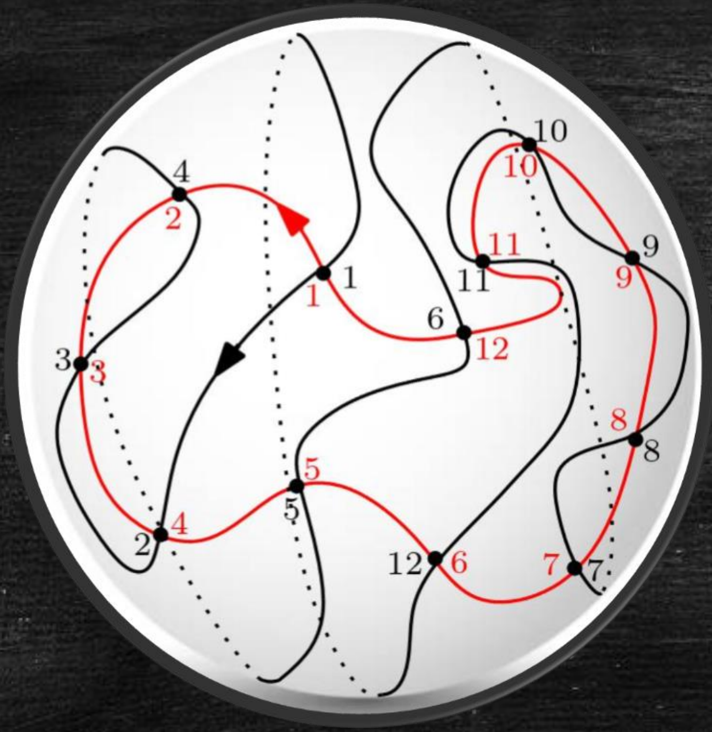
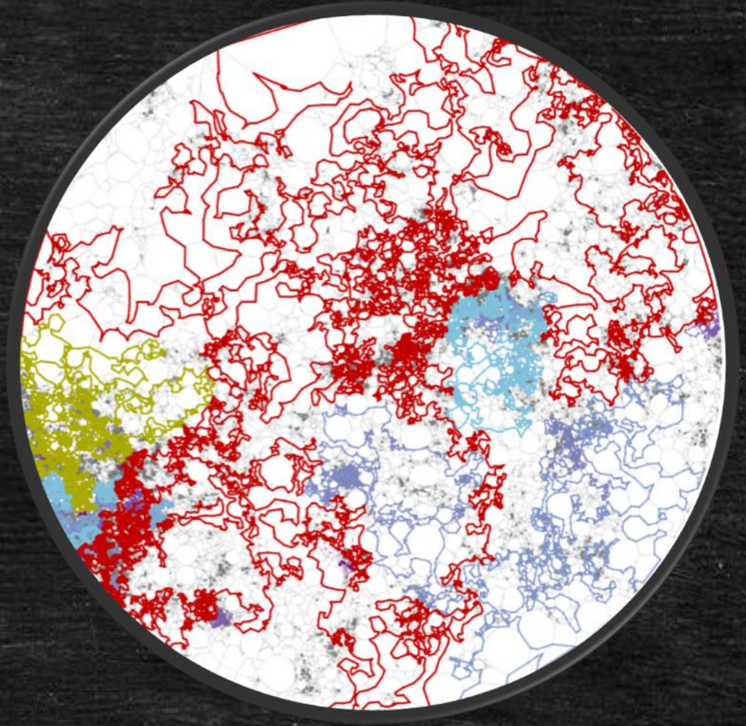




On the geometry of uniform meandric systems

(joint work with E. Gwynne and M. Park)

Jacopo Borga

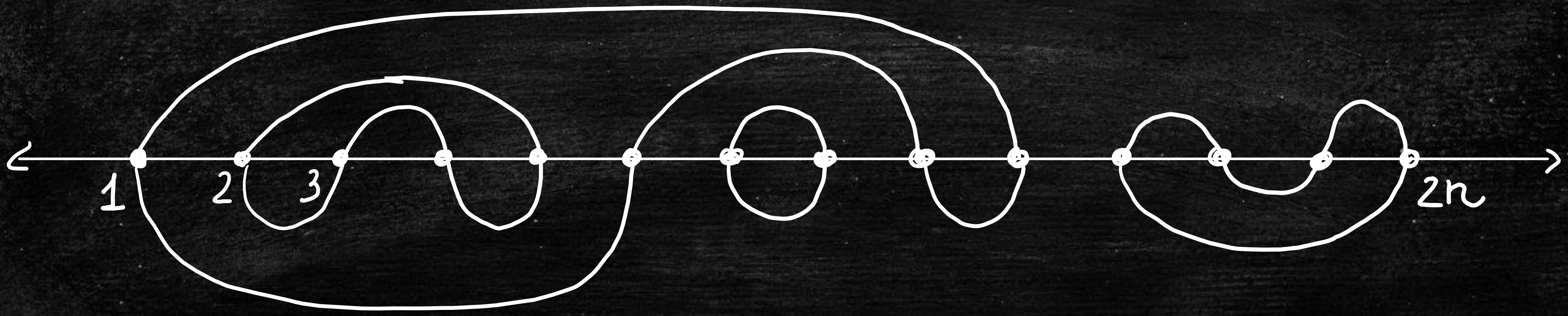
Stanford University

IPAM, May 23, 2024

The model: definition, motivations & previous work

Meandric systems

Def: A MEANDRIC SYSTEM of size $n \in \mathbb{N}$ is a collection of disjoint simple loops in $\mathbb{R}^2$ which orthogonally cross $\mathbb{R}$, precisely at the points $\{1, \dots, 2n\}$. Configurations are viewed modulo homeomorphisms fixing $\mathbb{R}$.

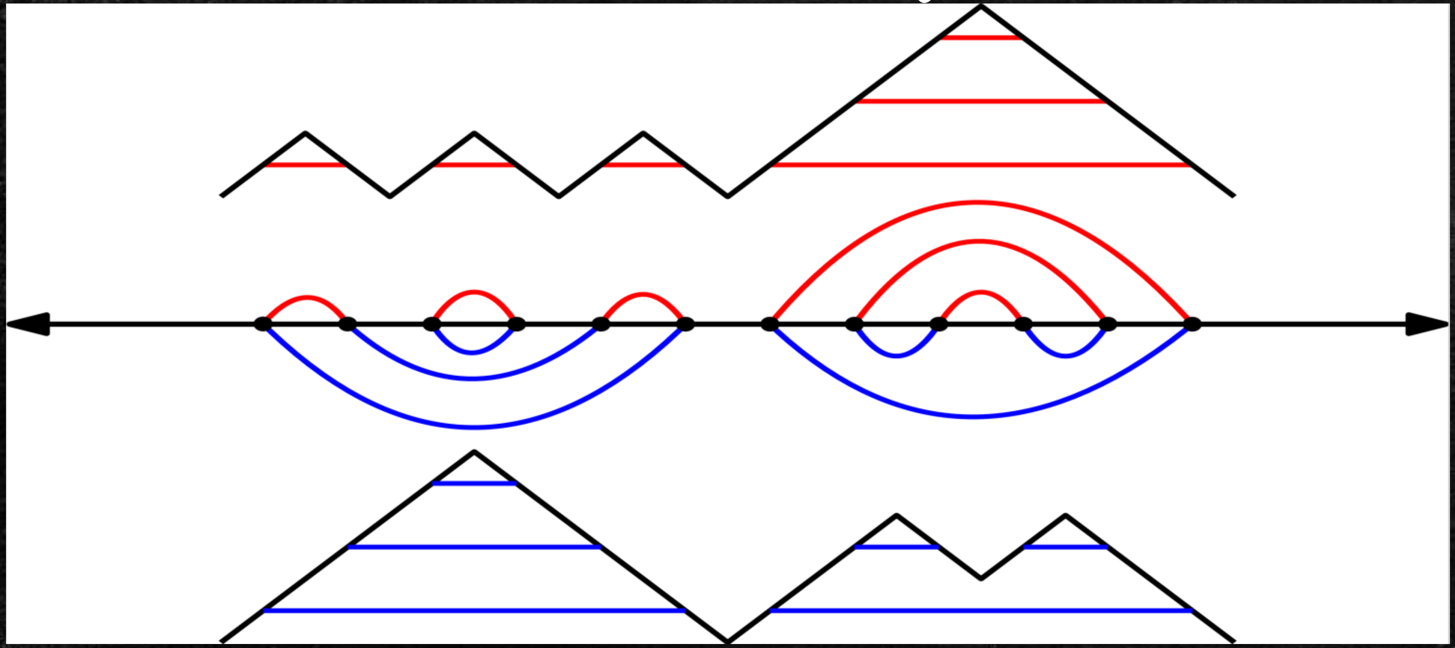


- This model is equivalent to:

CROSSING FULLY PACKED $O(0\times 1)$ loop model on PLANAR MAPS

- STUDIED BY : Di Francesco, Korgin, Féray-Thévenin, Curien-Kozma-Sidoravicious-Tournier, Golden-Nica-Puder, Fukuda-Nechita, Janson-Thévenin, etc...

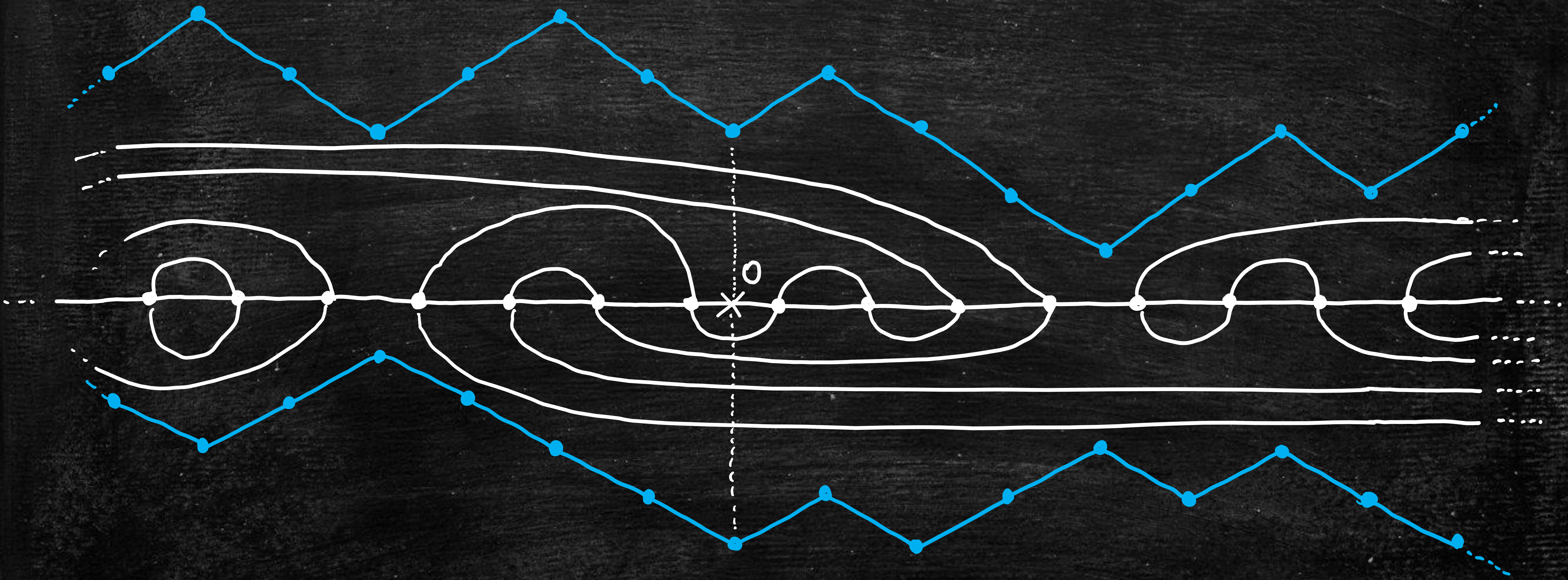
- How can we sample a uniform meandric system of size $2n$?



ISSUE: Loops are a very complicated functional of the 2 walks

BASIC QUESTIONS are still OPEN

We can also consider an <u>infinite volume version</u> of this model:



QUESTION: Is there an infinite loop? [Curien-Kozma-Sidoravicious-Tournier]

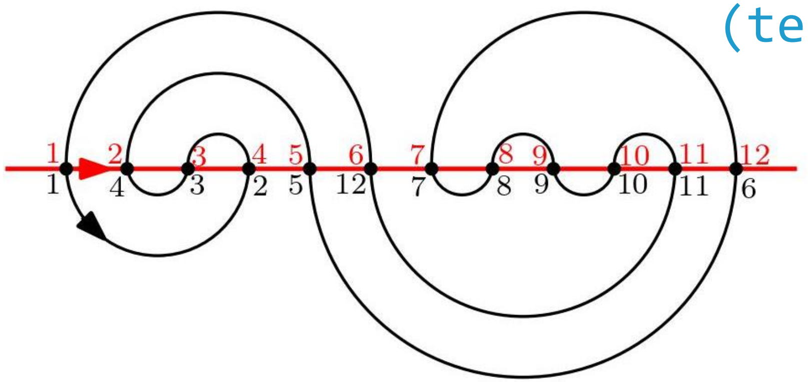
THE INFINITE NOODLE

1912

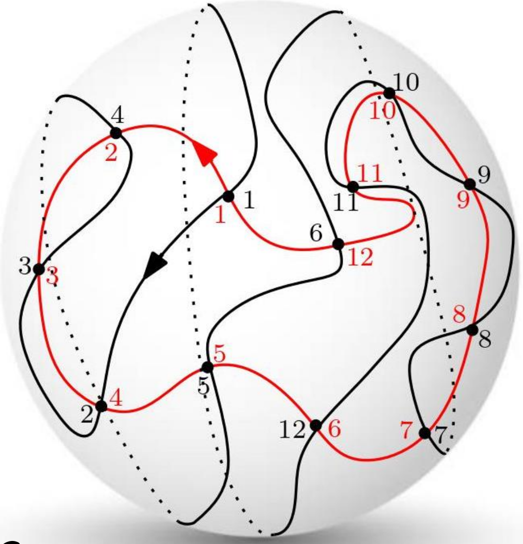
"In how many different ways a simple loop in the plane can cross a line a specified number of times?"

Meanders

(term coined by V.Arnold)



CROSSING FULLY PACKED $O(0\times 0)$ loop model on PLANAR MAPS



[B., Gwynne, Sun '22] → Conjectures for the scaling limits.

SOME QUESTIONS: (Loops are a complicated functional of the walks)

① How many loops?

- FÉRAY-THÉVENIN (2022, IMRN): $\# \text{loop} \sim c \cdot n$, where c is a complicated sum over meanders.

② What is the "SIZE" of the largest loop?

- Kargin (2022): The largest loop contains $\geq c \cdot \log(n)$ vertices

↳ Simulations suggest $\approx n^{\alpha}$ with $\alpha \approx 4/5$.

③ Does one loop dominate? Or, are there many large loops of similar "size"?

- Very related to the "infinite noodle" of Curien-Kozma-Sidoravicius-Tournier

↳ CONJ: There is NO INFINITE LOOP

④ What is the scaling limit as $n \to \infty$? ???

GOAL: • Conjectures for answers to the above questions;

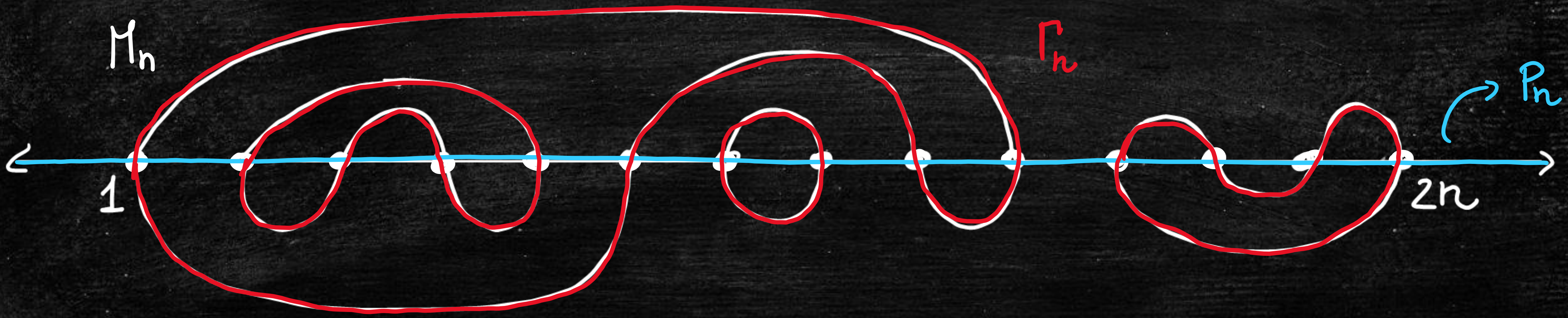
• Rigorous results in the direction of these conjectures.

• Some open questions.

We view a meandric system as a

PLANAR MAP + HAMILTONIAN PATH + LOOPS

M_n P_n Γ_n



The conjecture

The conjecture will involve 3 types of objects:

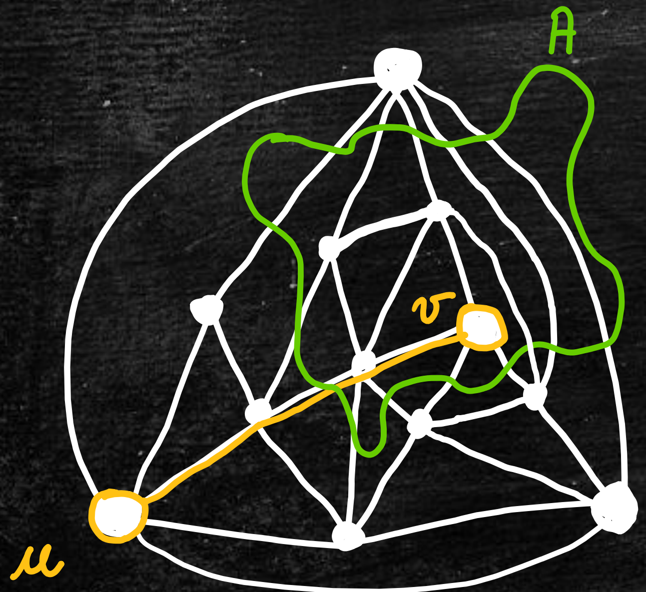
1. γ - LQG - measure / metric
2. SLE_κ
3. CLE_κ

I will define them as <u>scaling limits</u> of <u>natural discrete models</u>, but they can all be defined directly in the continuum.

[<u>Example</u>: Simple random walk $\longrightarrow$ Brownian motion]

γ - LQG - measure / metric

Random uniform triangulation T_n
with n vertices



$d_n(u,v) = 3$

$\mu_n(A) = 4$

Embedded in the sphere in a canonical way
[Circle packing / Tutte embedding / Smith embedding / Cardy embedding]

$$\left(\frac{\mu_n}{n}, \frac{d_n}{n^{1/4}}\right) \xrightarrow[n\to\infty]{d} (\mu, d)$$

(c=0) $\sqrt{8/3}$ - LQG - measure

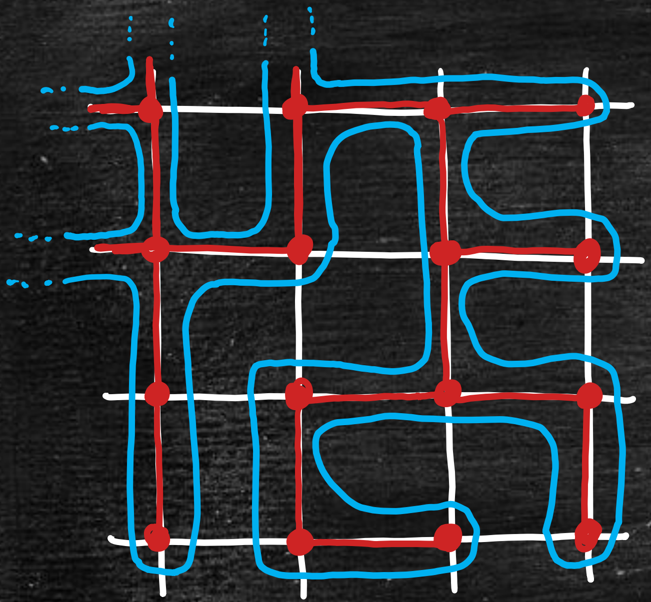
(c=0) $\sqrt{8/3}$ - LQG - metric

[Holden, Sun, '21]

NOTE:

$\mathrm{diam}(T_n) = n^{1/4}$

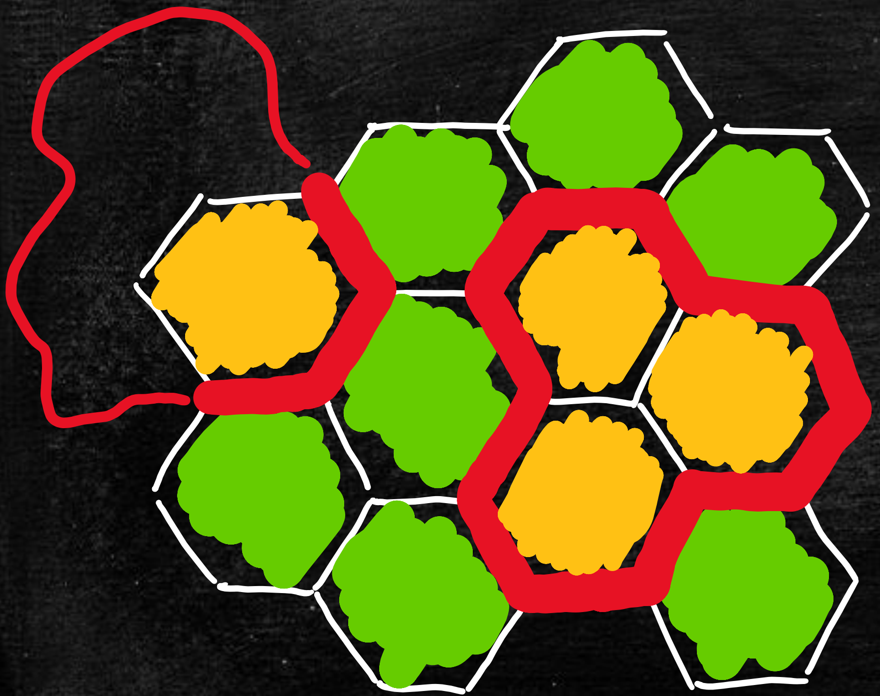
Hausdorff dimension$(\sqrt{8/3}$-LQG$) = 4$



$\xrightarrow[n\to\infty]{d}$ SLE_8

$(c=-2)$

[Lawler, Schramm, Werner, '01]



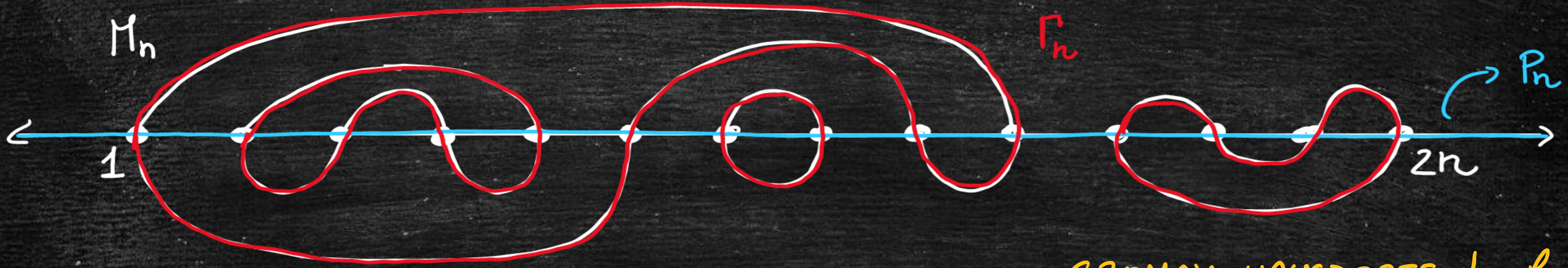
$\xrightarrow[n\to\infty]{d}$ CLE_6

$(c=0)$

[Smirnov '01, Camia & Newman '03]

PLANAR MAP + HAMILTONIAN PATH + LOOPS

M_n $\qquad$ P_n $\qquad$ Γ_n



CONJECTURE: (B., Gwynne, Park, '22)

- GROMOV-HAUSDORFF topology for metric spaces
- Using some EMBEDDING

(M_n, P_n, Γ_n) converges under an appropriate scaling limit to a

$$\underbrace{\sqrt{2}\text{-LQG-measure/metric} + SLE_8}_{\text{Same as planar maps + spanning tree}} + \underbrace{CLE_6}_{\text{Same as CRITICAL PERCOLATION}} \quad (\text{All } \perp\!\!\!\perp)$$

$(c=-2)$ $\qquad$ $(c=-2)$ $\qquad$ $(c=0)$

The conjecture is motivated by previous work of

Di Francesco, Golinelli, Guitter, 2000

on the Crossing fully packed $O(n \times m)$ loop model on planar maps

+ some LQG, SLE, CLE interpretation.

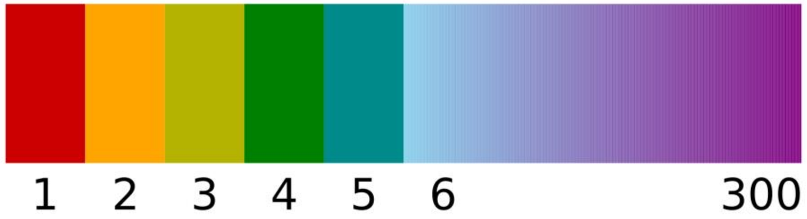
— o —

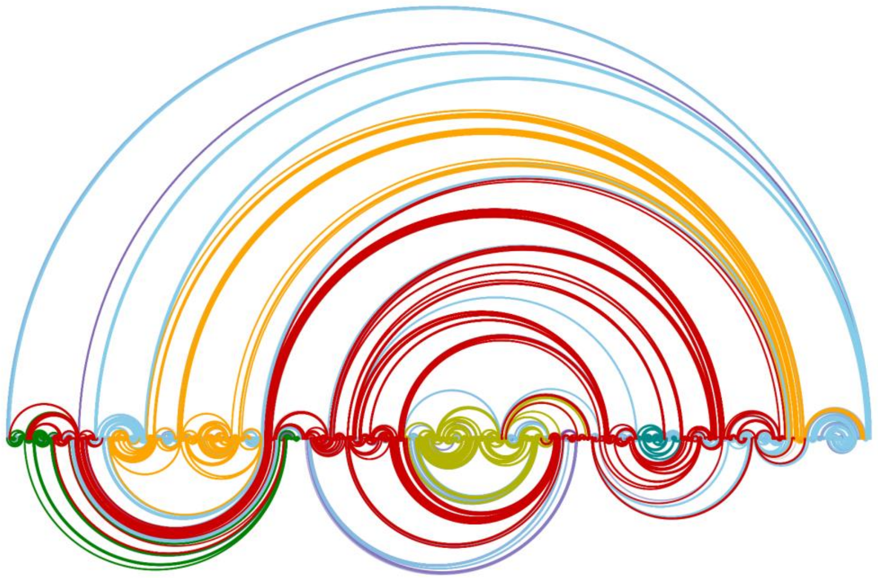
- I will show later an orthogonal justification of the conjecture
- In our paper, we have several numerical simulations consistent with our conjecture (see later slides).

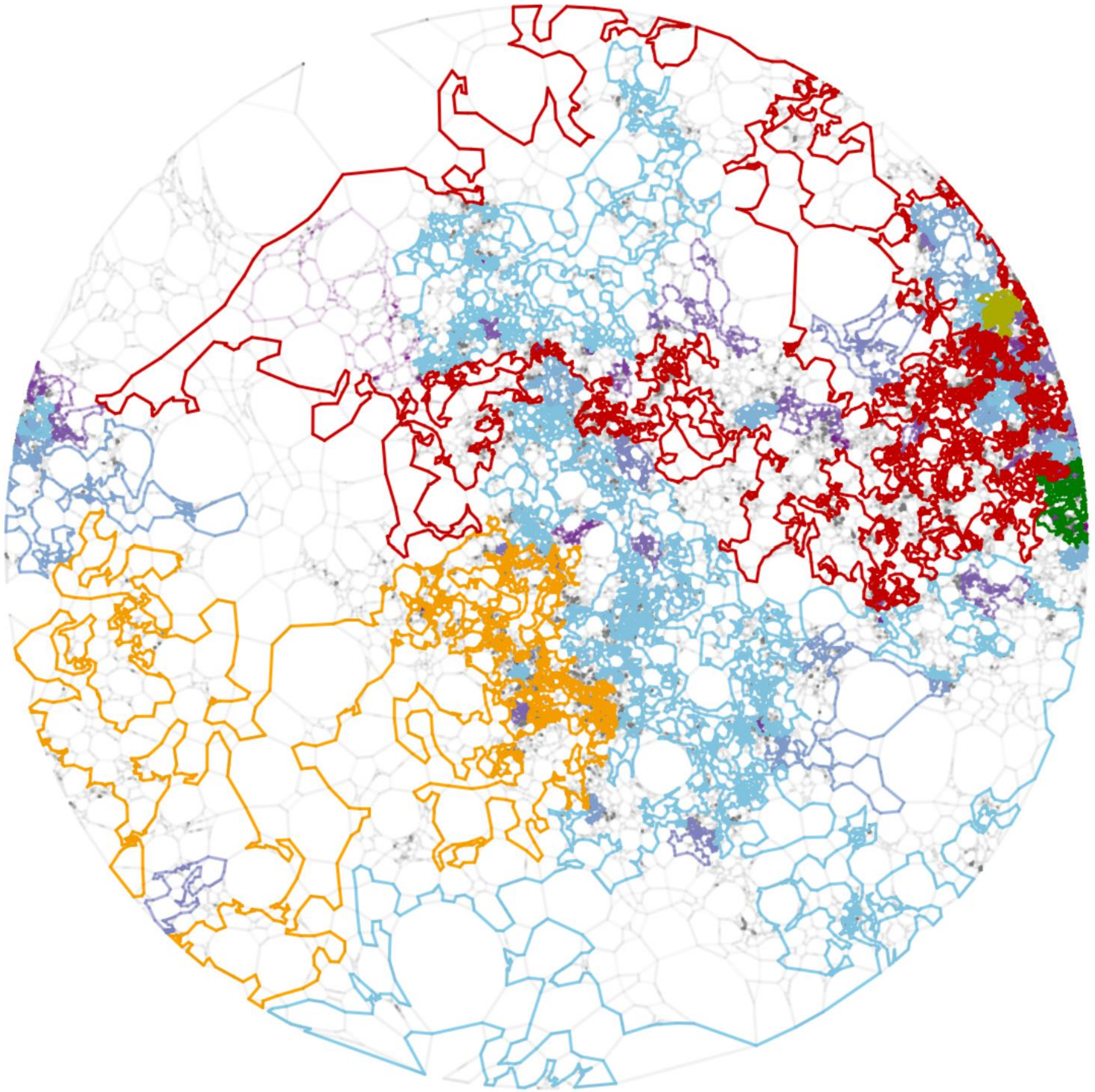
Planar map + loops

$\downarrow$

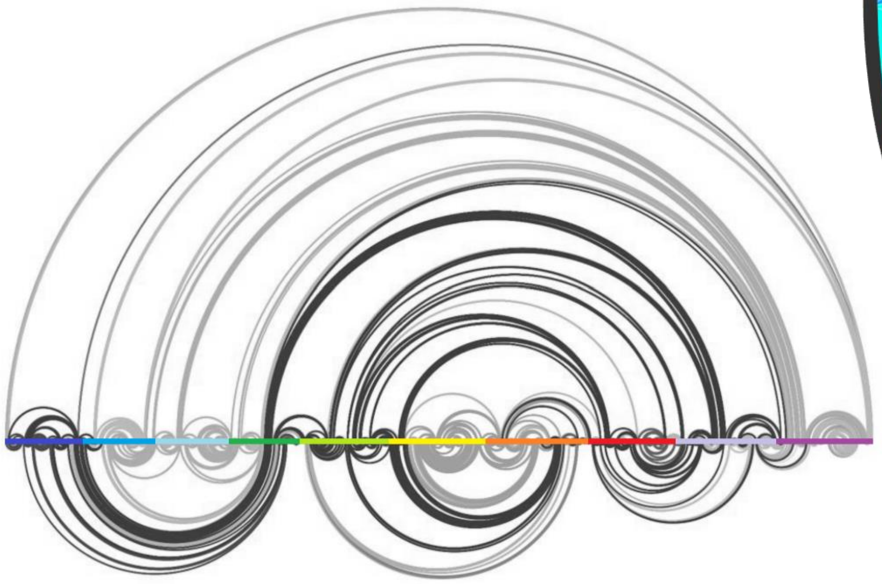
$\sqrt{2}$-LQG + CLE6

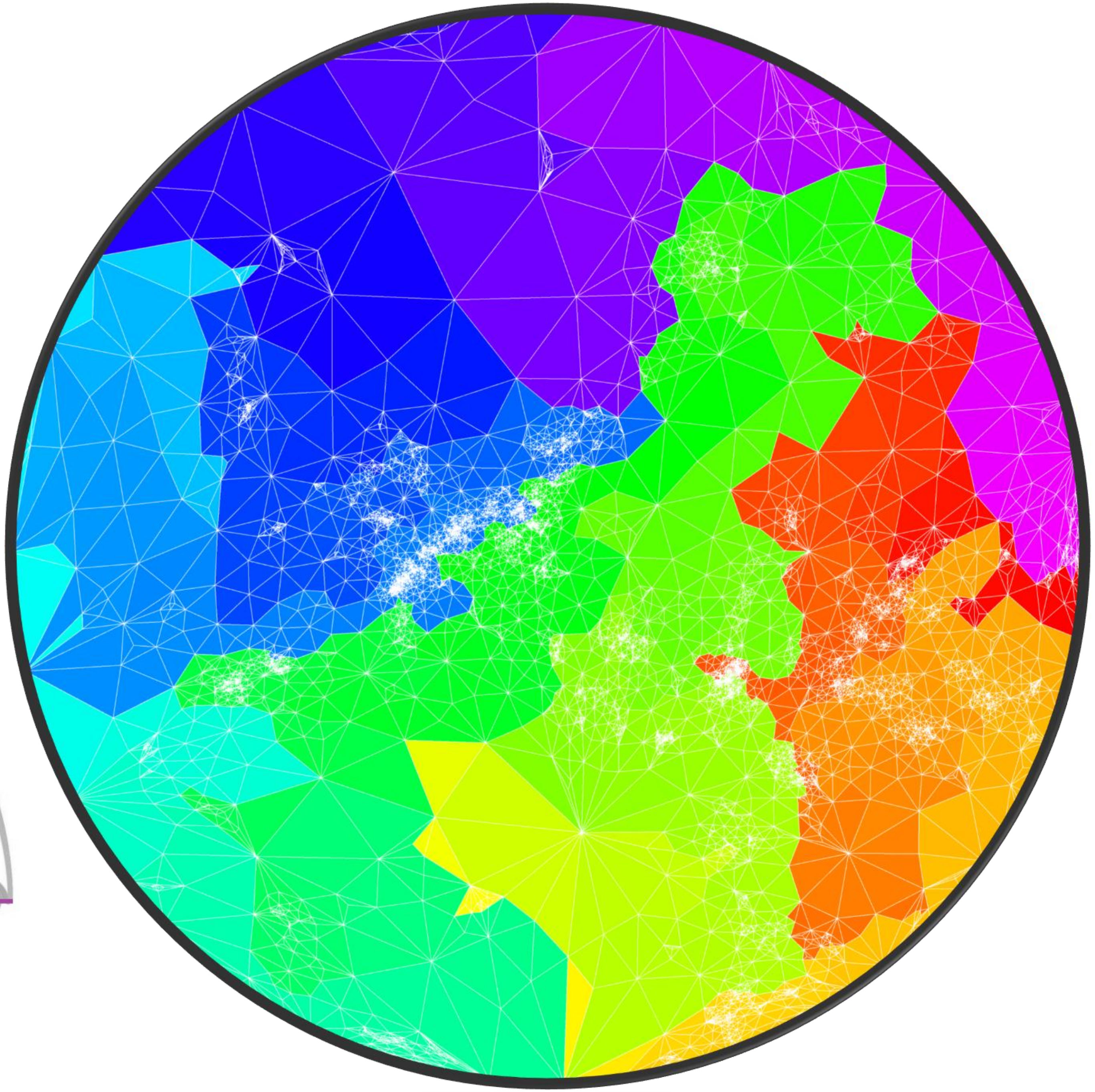






Planar map
+
Hamiltonian path
↓
√2-LQG + SLE8





The fact that

$$(M_n, P_n) \xrightarrow{n\to\infty} \sqrt{2}\text{-LQG} + \mathrm{SLE}_8$$

is known (at least in the Peanosphere sense).

A consequence of [Gwynne, Holden, Sun, 2020]:

Proposition: With probability tending to one as $n \to \infty$,

$$\mathrm{diameter}(M_n) = n^{1/d + o(1)}$$

where d is the Hausdorff dimension of the $\sqrt{2}$-LQG metric.

$$3.55 \leq d \leq 3.63$$

SOME QUESTIONS: (Loops are a complicated functional of the walks)

① How many loops?

- FÉRAY-THÉVENIN (2022, IMRN): $\# \text{loop} \sim c \cdot n$, where c is a complicated sum over meanders.

② What is the "SIZE" of the largest loop?

CONJECTURE (Borga-Gwynne-Park)

$$\# \text{ vertices of the } k\text{-th largest loop} \approx n^{\alpha + o(1)}, \text{ where } \alpha = \frac{3-\sqrt{2}}{2} \approx 0.7928$$

③ Does one loop dominate? Or, are there many large loops of similar "size"?

- Very related to the "infinite noodle" of Curien-Kozma-Sidoravicius-Tournier

↳ CONJ: There is NO INFINITE LOOP (confirmed + motivations)

④ What is the scaling limit as $n \to \infty$? CONJ from before

SEVERAL NUMERICAL SIMULATIONS (in our paper) CONFIRM the CONJECTURES.

The theorems

Theorem: (B., Gwynne, Park, CMP '23) $3.55 \le d \le 3.63$

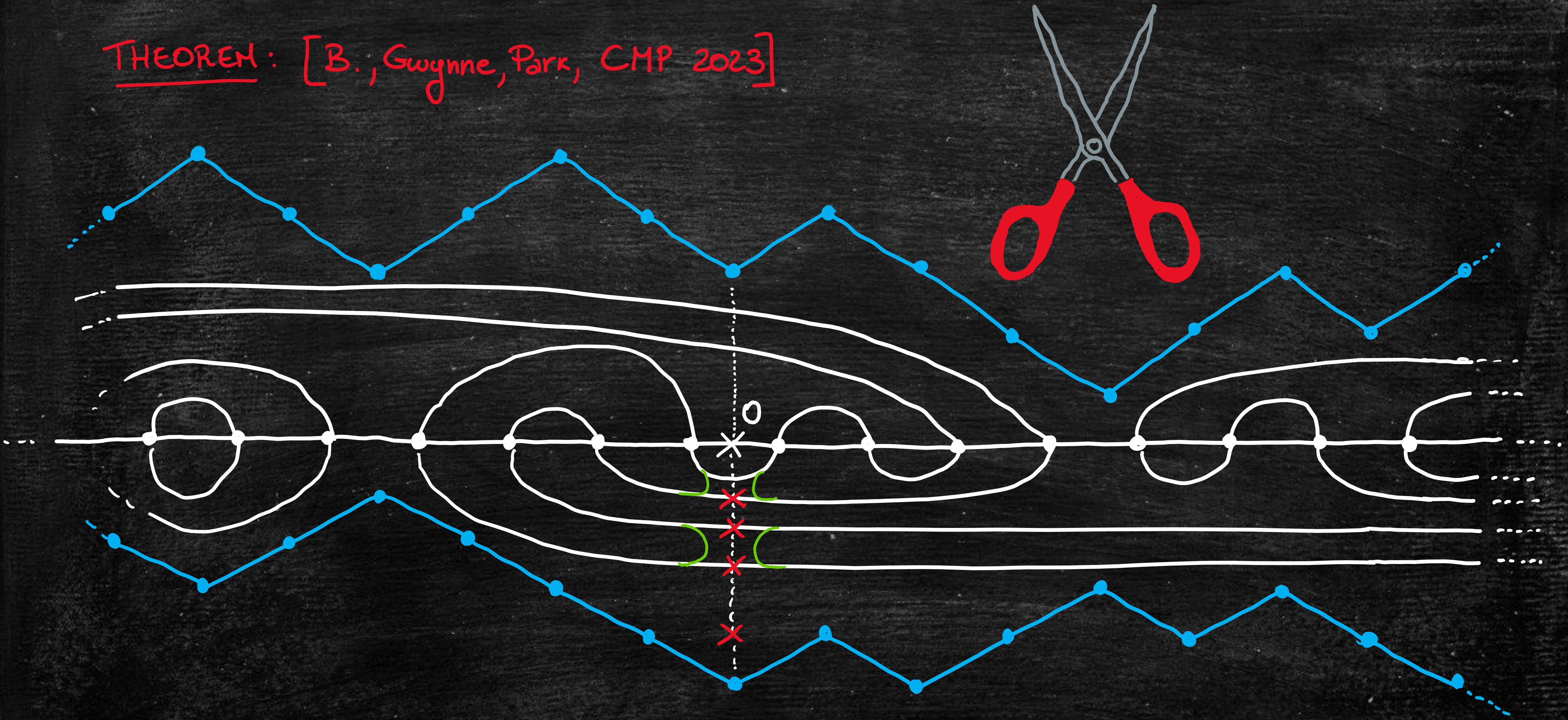
- Let d be the dimension of $\sqrt{2}$-LQG (just think of it as a constant)
- Let $(\mathcal{M}_n, \mathcal{P}_n, \Gamma_n)$ be a uniform meandric system of size $n \in \mathbb{N}$. Then

$$\#\text{vertices of largest loop in } \Gamma_n \geq n^{1/d + o(1)} \geq n^{0.275}.$$

Proof:
- STEP 1: Use a discrete parity argument to show that $\exists$ a large loop in Γ_n (w.r.t. the graph-metric induced by $\mathcal{M}_n$).
- STEP 2: SLE/LQG arguments to lower-bound graph-distances
  - ↳ tools: MATING-OF-TREES (Miller-Sheffield) / LQG-METRIC (Gwynne, Miller/Ding, Dubedat, Dunlap, Falconet)

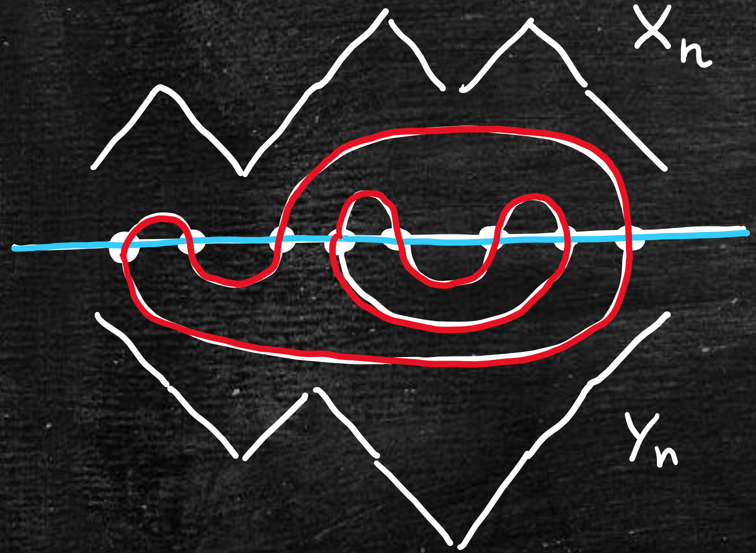
- Our theorem implies that there are (almost) macroscopic loops in Γ_n w.r.t. to $\mathcal{M}_n$ (which "surive in the scaling limit").
- Far from the conjecture ($\alpha \approx 0.793$) but better than previous results ($\log(n)$)

THEOREM: [B., Gwynne, Park, CMP 2023]



After cutting and rewiring, almost surely, there is no infinite noodle

Some further remarks on the conjecture



The fact that $\left(\frac{X_n}{\sqrt{n}}, \frac{Y_n}{\sqrt{n}}\right) \xrightarrow[n\to\infty]{d} (X,Y)$ → 2D Brownian motion

$\Downarrow$ Mating of trees

$\perp$

$$(M_n, P_n) \xrightarrow{d} (\sqrt{2}\text{-LQG}, SLE_8)$$

$\Updownarrow$ DETERMINE EACH OTHER

(X,Y)

$\perp$

Now, $(M_n, P_n, \Gamma_n) \xrightarrow[n\to\infty]{d} (\sqrt{2}\text{-LQG}, SLE_8, CLE_\kappa)$

$(c=-2)$ $(c=-2)$

What can be the value of κ? It should be the only value s.t. $c=0$

i.e. $\kappa=6$.

Why the CLE_6 should be independent of the $(\sqrt{2}\text{-LQG}, \mathrm{SLE}_8)$?

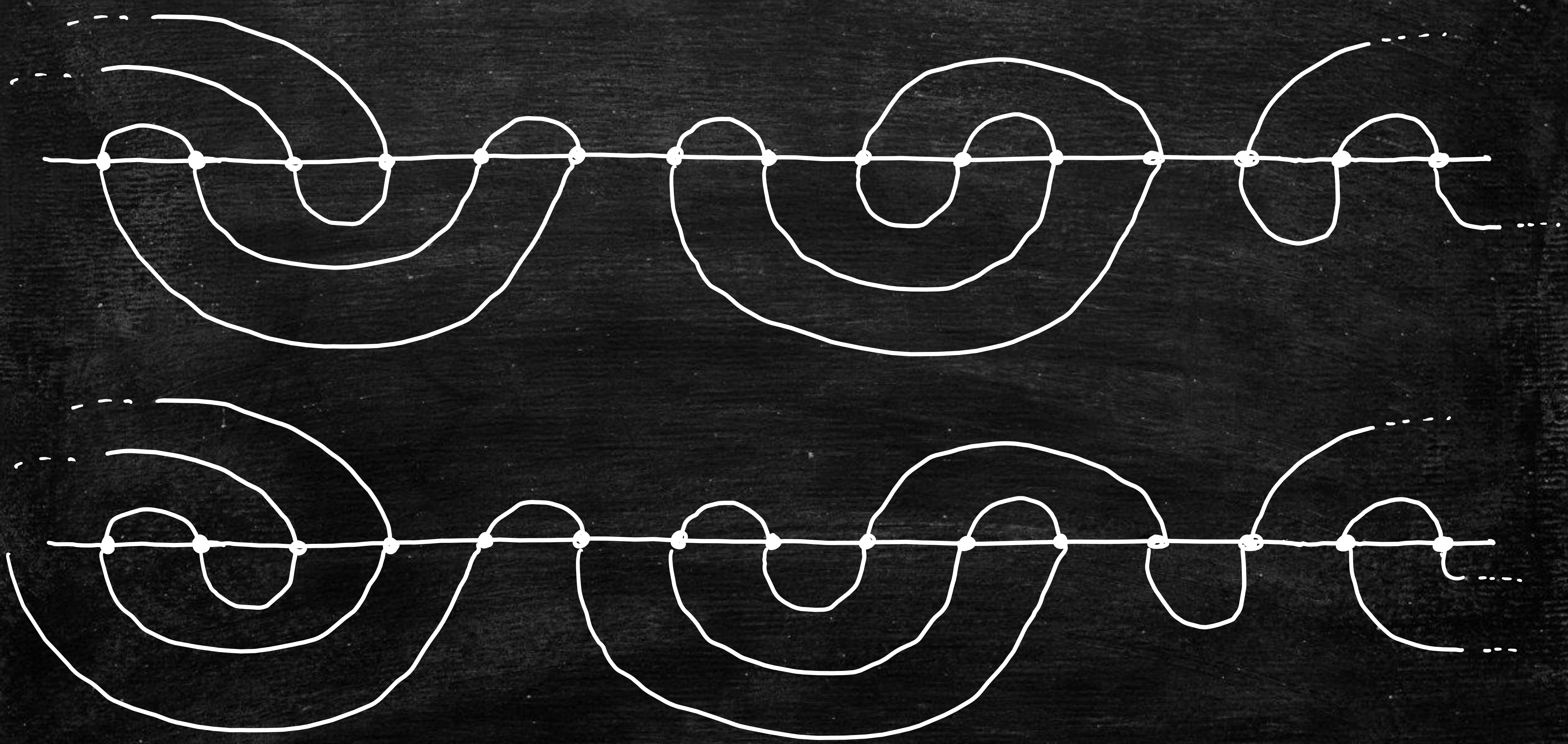
$(X, Y) \leftrightsquigarrow (\sqrt{2}\text{-LQG}, \mathrm{SLE}_8)$ determine each other

$\Longrightarrow$ CLE_6 is <u>NOT</u> at all determined by (X, Y)

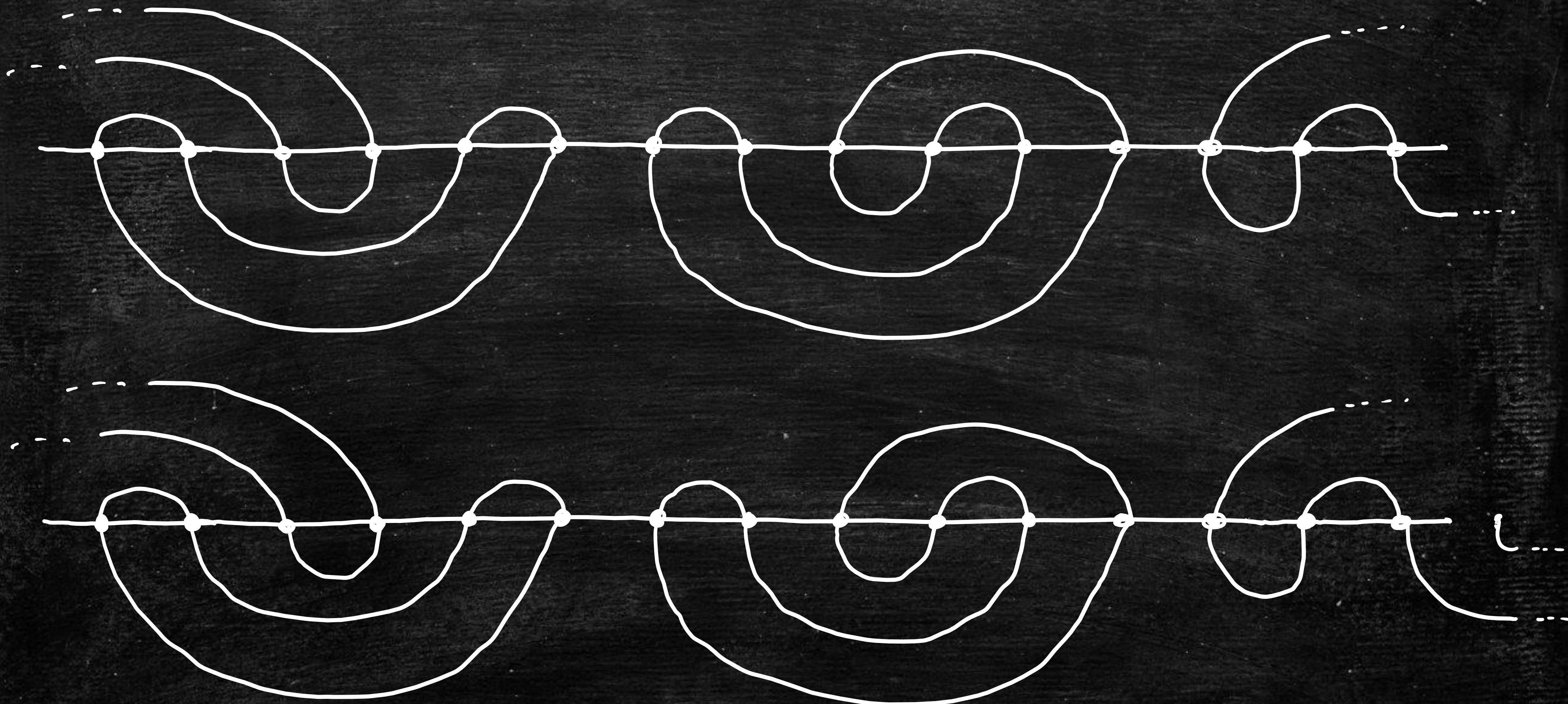
$\Longrightarrow$ We need to keep track of additional independent randomness!

But at the discrete level (X_n, Y_n) determines Γ_n ... 

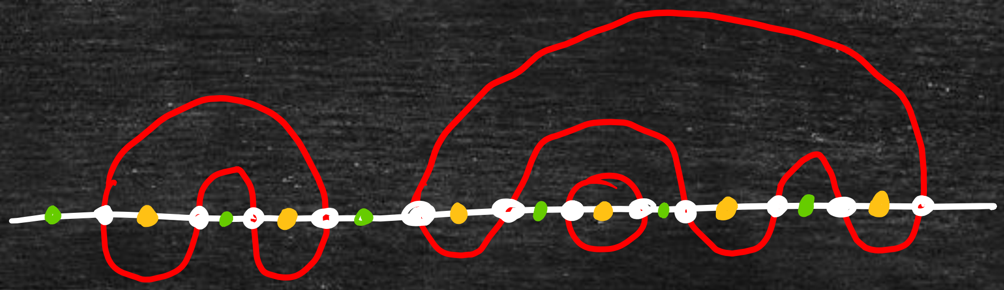
But there is hope !

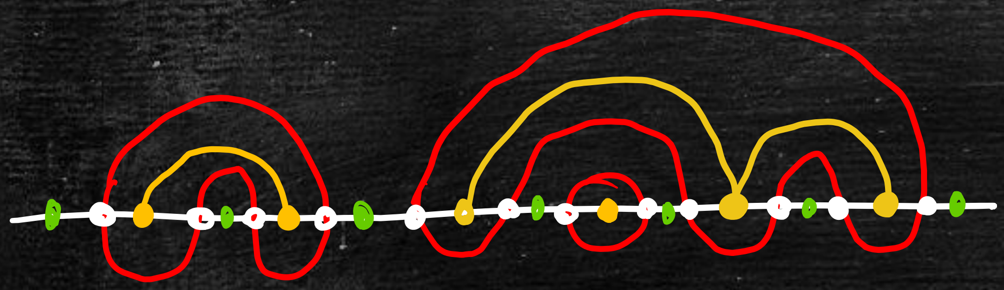


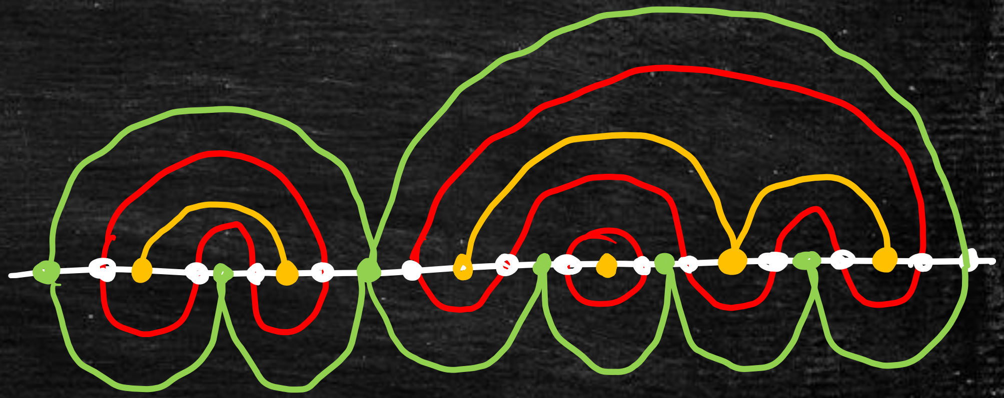
But there is hope!



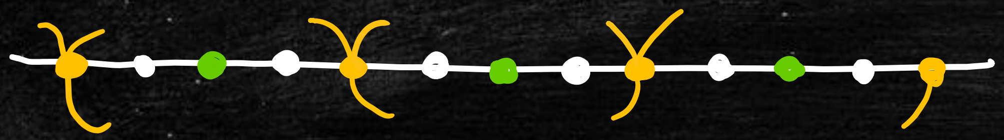
Percolation, where are you?

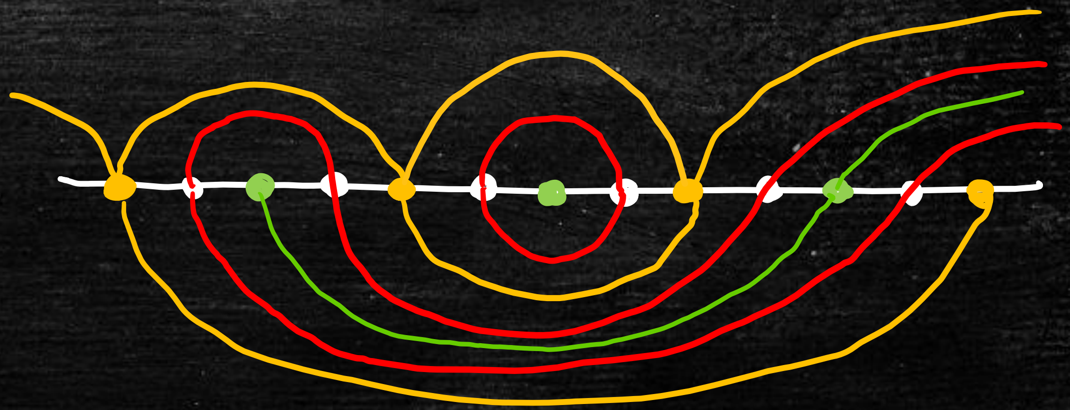






Each potential semi-arc appears independently with probability $1/2$.

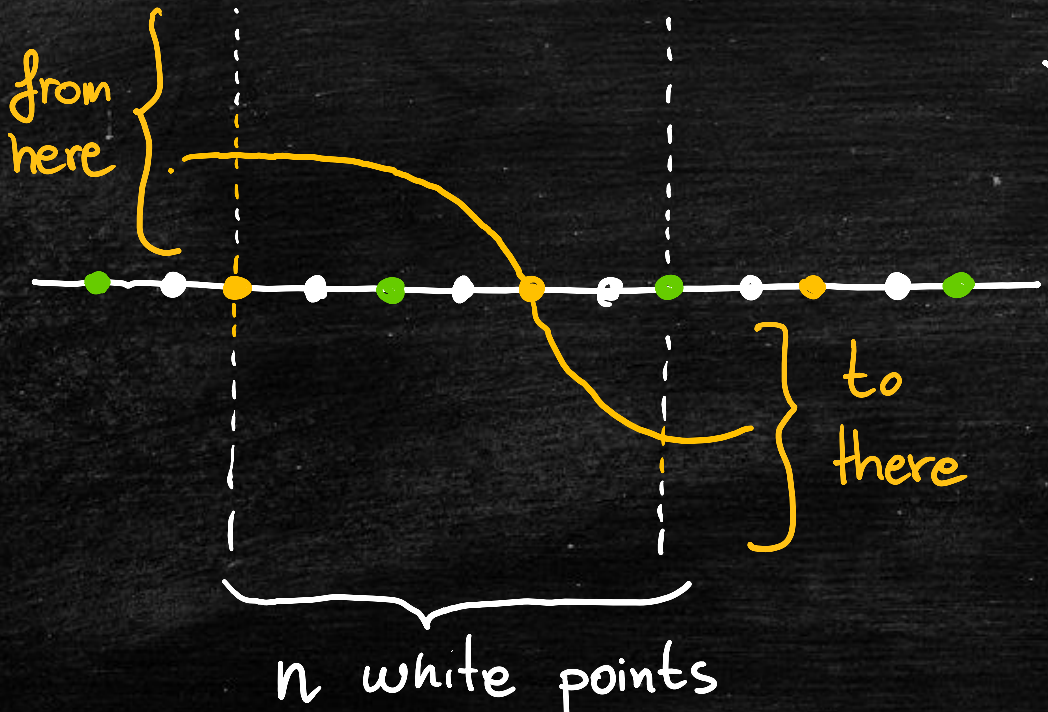




It looks like a PERCOLATION MODEL:

Can we say anything about crossing probabilities?

Proposition:

$\mathbb{P}\Big($  $\Big) = \frac{1}{2}$

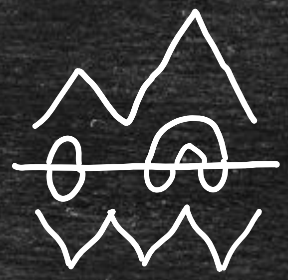
BUT ...

we do NOT have a good notion of monotonicity for FKG

Integrability, where are you?

[Di Francesco, 98]

$$M^n = \begin{array}{c|c} & d_1 \quad \cdots\cdots \quad d_{c_n} \\ \hline \begin{array}{c} d_1 \\ d_2 \\ \vdots \\ d_{c_n} \end{array} & \left(\begin{array}{ccc} & & \\ & & \\ & & \\ & & \end{array} \right) \end{array}$$



d_i

d_j

$$M^n_{ij} = q^{\#\mathrm{loop}(d_i,d_j)} = 2$$

THM: $\det(M^n) = \prod_{k=1}^{n} U_k(q/2)^{a_{n,k}}$, where

- $U_k(t)$ = Chebyshev polynomials of the 2^{nd} kind $\left[U_k(\cos\varphi) = \frac{\sin(k+1)\varphi}{\sin(\varphi)} \right]$
- $a_{n,k} = \binom{2n}{n-k} - 2 \cdot \binom{2n}{n-k-1} + \binom{2n}{n-k-2}$

Q: Can this result be used to establish "geometric properties"?

Thank you!