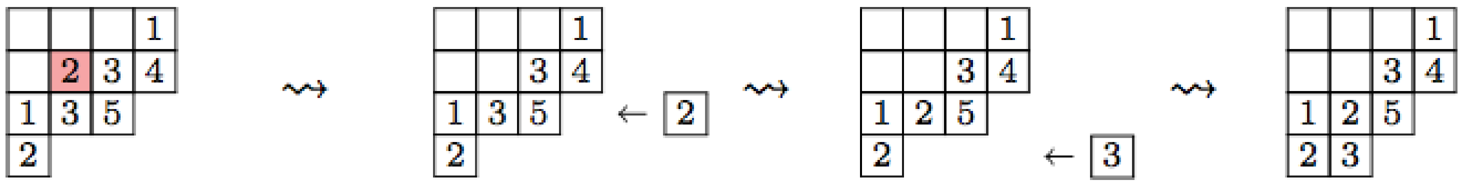
Q_1:

			2
	1	3	3
2	2	5	
3			

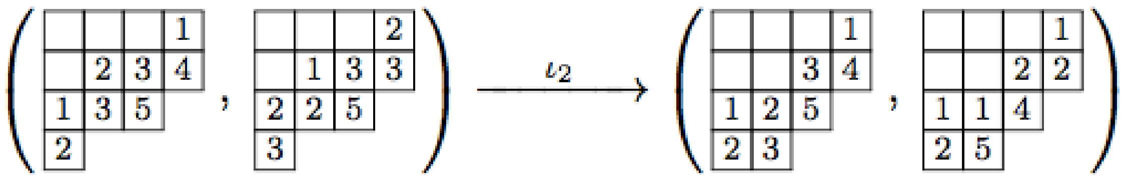
$\nu =$

Skew RSK map

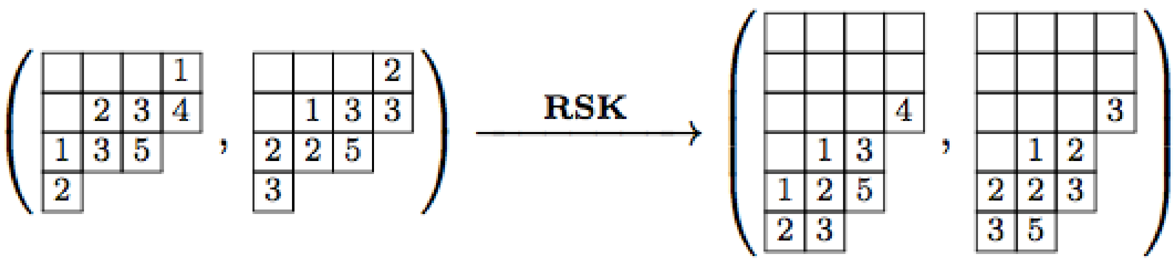
Internal insertion (Sagan-Stanley 1990)



Operation ι_2

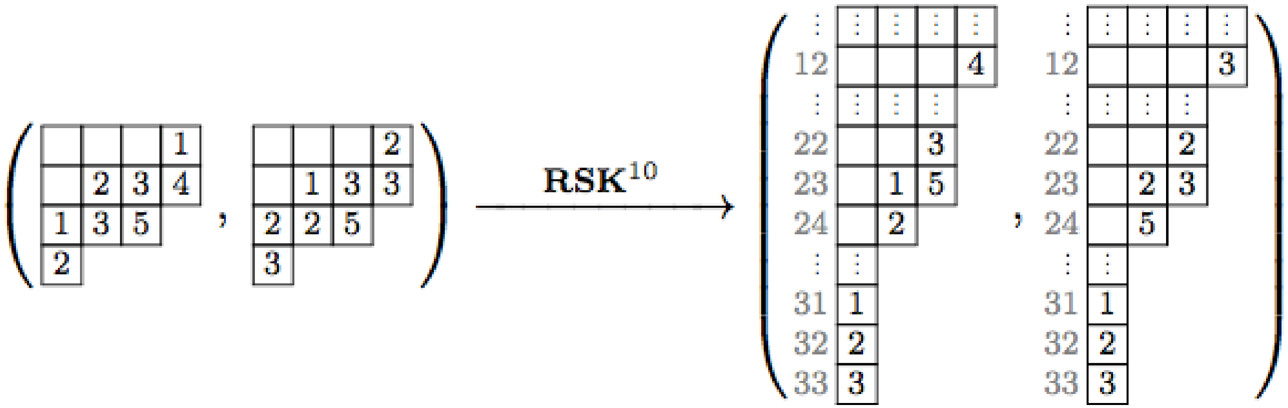


Skew RSK map: $\mathrm{RSK}(P, Q) = \iota_2^N(P, Q)$



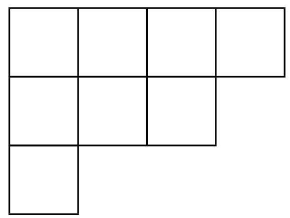
Skew RSK dynamics

Iterating skew RSK maps: $(P_{t+1}, Q_{t+1}) = \mathbf{RSK}(P_t, Q_t)$



Asymptotic tableaux and their shape

$$(V, W) = \left(\begin{array}{|c|c|c|c|} \hline 1 & 1 & 3 & 4 \\ \hline 2 & 2 & 5 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array} \; , \; \begin{array}{|c|c|c|c|} \hline 1 & 2 & 2 & 3 \\ \hline 2 & 5 & 3 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array} \right)$$

$\mu =$ 

$V, W \in \mathrm{VST}(\mu)$: "vertically strict tableaux" (VST) of same shape μ with elements increasing only in each column.

Similar to Box-Ball systems! (1990 Takahashi Satsuma, 2012 IKT)

Combinatorial formula for q-Whittaker function

- q-Whittaker function (e.g. 2012 Schilling Tingley)

$$P_\mu(x) = \sum_{V \in \mathrm{VST}(\mu)} q^{H(V)} x^V$$

1	2	2	3
2	5	3	
3			

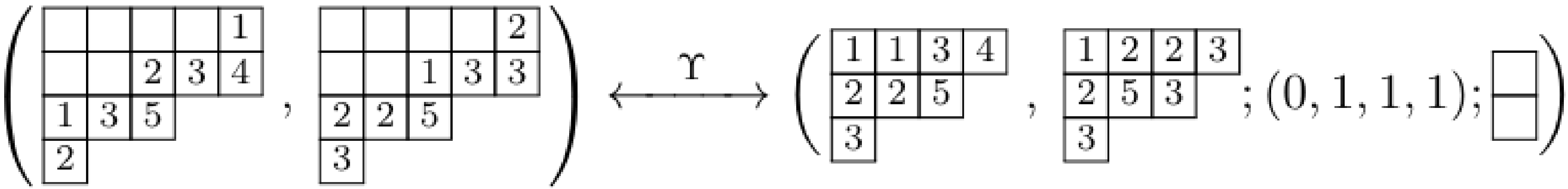
where H is energy function (e.g. 1997 Nakayashiki Yamada). In a way $H(V)$ measures how a VST V is far away from a semistandard tableaux. Note: P_μ tends to s_μ when $q \to 0$.

- Recall the identity

$$\sum_{\ell=0}^{N} \frac{q^\ell}{(q;q)_\ell} \sum_{\mu:\mu_1=N-\ell} b_\mu(q) P_\mu(a) P_\mu(b) = \sum_{\lambda,\rho:\lambda_1=N} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b)$$

- How do $H(V)$ and $b(\mu)$ appear?

Bijection $\Upsilon : (P,Q) \leftrightarrow (V,W,\kappa,\nu)$



(P,Q): A pair of skew SSTs with same shape λ/ρ

ν: partition obtained by "squeezing" (P,Q) to (P_1,Q_1).

(V,W): A pair of VSTs with same shape μ

$$\kappa \in \mathcal{K}(\mu) = \{\kappa = (\kappa_1,\ldots,\kappa_{\mu_1}) \in \mathbb{N}_0^{\mu_1} : \kappa_i \geq \kappa_{i+1} \text{ if } \mu'_i = \mu'_{i+1}\}$$

Theorem: There is a bijection Υ with weight preserving property

$$|\rho| = H(V) + H(W) + |\kappa| + |\nu|$$

Note $\sum_{\kappa\in\mathcal{K}(\mu)} q^{|\kappa|} = b_\mu(q)$, $\mathbb{P}[\nu_1 = \ell] = \frac{q^\ell}{(q;q)_\ell}(q;q)_\infty$.

A remark: Cauchy identities for three polynomials

Schur

$$\sum_{\lambda \in \mathcal{P}} s_\lambda(a) s_\lambda(b) = \prod_{i=1}^{N} \prod_{j=1}^{N} \frac{1}{1 - a_i b_j}$$

q-Whittaker

$$\sum_{\mu \in \mathcal{P}} P_\mu(a) Q_\mu(b) = \prod_{i=1}^{N} \prod_{j=1}^{N} \frac{1}{(a_i b_j; q)_\infty}$$

Skew Schur

$$\sum_{\substack{\lambda, \rho \in \mathcal{P} \\ \rho \subset \lambda}} q^{|\rho|} s_{\lambda/\rho}(a) s_{\lambda/\rho}(b) = \frac{1}{(q;q)_\infty} \prod_{i=1}^{N} \prod_{j=1}^{N} \frac{1}{(a_i b_j; q)_\infty}$$

Our bijection gives the first bijective proof of the Cauchy identity for q-Whittaker polynomials.

Symmetrized version

Littlewood identity for Schur function ($P = Q$ in RSK)

$$\sum_{\lambda:\lambda' \text{is even}} s_\lambda(x) = \prod_{1\le i<j\le n}^{n} \frac{1}{1 - x_i x_j}$$

Setting $P = Q$ in skew RSK dynamics, one can prove

Theorem: (2006 Warnaar)

$$\sum_\mu b_\mu(q;z) P_\mu(x;q^2) = \prod_{i=1}^{n} \frac{1}{(zx_i;q)_\infty} \prod_{1\le i<j\le n} \frac{1}{(x_i x_j;q^2)_\infty}$$

where

$$b_\mu(q;z) = \prod_{i=2,4,6\ldots} \frac{[qz^2+1]_{q^2}^{\mu_i-\mu_{i+1}}}{(q^2;q^2)_{\mu_i-\mu_{i+1}}} \prod_{i=1,3,5,\ldots} \frac{z^{\mu_i-\mu_{i+1}}}{(q;q)_{\mu_i-\mu_{i+1}}}$$

with

$$[A+B]_p^k = \sum_{j=0}^{k} A^j B^{k-j} \binom{k}{j}_p, \quad \binom{k}{j}_p = \frac{(p;p)_k}{(p;p)_j (p;p)_{k-j}}$$

A refined identity for the symmetrized version

Putting conditions on the length of the first rows gives an identity for restricted Littewood sums for q-Whittaker and skew Schur.

Theorem:

$$\sum_{\ell=0}^{k} g_\ell(z,q) \sum_{\mu:\mu_1=k-\ell} b_\mu(q;z) P_\mu(x;q^2) = \sum_{\lambda,\rho:\lambda_1=k} z^{\mathrm{odd}(\lambda')+\mathrm{odd}(\rho')} q^{|\rho|} s_{\lambda/\rho}(x)$$

where

$$g_\ell(z,q) = [qz^2+q^2]^{\ell}_{q^2}/(q^2;q^2)_\ell$$

This is useful for studying KPZ models in half-space.

4. Ideas of proof

- Proving properties of skew RSK dynamics based on its rules is difficult.
- Original Robinson's algorithm, which maps a permutation to a canonical one, can be understood as an application of crystal symmetry.
- We can use (affine) crystal to study skew RSK dynamics and prove our theorem. For a canonical object, skew RSK dynamics is linearized.

Affine Crystal for VST

VST(μ) is identified with $B^{\mu'_1,1} \otimes B^{\mu'_2,1} \otimes \cdots \otimes B^{\mu'_{\mu_1},1}$, the Kirillov-Reschetikhin crystals of type $A^{(1)}$.

Kashiwara operators: $\widetilde{e}_i, \widetilde{f}_i$ with $i = 1, \cdots, n-1$ and

$$\widetilde{e}_0 = \mathrm{pr}^{-1} \circ \widetilde{e}_1 \circ \mathrm{pr}, \ \widetilde{f}_0 = \mathrm{pr}^{-1} \circ \widetilde{f}_1 \circ \mathrm{pr}$$

where pr is the promotion operator.

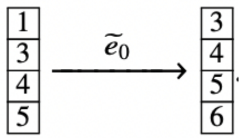
Kashiwara operators

$i \neq 0$ on words ("signature rule")

$\pi =$	4	2	3	2	1	2	3	1	4	3	3	2	1	2	4	1	2	3	3
		)	(	)		)	(			(	(	)		)			)	(	(
		)	(	)		)	(			(	(	)		)			)	(	(
$\widetilde{e}_2(\pi) =$	4	2	3	2	1	2	3	1	4	3	3	2	1	2	4	1	2	**2**	3
$\widetilde{f}_2(\pi) =$	4	2	3	2	1	**3**	3	1	4	3	3	2	1	2	4	1	2	3	3.

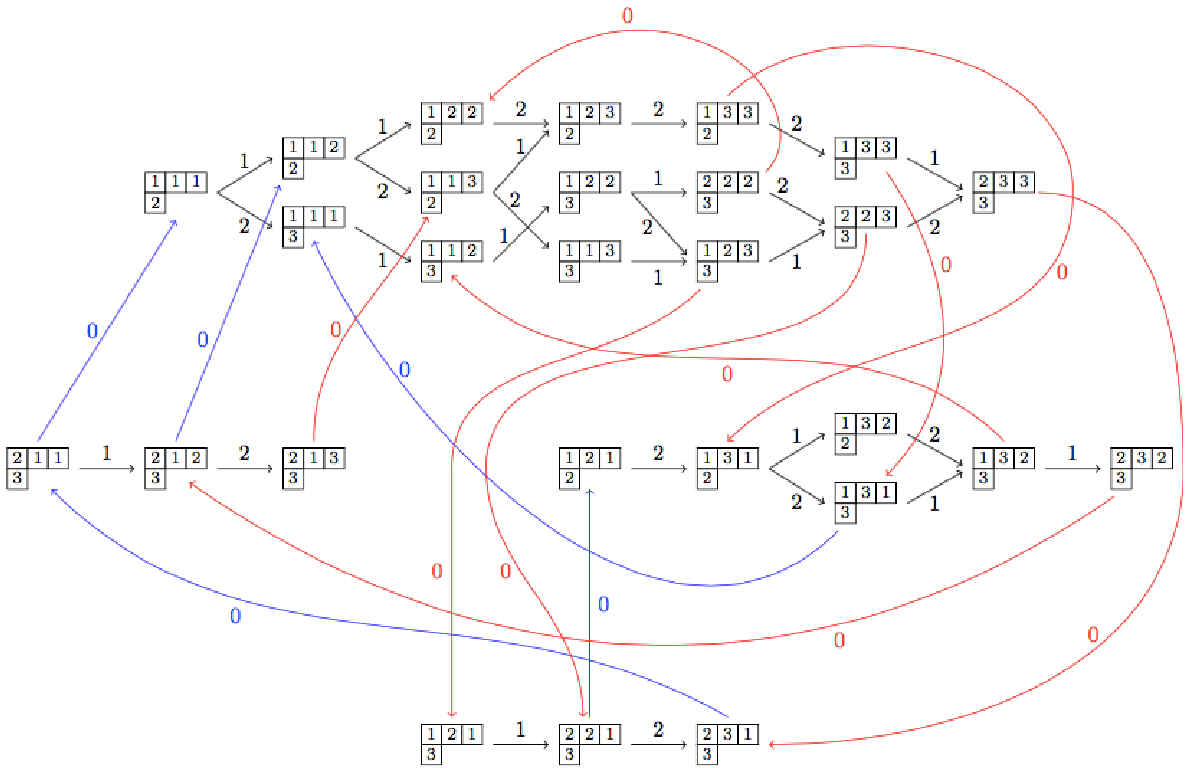
For tableaux, use the column reading words.

e_0: on a single column tableau, replace the 1-cell with an n-cell

and reorder. 

On VST, use $\mathrm{pr}(b_1 \otimes \cdots \otimes b_N) = \mathrm{pr}(b_1) \otimes \cdots \otimes \mathrm{pr}(b_N)$.

Example of affine crystal graph



Edge $\xrightarrow{i}$ is $\widetilde{f}_i$. Blue arrows are 0-Demazure arrows.

Here energy is $H = \#\widetilde{f}_0 - \#\widetilde{e}_0$.

Leading map for VST

Affine bicrystal structure for (V, W)

$$\widetilde{e}_i \times \mathbf{1}, \qquad \mathbf{1} \times \widetilde{e}_i, \qquad \widetilde{f}_i \times \mathbf{1}, \qquad \mathbf{1} \times \widetilde{f}_i.$$

$$(V, W) \xrightarrow{\mathcal{L}_V \times \mathcal{L}_W} (Y, Y)$$

Example

$$\left(\begin{array}{cccc} 1 & 1 & 3 & 4 \\ 2 & 2 & 5 & \\ 3 & & & \end{array} , \begin{array}{cccc} 1 & 2 & 2 & 3 \\ 2 & 5 & 3 & \\ 3 & & & \end{array} \right) \xrightarrow{\mathcal{L}_V \times \mathcal{L}_W} \left(\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & \\ 3 & & & \end{array} , \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & \\ 3 & & & \end{array} \right)$$

where

$$\mathcal{L}_V = \widetilde{e}_2 \circ \widetilde{e}_3 \circ \widetilde{e}_4 \circ \widetilde{e}_1 \circ \widetilde{e}_2 \circ \widetilde{e}_3 \circ \widetilde{e}_1 \circ \widetilde{e}_2,$$

$$\mathcal{L}_W = \widetilde{e}_3 \circ \widetilde{e}_4 \circ \widetilde{e}_1 \circ \widetilde{f}_0 \circ \widetilde{f}_4 \circ \widetilde{f}_3 \circ \widetilde{f}_1{}^2 \circ \widetilde{e}_2 \circ \widetilde{e}_1{}^3 \circ \widetilde{e}_2$$

Note $H(V) = 0, H(W) = 1$.

Affine Crystal for (P, Q)

Affine bicrystal structure for (V, W) can be lifted to (P, Q).

$$\widetilde{E}_0^{(2)} = \iota_2 \circ (\mathbf{1} \times \widetilde{e}_1) \circ \iota_2^{-1}, \qquad \widetilde{F}_0^{(2)} = \iota_2 \circ (\mathbf{1} \times \widetilde{f}_1) \circ \iota_2^{-1},$$

$$\widetilde{E}_0^{(1)} = \iota_1 \circ (\widetilde{e}_1 \times \mathbf{1}) \circ \iota_1^{-1}, \qquad \widetilde{F}_0^{(1)} = \iota_1 \circ (\widetilde{f}_1 \times \mathbf{1}) \circ \iota_1^{-1}.$$

This is consistent with the projection $(P, Q) \to (V, W)$.

Theorem: Skew RSK map commute with $\widetilde{E}_i^{(\epsilon)}, \widetilde{F}_i^{(\epsilon)}$ for all $i = 0, 1, .., n-1$ and $\epsilon = 1, 2$.

Leading map and leading tableaux

By replacing $\widetilde{e}_i, \widetilde{f}_i$ by $\widetilde{E}_i^{(\epsilon)}, \widetilde{F}_i^{(\epsilon)}, \epsilon = 1, 2$, one can define $\mathcal{L}$, which sends (P, Q) to a pair of "leading tableaux" (T, T), where whenever T has k i-cells at row r, then it has at least k $(i-1)$-cells at row $r-1$ for all r and $i = 2, 3, \ldots$.

$$(P, Q) \xrightarrow{\quad\mathcal{L}\quad} (T, T)$$

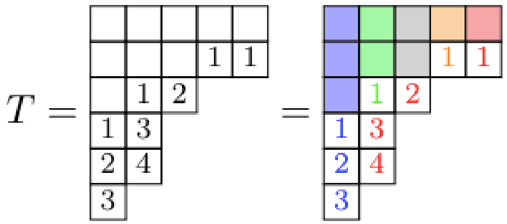
Example

$$\left(\begin{array}{|c|c|c|c|}\hline & & & 1 \\ \hline & 2 & 3 & 4 \\ \hline 1 & 3 & 5 \\ \cline{1-3} 2 \\ \cline{1-1} \end{array} \; , \; \begin{array}{|c|c|c|c|}\hline & & & 2 \\ \hline & 1 & 3 & 3 \\ \hline 2 & 2 & 5 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array} \right) \xrightarrow{\quad\mathcal{L}\quad} \left(\begin{array}{|c|c|c|c|}\hline & & & 1 \\ \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 \\ \cline{1-3} \end{array} \; , \; \begin{array}{|c|c|c|c|}\hline & & & 1 \\ \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 \\ \cline{1-3} \end{array} \right)$$

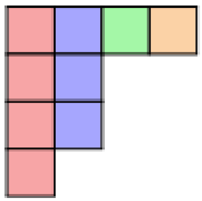
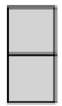
Note that the change of $\#$ empty boxs $= H(V) + H(W)$.

Finding κ

Prop. There is a bijection $\mathrm{LdT}(\mu) \longleftrightarrow \mathcal{K}(\mu) \times \mathcal{P}$



$$\begin{pmatrix} 0 & 2 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \end{pmatrix} = \begin{pmatrix} 0 & 1+1 & 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \cdots \end{pmatrix}$$

$\mu =$  $\kappa = ((1), (3), (2, 1))$ $\nu =$ 

This completes the construction of the bijection.

Can define $T = T(\mu, \kappa; \nu)$.

Proof: Linearization

Map $\mathcal{L}$ commutes with RSK map and linearizes it.

$$\begin{array}{ccc}
(P,Q) & \xrightarrow{\mathcal{L}} & (T,T) \\
\mathbf{RSK}\downarrow & & \downarrow\mathbf{RSK} \\
(P',Q') & \xrightarrow{\mathcal{L}} & (T',T')
\end{array}$$

Theorem: If $T = T(\mu, \kappa; \nu)$, then $T' = T(\mu, \kappa + \mu'; \nu)$.

5. Column skew RSK

IMS+Scrimshaw 2024+

- Horizontally weak tableaux (HWT) instead of VST
- Modified Hall-Littlewood polynomials
- The standard Box-Ball system appears
- KKR(Kerov-Kirillov-Reshetikhin) bijection linearizes the cRSK dynamics
- Needed to prove a new property of a crystal

Example

$$
\begin{array}{cc}
\cdots 0000000200000000000 \cdots & \cdots 0000000200000000000 \cdots \\
\cdots 0000220000000000000 \cdots & \cdots 0000220000000000000 \cdots \\
\cdots 1130000000000000000 \cdots & \cdots 1330000000000000000 \cdots \\
\multicolumn{2}{c}{\downarrow} \\
\cdots 0000000022000000000 \cdots & \cdots 0000000022000000000 \cdots \\
\cdots 0000002000000000000 \cdots & \cdots 0000002000000000000 \cdots \\
\cdots 0001130000000000000 \cdots & \cdots 0001330000000000000 \cdots \\
\multicolumn{2}{c}{\downarrow} \\
\cdots 0000000000220000000 \cdots & \cdots 0000000000220000000 \cdots \\
\cdots 0000000120000000000 \cdots & \cdots 0000000230000000000 \cdots \\
\cdots 0000001300000000000 \cdots & \cdots 0000001300000000000 \cdots \\
\multicolumn{2}{c}{\downarrow} \\
\cdots 0000000000002220000 \cdots & \cdots 0000000000002220000 \cdots \\
\cdots 0000000001100000000 \cdots & \cdots 0000000001300000000 \cdots \\
\cdots 0000000030000000000 \cdots & \cdots 0000000030000000000 \cdots \\
\multicolumn{2}{c}{\downarrow} \\
\cdots 0000000000000002220 \cdots & \cdots 0000000000000002220 \cdots \\
\cdots 0000000000011000000 \cdots & \cdots 0000000000013000000 \cdots \\
\cdots 0000000003000000000 \cdots & \cdots 0000000003000000000 \cdots
\end{array}
$$

HWTs

$$
H_1 = \begin{array}{l} 222 \\ 11 \\ 3 \end{array} \qquad H_2 = \begin{array}{l} 222 \\ 13 \\ 3. \end{array}
$$

Theorem.

Theorem. There exists a bijection:

$$\bigsqcup_{\lambda,\rho} \mathrm{SST}(\lambda/\rho, m) \times \mathrm{SST}(\lambda/\rho, n) \Leftrightarrow \left(\bigsqcup_{\mu} \mathrm{HWT}(\mu, m) \times \mathrm{HWT}(\mu, n) \times \tilde{\mathcal{K}}(\mu) \right) \times \mathcal{P}$$

with

$\tilde{\mathcal{K}}(\mu) = \{\kappa = (\kappa_1, \cdots, \kappa_{\ell(\mu)}) \in \mathbb{Z}_{\geq 0}^{\ell(\mu)} : \kappa_i \geq \kappa_{i+1}$ if $\mu_i = \mu_{i+1}\}$.

Furthermore for each correspondence $(P, Q) \mapsto (H_1, H_2, \tilde{\kappa}, \nu)$, let λ/ρ and μ be the shape of (P, Q) and (H_1, H_2) respectively.

Then we have

$$|\rho| = D(H_1) + D(H_2) + |\tilde{\kappa}| + |\nu|,$$
$$\ell(\lambda) = \ell(\mu) + \ell(\nu),$$

where $D(H)$ is the energy of the HWT H.

Modified Hall-Littlewood function

The modified Hall–Littlewood polynomials are defined by

$$H_\mu(x;q) = \sum_\lambda K_{\lambda,\mu}(q) s_\lambda(x),$$

where $K_{\lambda,\mu}(q)$ is the Kostka–Foulkes polynomial.

The Cauchy identity for $H_\mu(x;q)$ is

$$\sum_\mu \frac{1}{c_\mu(q)} H_\mu(x;q) H_\mu(y;q) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{(x_i y_j; q)_\infty},$$

where $c_\mu(q) = \prod_{i=1}^{\mu_1} (q;q)_{m_i}$ and $m_i,\ i = 1, 2, \cdots$ is defined by $\mu = 1^{m_1} 2^{m_2} \cdots$. This can be proved in a bijective manner.

Restricted Cauchy sum identity

Theorem.

$$\sum_{\substack{\mu,\nu \\ \ell(\mu)+\ell(\nu)=k}} \frac{q^{|\nu|}}{c_\mu(q)} H_\mu(x;q) H_\mu(y;q) = \sum_{\substack{\lambda,\rho \\ \ell(\lambda)=k}} q^{|\lambda/\rho|} s_{\lambda/\rho}(x) s_{\lambda/\rho}(y),$$

for $k = 0, 1, 2, \cdots$.

With our column skew RSK, this identity can proved in a bijective manner. The refined identity may be proved in a few different ways

On the proof

- Basic Ideas are similar to the previous case but some differences.
- Leading map transforms tableaux to the one with only 1's, which can be identified with particle configuration on $\mathbb{Z}$.
- Time evolution is identical to Box and Ball system (BBS), which can be linearlized by KKR algorithm.
- Demazure crystal does not exist but one can prove some necessary properties of affine crystals related to our column skew RSK.

Example

$$P = \begin{array}{l} 00000000000000 \\ 000000000000 \\ 000000000455 \\ 0000000005 \\ 000000044 \\ 000003 \\ 000 \\ 002 \end{array}, \quad Q = \begin{array}{l} 00000000000000 \\ 000000000000 \\ 000000000445 \\ 0000000005 \\ 000000055 \\ 000003 \\ 000 \\ 003 \end{array}, \quad n = 5$$

Leading tableau

$$\mathrm{Ld} = \begin{array}{l} 000000000000111 \\ 0000000111 \\ 000001 \\ 001 \end{array}$$

Corresponding BBS configuration:

001001011100111

Summary

- Stochastic vertex models are related to q-Whittaker measures, which are not free fermionic.

  We have found a bijective relation to the periodic Schure measure, which is free fermionic.

- This was achieved by our skew RSK dynamics.

  The proof uses the theory of (affine) crystal.

- We have introduced a column version of skew RSK dynamics.

  It shows a direct connection to BBS.

Free fermion and its correlation kernel

- A free fermion is a quantum many (infinite) particle system for which each one particle state $\phi_n(x)$ ($n \geq 1$, energy ϵ_n) can be either occupied or empty (Pauli principle).

- At $T = 0$, for N particles, the ground state filling $n = 1, \ldots, N$ is realized. The pdf of particle positions is

$$\frac{1}{Z} \left(\det(\phi_n(x_m))_{n,m=1}^{N} \right)^2$$

  Correlations and gap dist. are (Fredholm) deteterminants with the kernel $K(x,y) = \sum_{n=1}^{N} \phi_n(x)\phi_n(y)$.

- For $T > 0$, state n is filled with prob $\frac{1}{1+e^{\beta(\mu-\epsilon_n)}}, \beta = \frac{1}{k_B T}$ (Fermi-Dirac factor). Kernel is $K(x,y) = \sum_{n=1}^{\infty} \frac{\phi_n(x)\phi_n(y)}{1+e^{\beta(\mu-\epsilon_n)}}$.

- Both cases are determinantal point process (DPP).