# Fermionic Gaussian free field and connections to random lattice models

joint work with L. Chiarini (U Dur), A. Cipriani (UCL) and A. Rapoport (UU)

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Workshop IV: Vertex Models: Algebraic and Probabilistic Aspects of Universality - 21st May 2024

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# Plan of the talk

1. Abelian sandpile model and uniform spanning trees
2. Grassmannian algebra and fermionic Gaussian free field
3. Cumulants of observables of ASM and UST in terms of fermionic GFF

# Introduction (Abelian sandpile model)

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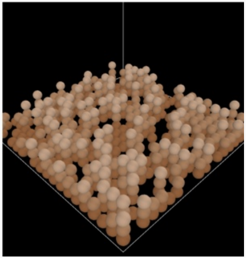

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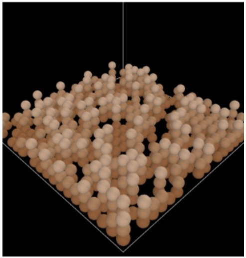

Many contributions: Athreya, Bak, Chiarini, Cipriani, Dhar, Dürre, Fey, Frometa, Hazra, Jara, Járai, Levine, Maes, Majumdar, Meester, Murugan, Pegden, Piroux, Poncelet, Rapoport, Peres, Redig, Ruelle, Saada, Smart, Tang, Werning, Wiesenfeld...

# Example: ASM on $\Lambda \subset \mathbb{Z}^2$

Choose a site uniformly at random.

| | | | |
|---|---|---|---|
| 4 | 3 | 1 | 2 |
| 4 | 4 | 3 | 3 |
| 1 | 4 | 2 | 4 |
| 2 | 3 | 4 | 2 |

# Example

The configuration is unstable, so we topple...

| | | | |
|---|---|---|---|
| 4 | 3 | 1 | 2 |
| 4 | 5 | 3 | 3 |
| 1 | 4 | 2 | 4 |
| 2 | 3 | 4 | 2 |

# Example

| | | | |
|---|---|---|---|
| 4 | 4 | 1 | 2 |
| 5 | 1 | 4 | 3 |
| 1 | 5 | 2 | 4 |
| 2 | 3 | 4 | 2 |

# Example

| | | | |
|---|---|---|---|
| 4 | 4 | 1 | 2 |
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| | | | |
|---|---|---|---|
| 4 | 4 | 1 | 2 |
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Toppling order does not matter!

# Example

Border acts as a sink.

| | | | |
|---|---|---|---|
| 5 | 4 | 1 | 2 |
| 1 | 2 | 4 | 3 |
| 2 | 5 | 2 | 4 |
| 2 | 3 | 4 | 2 |

# Example

| | | | |
|---|---|---|---|
| 1 | 5 | 1 | 2 |
| 2 | 2 | 4 | 3 |
| 2 | 5 | 2 | 4 |
| 2 | 3 | 4 | 2 |

# Example

| | | | |
|---|---|---|---|
| 2 | 1 | 2 | 2 |
| 2 | 3 | 4 | 3 |
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# Example

We obtain a *unique* stable configuration!

| | | | |
|---|---|---|---|
| 2 | 1 | 2 | 2 |
| 2 | 4 | 4 | 3 |
| 3 | 1 | 3 | 4 |
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# Example

Adding a particle produced an avalanche of size 5!

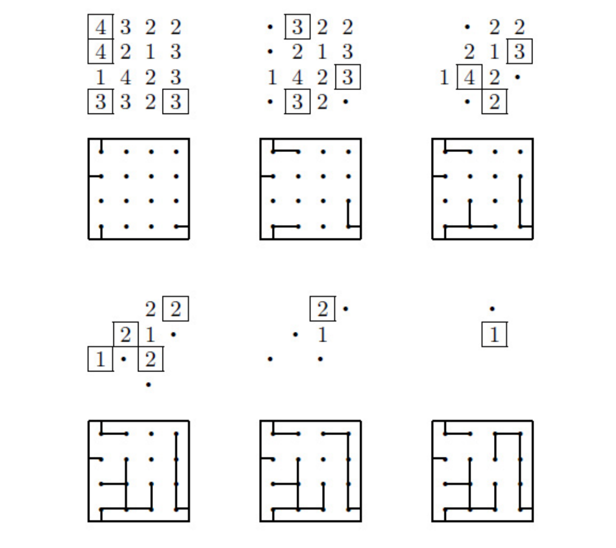
| | | | |
|---|---|---|---|
| 2 | 1 | 2 | 2 |
| 2 | 4 | 4 | 3 |
| 3 | 1 | 3 | 4 |
| 2 | 4 | 4 | 2 |

# Introduction (Abelian sandpile model)

- $G = (W \cup \{s\}, E)$ finite connected graph with graph Laplacian $\Delta$, wired boundary conditions
- configuration $\eta : W \to \mathbb{N}$, stable if $\forall v \in W : \eta(v) \leq \deg(v)$
- dynamics: add a particle uniformly at random and topple, give each neighbour one particle
- model can be described by a Markov chain
- unique stationary measure $\mu$, uniform on set of recurrent configurations $\mathcal{R}$ for Markov chain
- Abelian group $(\mathcal{R}, \oplus)$
- matrix tree theorem: $|\mathcal{R}| = \det(-\Delta) = |\text{spanning trees on} G|$

Questions: stationary measures, avalanche size distributions, stabilization, odometer functions, scaling limits...?

# Connection to spanning trees: Dhar's burning algorithm



# Grassmannian algebras and fermionic Gaussian free field

# Grassmannian algebra and framework

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- $F \in \Omega^{2\Lambda}$ can be written as $F = \sum_{I \subset [2|\Lambda|]} a_I \xi_I$ where $a_I \in \mathbb{R}$, $\xi_I = \xi_{i_1} \cdot \ldots \xi_{i_p}$ for $I = \{i_1, \ldots, i_p\}$

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- generators $\xi_i' s$ satisfy $\xi_i^2 = 0$ and $\xi_i \cdot \xi_j = -\xi_j \cdot \xi_i$
- derivation:

$$\partial_{\xi_j} \xi_I = \begin{cases} (-1)^{\alpha-1} \xi_{i_1} \cdot \ldots \xi_{i_{\alpha-1}} \cdot 1 \cdot \xi_{i_{\alpha+1}} \cdot \ldots \cdot \xi_{i_p} & \text{if } i_\alpha = j \\ 0 & \text{else} \end{cases}$$

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- integration: $\int F d\xi := \partial_{\xi_{2|\Lambda|}} \ldots \partial_{\xi_1} F$

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Let the state $[\cdot]_\Lambda : \Omega^{2|\Lambda|} \to \mathbb{R}$, be defined by

$$[F]_\Lambda = \prod_{v \in \Lambda} \partial_{\bar{\Psi}_v} \partial_{\Psi_v} \exp(\langle \Psi, -\Delta_\Lambda \bar{\Psi} \rangle) F.$$

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Wick's theorem: Let $A$ be an $m \times m$ matrix with real coefficients, then

$$\prod_{i=1}^{m} \partial_{\bar{\Psi}_i}\partial_{\Psi_i} \exp(\langle \Psi, A\bar{\Psi}\rangle) = \det(A).$$

# Properties

- let $F = \Psi_v \bar{\Psi}_v$ then $\langle \Psi_v \bar{\Psi}_v \rangle_\Lambda = G_\Lambda(v, v)$ where $G_\Lambda$ is the Green's function with Dirichlet boundary conditions

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$$\begin{aligned} \langle \Psi_v \bar{\Psi}_v \Psi_w \bar{\Psi}_w \rangle_\Lambda &= \det \begin{pmatrix} G_\Lambda(v, v) & G_\Lambda(v, w) \\ G_\Lambda(w, v) & G_\Lambda(w, w) \end{pmatrix} \\ &= G_\Lambda(v, v) G_\Lambda(w, w) - G_\Lambda^2(v, w) \end{aligned}$$

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- $\langle \left(\Psi_v \bar{\Psi}_v - \langle \Psi_v \bar{\Psi}_v \rangle_\Lambda\right) \left(\Psi_w \bar{\Psi}_w - \langle \Psi_w \bar{\Psi}_w \rangle_\Lambda\right) \rangle_\Lambda = -G_\Lambda^2(v, w)$

# Connections to DGFF

Remember the fGFF was defined as

$$\langle F\rangle_\Lambda = \frac{1}{\det(-\Delta_\Lambda)} \prod_{v\in\Lambda} \partial_{\bar{\Psi}_v}\partial_{\Psi_v} \exp(\langle \Psi, -\Delta_\Lambda \bar{\Psi}\rangle) F.$$

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The discrete GFF on $\Lambda \subset \mathbb{Z}^d$ is defined as by the density

$$\mu_\Lambda(d\phi) = \frac{1}{\sqrt{\det(-\Delta_\Lambda)(2\pi)^d}} \exp(\langle \phi, -\Delta_\Lambda \phi \rangle) \prod_{i \in \Lambda} d\phi_i.$$

# Cumulants and correlation functions

- cumulant generating function for vector $\mathbf{X} = (X_1, \ldots, X_n)$

$$K(\mathbf{t}) = \log(\mathbb{E}(e^{\mathbf{tX}})) = \sum_{m \in \mathbb{N}^n} k_m(\mathbf{X}) \prod_{j=1}^{n} \frac{t_j^m}{m_j!}$$

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- joint cumulant of $X_1, \ldots, X_n$:

$$\mathcal{K}(X_1, \ldots, X_n) = \frac{\partial^n}{\partial t_1 \ldots \partial t_n} K(\mathbf{t})|_{t_1 = \ldots = t_n = 0}$$

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  taking $m = (1, \ldots, 1)$
- Example: $\mathcal{K}(X_i, X_j) = Cov(X_i, X_j)$
- $\mathcal{K}(X_i; i \in A) = \sum_{\pi \in \Pi(A)} (|\pi| - 1)!(-1)^{|\pi|-1} \prod_{B \in \pi} \mathbb{E}(\prod_{i \in B} X_i)$

## Local observables

Probabilistic observables:

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Fermionic observables:

- fermionic field: $X_v = \frac{1}{\deg(v)} \sum_{i=1}^{\deg(v)} \nabla_{e_i}\Psi(v)\nabla_{e_i}\bar{\Psi}(v)$ and

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- fermionic "corrector": $Y_v = \prod_{i=1}^{\deg(v)} (1 - \nabla_{e_i} \Psi(v) \nabla_{e_i} \bar{\Psi}(v))$

# Height-1 field and UST's

On the one hand we have that

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- $\mathbb{E}(h(o)) = \mathbb{P}(f \notin T, f \neq e)$, where $e = (v, \cdot)$
- let wlog $e = e_1 = (1, 0)$ then by inclusion-exclusion

$$\begin{aligned}
&\mathbb{P}(e_2 \notin T, e_3 \notin T, e_4 \notin T) \\
&= \mathbb{P}(e_1 \in T, e_2 \notin T, e_3 \notin T, e_4 \notin T) \\
&= \mathbb{P}(e_1 \in T) - \sum_{i \neq 1} \mathbb{P}(e_1 \in T, e_i \in T) + \sum_{i \neq j \neq 1} \mathbb{P}(e_1 \in T, e_i \in T, e_j \in T) \\
&- \mathbb{P}(e_1 \in T, e_2 \in T, e_3 \in T, e_4 \in T)
\end{aligned}$$

# Height-1 field, UST's and fGFF's

and on the other

$$Y_o = \prod_{i=1}^{4}(1 - \nabla_{e_i}\Psi(o)\nabla_{e_i}\bar{\Psi}(o)) = \prod_{i=1}^{4}(1 - a_i)$$

$$= 1 - \sum_{i=1}^{4} a_i + \sum_{i\neq j} a_i a_j - \sum_{i\neq j\neq k} a_i a_j a_k + a_1 a_2 a_3 a_4$$

and (use $a_i^2 = 0$)

$$X_o Y_o = \frac{1}{4}\sum_{i=1}^{4}\nabla_{e_i}\Psi(o)\nabla_{e_i}\bar{\Psi}(o)\cdot\left(\prod_{i=1}^{4}(1 - \nabla_{e_i}\Psi(o)\nabla_{e_i}\bar{\Psi}(o))\right)$$

$$= \frac{1}{4}\left(\sum_{i=1}^{4} a_i - \sum_{i\neq j} a_i a_j + \sum_{i\neq j\neq k} a_i a_j a_k - a_1 a_2 a_3 a_4\right)$$

# Height-1 field, UST's and fGFF's

Connecting those two observations, together with (Bauerschmidt, Crawford, Helmuth, Swan '21)

$$\mathbb{P}(e_1 \in T) = \left\langle \nabla_{e_1} \Psi(o) \nabla_{e_1} \bar{\Psi}(o) \right\rangle_\Lambda$$

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we obtain:

$$\mathbb{E}(h(o)) = \langle X_o Y_o \rangle_\Lambda .$$

# Results: Height-1 and degree field as fermionic observables

**Theorem (2)**
*For some good set $V \subset \Lambda$ we have that*

1. *Height-1 field of the ASM:* $\mathbb{E}\left(\prod_{v \in V} h(v)\right) = \left\langle \prod_{v \in V} X_v Y_v \right\rangle_\Lambda$
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The height-1 field and the degree field can be expressed as squares of local observables w.r.t. the fermionic Gaussian free field state in the Grassmann algebra formalism!

# Ideas of the proof

Main ingredients:

1. (Bauerschmidt, Crawford, Helmuth, Swan '21)

$$\mathbb{P}(S \subset T) = \left\langle \prod_{f \in S} \nabla_f \Psi(f^-) \nabla_f \bar{\Psi}(f^-) \right\rangle_\Lambda$$

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2. (Caracciolo, Sokal, Sportiello '13) Wick's formula

$$\left[ \prod_{f \in S} \nabla_f \Psi(f^-) \nabla_f \bar{\Psi}(f^-) \right]_\Lambda = \det(-\Delta_\Lambda) \det(M_S)$$

where $M_S = (M(e,f))_{e,f \in S}$ and $M(e,f) = \nabla_e^{(1)} \nabla_f^{(2)} G_\Lambda(e^-, f^-)$

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3. $\mathbb{E}(\prod_{v \in V} h(v)) = \mathbb{P}(e \notin T$ for each edge incident to v)

## Results: Cumulants of the height-1 field as a fermionic observables

**Theorem (2)**
*For some good set $V \subset \Lambda$, $|V| = n$, we have that*

$$\mathcal{K}(X_v Y_v; v \in V) = c_L^n \sum_{E:|E_v|\geq 1, v\in V} K(E) \sum_{\tau \in S_{co}(E)} sign(\tau) \prod_{f\in E} \nabla_f^{(1)} \nabla_{\tau(f)}^{(2)} G_\Lambda(f^-, \tau(f)^-)$$

*where $c_L = -\frac{1}{2d}$ for $L = \mathbb{Z}^d$ resp. $c_L = -\frac{1}{6}$ for $L = \mathbb{T}$*

# Results: Cumulants of the degree field as a fermionic observables

**Theorem (3)**
*For some good set $V \subset \Lambda$, $|V| = n$, we have that*

$$\mathcal{K}(X_v; v \in V)$$
$$= -\tilde{c}_L^n \sum_{\sigma \in S_{cycl}(V)} \sum_{\eta: V \to \{e_1, \ldots, e_{\deg(o)}\}} \prod_{v \in V} \nabla^{(1)}_{\eta(v)} \nabla^{(2)}_{\eta(\sigma(v))} G_\Lambda(v, \sigma(v))$$

*where $\tilde{c}_L = -\frac{1}{2d}$ for $L = \mathbb{Z}^d$ resp. $\tilde{c}_L = -\frac{1}{6}$ for $L = \mathbb{T}$*

# Results: Scaling limit of the cumulants of the height-1 field as a fermionic observables

Let $U \subset \mathbb{R}^d$, $\epsilon > 0$ and $U_\epsilon = U/\epsilon \cap L$ and $v_\epsilon$ the discrete approximation of $v$ in $U_\epsilon$.

**Theorem (4)**
*For some good set $V \subset U$, $|V| = n$, we have that*

$$\lim_{\epsilon \to 0} \epsilon^{-n \cdot d_L} \mathcal{K}(X_v^\epsilon Y_v^\epsilon; v \in V)$$
$$= -C_L^n \sum_{\sigma \in S_{cycl}(V)} \sum_{\eta : V \to \{e_1, \ldots, e_{\deg(o)}/2\}} \prod_{v \in V} \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U(v, \sigma(v))$$

*where $d_L = d$ for $L = \mathbb{Z}^d$ resp. $d_L = 2$ for $L = \mathbb{T}$ and explicit formula for $C_L$.*

Special values: $C_{\mathbb{Z}^2} = \pi \mathbb{P}(h(o) = 1) = \frac{2}{\pi^2} - \frac{4}{\pi^3}$ and
$C_{\mathbb{T}} = \left(\frac{1}{18} + \frac{1}{\sqrt{3}\pi}\right)^{-1} \mathbb{P}(h(o) = 1)$

# Results: Scaling limit of the cumulants of the degree field as a fermionic observables

Let $U \subset \mathbb{R}^d$, $\epsilon > 0$ and $U_\epsilon = U/\epsilon \cap L$ and $v_\epsilon$ the discrete approximation of $v$ in $U_\epsilon$.

**Theorem (5)**
*For some good set $V \subset U$, $|V| = n$, we have that*

$$\lim_{\epsilon \to 0} \epsilon^{-n \cdot d_L} \mathcal{K}(X_v^\epsilon; v \in V)$$
$$= -\tilde{C}_L^n \sum_{\sigma \in S_{cycl}(V)} \sum_{\eta : V \to \{e_1, \dots, e_{\deg(o)}/2\}} \prod_{v \in V} \partial^{(1)}_{\eta(v)} \partial^{(2)}_{\eta(\sigma(v))} g_U(v, \sigma(v))$$

*where $d_L = d$ for $L = \mathbb{Z}^d$ resp. $d_L = 2$ for $L = \mathbb{T}$ and explicit formula for $\tilde{C}_L$.*

Special values: $\tilde{C}_{\mathbb{Z}^2} = -\frac{1}{d}$ and $\tilde{C}_{\mathbb{T}} = -\frac{1}{2}$

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- we were able to find an explicit formula for the cumulants and its scaling limit for the height-1 field and degree field in terms of Grassmannian variables

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- support conjecture of Ruelle that the height-1 field can be described by a sympletic fermion theory (constants match); primary field of Ruelle has form:

$$\Phi = -\frac{C}{2}(\partial_z \theta \partial_{\bar{z}} \bar{\theta} + \partial_{\bar{z}} \bar{\theta} \partial_z \theta)$$

# Open questions and future work

- universality in the scaling limit, other graphs
- scaling limit of cumulants of higher heights and general observables
- construct a lattice symplectic fermion theory and match the cumulants
- important work in this direction: ([Hongler, Kytölä and Viklund '23] and [Adame-Carrillo, Behzad and Kytölä '24])
- other lattice models and lattice CFT's

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Thank you for your attention!