Vertex Models: Algebraic and Probabilistic Aspects of Universality
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Frozen boundaries and their fluctuations in the square-ice model

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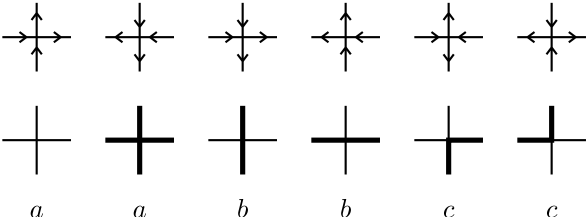
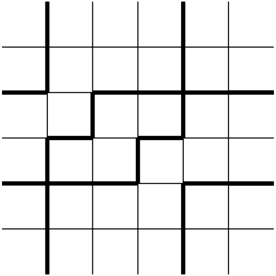
Based on arXiv:2405.04358
Joint work with A. Pronko (Steklov Mathematical Institute, St. Petersburg)

The six-vertex model

[Lieb'67] [Sutherland'67]





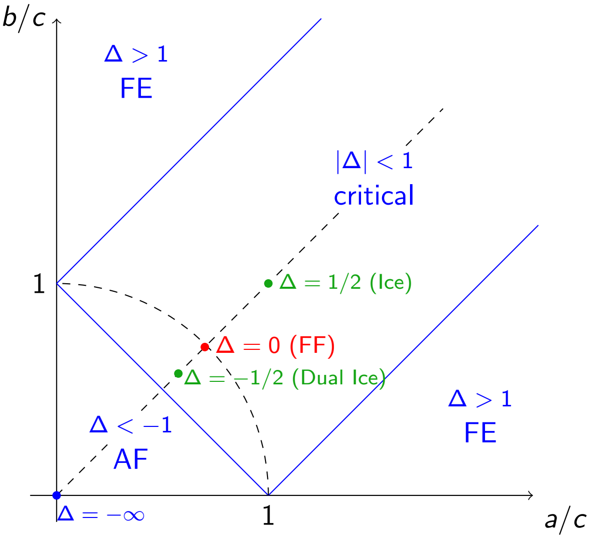
$$\Delta = \frac{a^2 + b^2 - c^2}{2ab}$$

$$t = b/a$$

square ice:

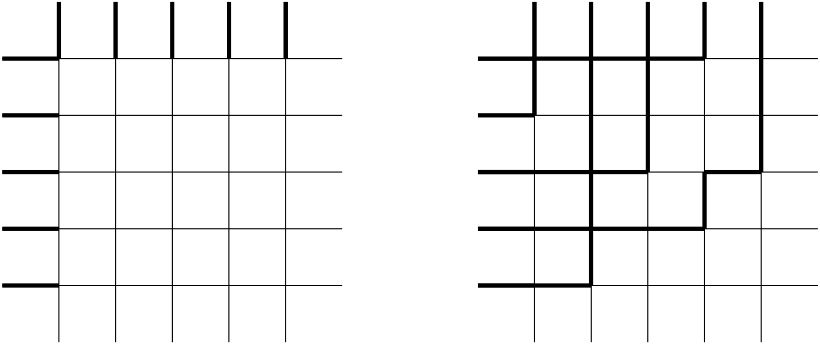
$a = b = c$ or $\Delta = \frac{1}{2}$, $t = 1$

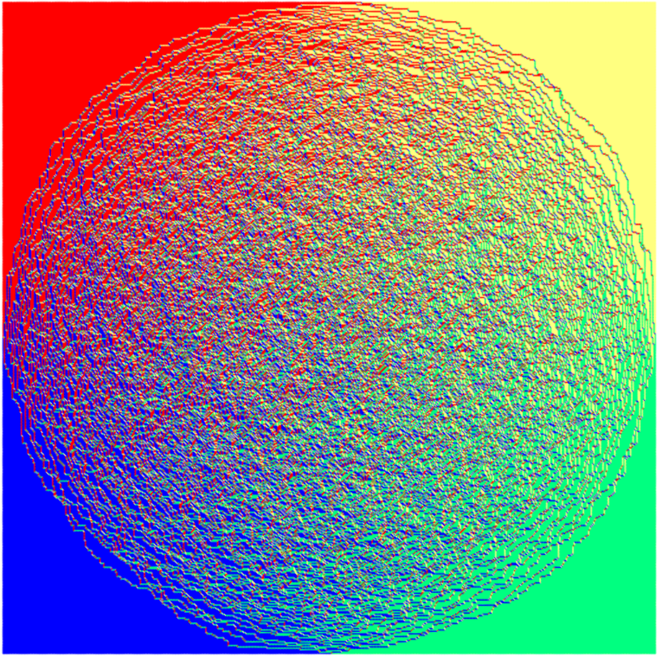
Phase diagram



The Domain Wall boundary conditions

[Korepin'82]



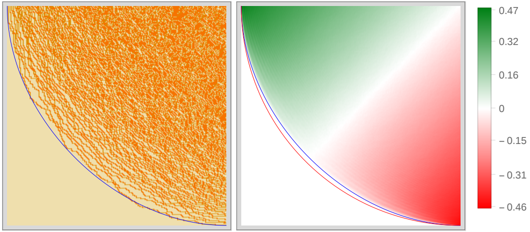


$\Delta = 1/2$

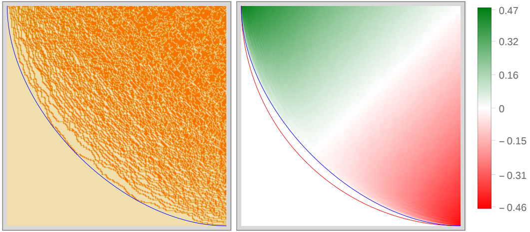
$N = 500$

Arctic curves

$$\Delta = -1/2$$



$$\Delta = -1$$



[Lyberg,Korepin,Viti'18]

Arctic curves

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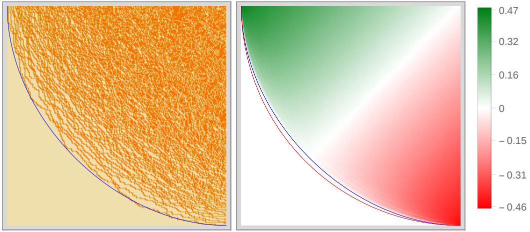
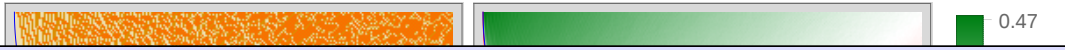


$\Delta = -1$

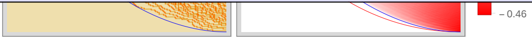


- Conjectural analytic expressions have been around for some time

  [FC-Pronko'09]

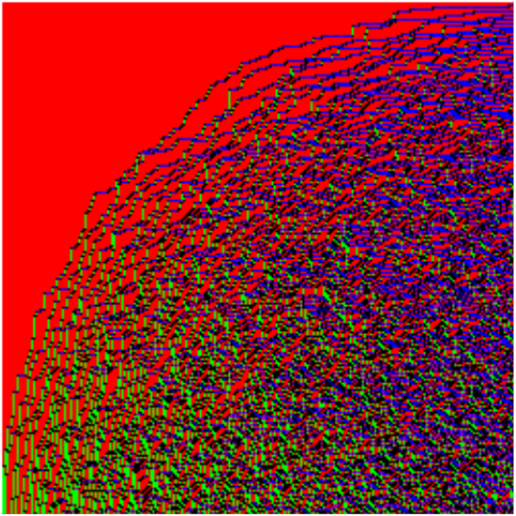
- Rigorous proof provided for the sole $\Delta = 1/2$ case

  [Aggarwal'19]



[Lyberg,Korepin,Viti'18]

Interface fluctuations



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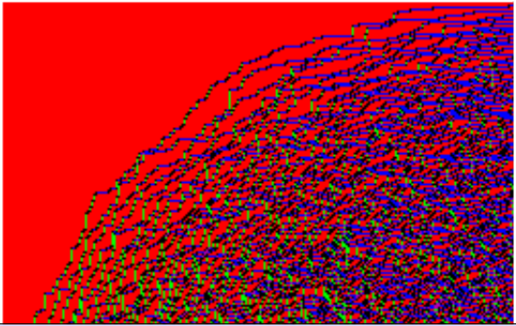


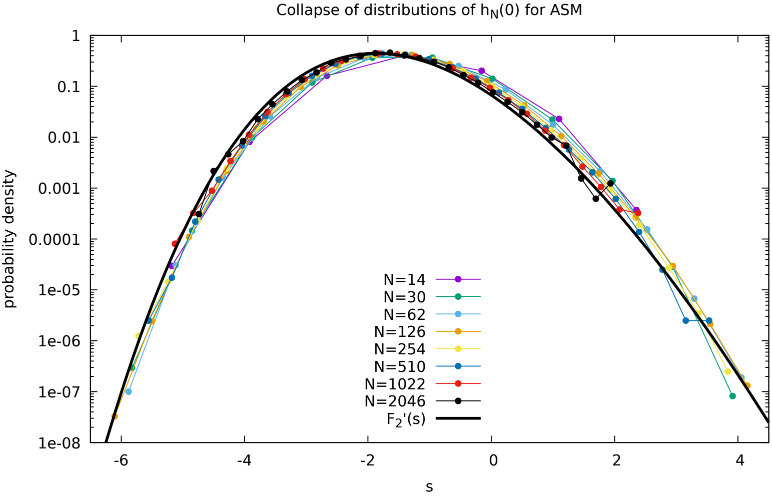
- two different statistics:
  - intersection of most external path with diagonal
  - maximum deviation of most external path
- for $\Delta = 0$, the model is in correspondence with Airy_2 process; first statistics is governed by GUE TW [Johansson'00], and consequently [Corwin-Quastel-Remenik'13] second statistic is governed by GOE TW

Interface fluctuations ($\Delta = 1/2$)

Strong numerical evidence that interface fluctuations follow GUE TW

[Prauhofer-Spohn'19](private communication)

[Korepin-Lyberg-Viti'23][Prauhofer-Spohn'24]



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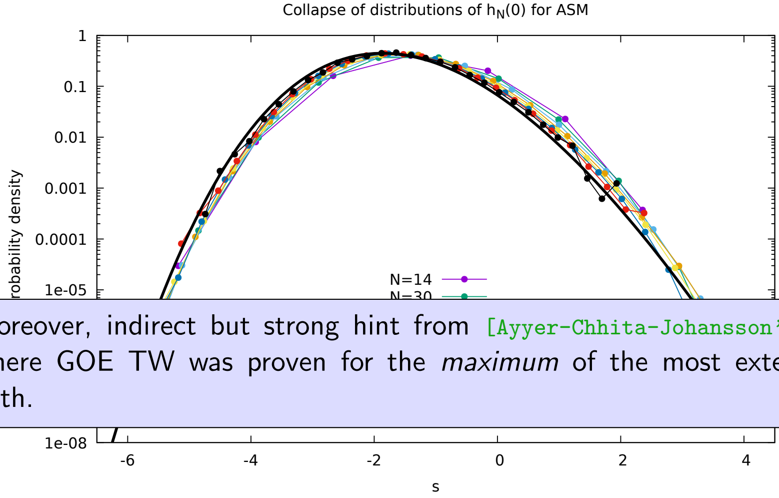
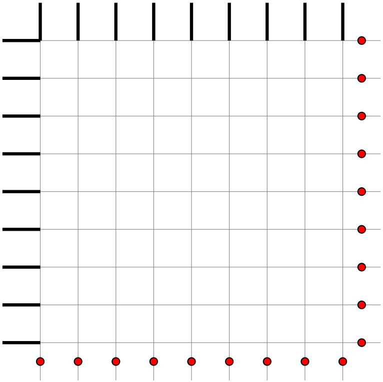


Moreover, indirect but strong hint from [Ayyer-Chhita-Johansson'23], where GOE TW was proven for the *maximum* of the most external path.

Partition function

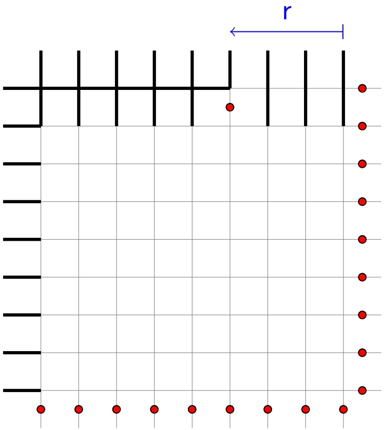


$$Z_N := \sum_{\{\mathcal{C}\}} a^{n_a} b^{n_b} c^{n_c}$$

$$n_a + n_b + n_c = N^2$$

Z_N evaluated as an I-K or Hankel determinant [Korepin'82][Izergin'87]

One-point boundary correlation function $H_N^{(r)}$



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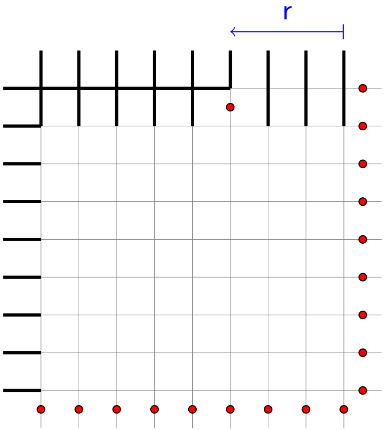
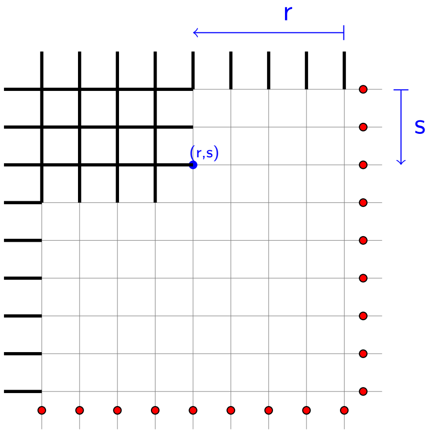


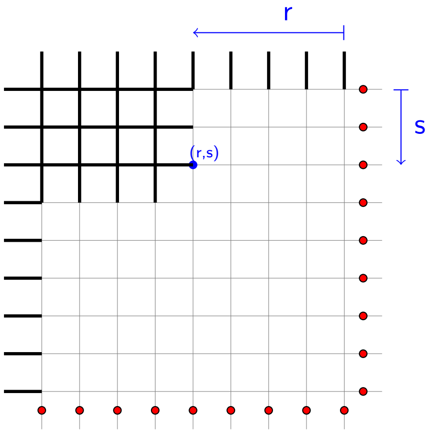
$H_N^{(r)}$ evaluated as an I-K or Hankel determinant with one modified column

[Bogoliubov-Pronko-Zvonarev'02]

Emptiness Formation Probability (EFP) $F_N^{(r,s)}$



Emptiness Formation Probability (EFP) $F_N^{(r,s)}$



- dicriminates the transition between top-left ordered region and central disordered region of the curve
- expected stepwise behaviour in correspondence of the Arctic curve
- Multiple Integral Representations (MIRs) provided [FC-Pronko'08]['21]

Multiple Integral Representation for EFP

Generating function of the one-point boundary correlator:

$$h_N(z) := \sum_{r=1}^{N} H_N^{(r)} z^{r-1}, \qquad h_N(1) = 1$$

Now define:

$$h_{N,s}(z_1,\dots,z_s) := \frac{1}{\Delta_s(z_1,\dots,z_s)} \det\left[(z_j-1)^k\, z_j^{s-k}\, h_{N-s+k}(z_j)\right]_{j,k=1}^{s}$$

- symmetric polynomials of order $N-1$.

- they provide a new, alternative representation (wrt Izergin-Korepin'one) for the partially inhomogeneous partition function $Z_N(\lambda_1,\dots,\lambda_s)$.

- two important properties:

$$h_{N,s}(z_1,\dots,z_{s-1},1) = h_{N,s-1}(z_1,\dots,z_{s-1})$$
$$h_{N,s}(z_1,\dots,z_{s-1},0) = h_N(0)\, h_{N-1,s-1}(z_1,\dots,z_{s-1})$$

Multiple Integral Representation for EFP

[FC-Pronko'08]

$$F_N^{(r,s)} = (-1)^s \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{s} \frac{[(t^2 - 2\Delta t) z_j + 1]^{s-j}}{z_j^r (z_j - 1)^{s-j+1}}$$

$$\times \prod_{1 \le j < k \le s} \frac{z_j - z_k}{t^2 z_j z_k - 2\Delta t z_j + 1} \, h_{N,s}(z_1, \ldots, z_s) \, \frac{\mathrm{d}^s z}{(2\pi \mathrm{i})^s}.$$

Remark: Similar (but somewhat simpler) expressions occur for various correlation functions of ASEP [Tracy-Widom'08-11], or also of the six-vertex model (possibly with higher spin, or coloured), but only in its stochastic version [Borodin-Corwin-Gorin'14] [Borodin-Petrov'16] [Aggarwal-Borodin'16] [Borodin-Bufetov-Wheeler'16] [Borodin-Corwin-Ferrari'16] [Dimitrov'16] [Barraquand-Borodin-Corwin'20] [Borodin-Wheeler'20]...

1) restrict to $t = 1$, and change variables: $z_j \mapsto z_j^{-1}, \quad j = 1, \dots, s$:

$$F_N^{(r,s)} = \oint_{C_\infty} \cdots \oint_{C_\infty} J_N^{(r,s)}(z_1, \dots, z_s)\, \mathrm{d}^s z,$$

where

$$J_N^{(r,s)}(z_1, \dots, z_s) = \frac{1}{(2\pi \mathrm{i})^s} \prod_{j=1}^{s} \frac{[1 - 2\Delta + z_j]^{s-j}}{z_j^{N-r}(z_j - 1)^{s-j+1}} \times \prod_{1 \le j < k \le s} \frac{z_j - z_k}{1 - 2\Delta z_k + z_j z_k}\, h_{N,s}(z_1, \dots, z_s).$$

2) deform integration contours. Miracolously, poles from double products give vanishing contribution [FC-Di Giulio-Pronko'21]. Thus

$$F_N^{(r,s)} = \oint_{C_1 \cup C_0} \cdots \oint_{C_1 \cup C_0} J_N^{(s)}(z_1, \dots, z_s)\, \mathrm{d}^s z.$$

that is:

$$F_N^{(r,s)} = \sum_{k=0}^{s} I_k, \qquad I_k := \sum_{|S|=k} \prod_{i \in S} \oint_{C_0} \mathrm{d}z_i \prod_{j \in S^c} \oint_{C_1} \mathrm{d}z_j J_N^{(r,s)}(z_1, \dots, z_s)$$

Two lemmas

Lemma

For arbitrary values of parameters r, s, Δ,

$$I_0 \equiv \operatorname*{res}_{z_1=1} \dots \operatorname*{res}_{z_s=1} J_N^{(r,s)}(z_1,\dots,z_s) = 1.$$

(Actually holds for generic values of t as well).

Lemma

At the ice point, $\Delta = 1/2$, $t = 1$, and for $r = N - s$ (square EFP)

$$I_s \equiv \operatorname*{res}_{z_1=0} \dots \operatorname*{res}_{z_s=0} J_N^{(N-s,s)}(z_1,\dots,z_s) = (-1)^s h_N \cdots h_{N-s+1},$$

where $h_N \equiv h_N(0)$, etc.

Proof is elementary

And when $k \neq 0, s$?

Recall:

$$I_k := \sum_{|S|=k} \prod_{i \in S} \oint_{C_0} \frac{\mathrm{d}z_i}{2\pi\mathrm{i}} \prod_{j \in S^c} \oint_{C_1} \frac{\mathrm{d}z_j}{2\pi\mathrm{i}} \prod_{j=1}^{s} \frac{[1 - 2\Delta + z_j]^{s-j}}{z_j^{N-r}(z_j - 1)^{s-j+1}}$$

$$\times \prod_{1 \leq j < k \leq s} \frac{z_j - z_k}{1 - 2\Delta z_k + z_j z_k} \; h_{N,s}(z_1, \dots, z_s)$$

where

$$h_{N,s} := \frac{1}{\Delta_s(z_1, \dots, z_s)} \det \left[(z_j - 1)^k \, z_j^{s-k} \, h_{N-s+k}(z_j) \right]_{j,k=1}^{s}$$

Two types of identities (type I)

$$h'_{N-1}(1) = \frac{1}{1-2\Delta t+t^2}\left\{\frac{h'_N}{h_N} - t^2\right\},$$

$$h''_{N-2}(1) = \frac{1}{(1-2\Delta t+t^2)^2}\left\{-\frac{h''_N}{h_N} + 2\frac{h'_{N-1}h'_N}{h_{N-1}h_N} - 2\left(1-2\Delta t+2t^2\right)\frac{h'_{N-1}}{h_{N-1}} + 2\frac{h'_N}{h_N} - 2t^2 + 2t^4\right\},$$

$$\begin{aligned} h'''_{N-3}(1) = \frac{1}{(1-2\Delta t+t^2)^2}\Bigg\{ & \frac{h'''_N}{h_N} - 3\frac{h'_{N-2}h''_N}{h_{N-2}h_N} - 3\frac{h''_{N-1}h'_N}{h_{N-1}h_N} \\ & + 3\left(2+3t^2-4t\Delta\right)\frac{h''_{N-1}}{h_{N-1}} - 6\frac{h''_N}{h_N} + 6\frac{h'_{N-2}h'_{N-1}h'_N}{h_{N-2}h_{N-1}h_N} \\ & - 6\left(2+3t^2-4t\Delta\right)\frac{h'_{N-2}h'_{N-1}}{h_{N-2}h_{N-1}} + 6\frac{h'_{N-2}h'_N}{h_{N-2}h_N} + 6\frac{h'_{N-1}h'_N}{h_{N-1}h_N} \\ & + 6\left(1+2t^2+3t^4-4t\Delta-6t^3\Delta+4t^2\Delta^2\right)\frac{h'_{N-2}}{h_{N-2}} \\ & - 6\left(2+3t^2-4t\Delta\right)\frac{h'_{N-1}}{h_{N-1}} + 6\frac{h'_N}{h_N} + 18t^4 - 6t^6 - 12t^3\Delta\Bigg\}. \end{aligned}$$

$$h''''_{N-4}(1) = \ldots$$

Two types of identities (type I)

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$$h'''_{N-3}(1) = \frac{1}{(1-2\Delta t+t^2)^2}\left\{\frac{h'''_N}{h_N} - 3\frac{h'_{N-2}h''_N}{h_{N-2}h_N} - 3\frac{h''_{N-1}h'_N}{h_{N-1}h_N}\right.$$

- valid for any Δ and t
- follows from availability of different MIR's for EFP (see [FC-Di Giulio-Pronko'21] for details)
- relate sums over the set of functions $H_N^{(r)}$, $r = 1, \ldots, N$ to the first few values of them (sum rules identities)
- allow to express the result of integration of our MIRs in terms of the sole value of $h_N(z)$ (and derivatives) at the origin

$$h''''_{N-4}(1) = \ldots$$

Two types of identities (type II)

When $\Delta = 1/2$ and $t = 1$ [Zeilberger'96]

$$h_N(z) = \frac{(N)_{N-1}}{(2N)_{N-1}} \, {}_2F_1\left(\left. \begin{matrix} -N+1,\ N \\ -2N+2 \end{matrix} \right| z \right).$$

It is easy to derive:

$$\frac{h'_N}{h_N} - \frac{h'_{N-1}}{h_{N-1}} - \frac{1}{2} = 0$$

$$\frac{h''_N}{h_N} - \frac{h''_{N-1}}{h_{N-1}} - \frac{h'_N}{h_N} - 2\frac{h_{N-2}}{h_{N-1}} + \frac{7}{2} = 0$$

$$\frac{h'''_N}{h_N} - \frac{h'''_{N-1}}{h_{N-1}} - \frac{3}{2}\left(\frac{h'_N}{h_N}\right)^2 - \frac{21}{2}\left(\frac{h_{N-2}}{h_{N-1}} - \frac{7}{4}\right) = 0$$

$$\frac{h''''_N}{h_N} - \qquad \ldots \qquad = 0$$

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$$h''_N \quad h''_{N-1} \quad h'_N \quad h_{N-2} \quad 7$$

- valid only at ice-point
- follow from standard relation for Gauss hypergeometric functions
- involves only functions $h_N(z)$ and derivatives, evaluated at $z = 0$
- allow to express the result of integration of our MIRs in terms of just $2s - 1$ formally independent objects, $h_{N-s+1}, \ldots, h_N, h'_N, \ldots, h_N^{(s-1)}$

Determinant structure

Inspired by [Tracy-Widom'08] [Saenz-Tracy-Widom'22] we assume that, for each s, an $s \times s$ matrix $A = A(N, s)$ exists, such that

$$\sum_{k=0}^{s} \lambda^k I_k = \det{}_s(I - \lambda A)$$

Clearly, from last lemma, $\det_s A = I_s$, for any s.
Below, we shall also observe that

- A is such that by eliminating its last row and colum, the reduction $s \mapsto s - 1, \;\; N \mapsto N - 1$ is made;
- A can be given explicitely in a factorized form $A = DLU$.

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To proceed, it is convenient to introduce the abbreviated notations

$$b_i \equiv h_{N-i}, \qquad i = 0, 1, 2, \ldots, s-1,$$

and

$$\kappa_i' = \frac{h_{N-i}'}{h_{N-i}}, \quad \kappa_i'' = \frac{h_{N-i}''}{h_{N-i}}, \quad \kappa_i''' = \frac{h_{N-i}'''}{h_{N-i}}, \quad \ldots$$

Recall that $h_N \equiv h_N(0)$, $h_N' \equiv h_N'(0)$, etc.

Case s=1

$$F_N^{(N-1,1)} = \oint_{C_1 \cup C_0} \frac{1}{z(z-1)} h_N(z) \frac{\mathrm{d}z}{2\pi \mathrm{i}}$$

$$= h_n(1) - h_N(0)$$
$$= 1 - b_0$$

That is

$$I_0 = 1 \qquad I_1 = -b_0$$

as we already knew from our two lemmas.

We are looking for 1×1 matrix A such that

$$\det_1(1 - A) = 1 - b_0$$

Thus:

$$A = b_0$$

Case s=2

$$I_0 = 1$$
$$I_1 = -b_0 k_0' - b_1 - b_0 k_0' h_{N-1}'(1) \qquad = -\operatorname{tr} A$$
$$I_2 = b_0 b_1 \qquad = \det A$$

Use first identity of type I, namely $h_{N-1}'(1) = k_0' - 1$, and get

$$I_0 = 1$$
$$I_1 = -b_1 - b_0 (k_0')^2 \qquad = -\operatorname{tr} A$$
$$I_2 = b_0 b_1 \qquad = \det A$$

If 2×2 matrix A exists , it must be such that when $b_0 = 0$ its top-left entries is b_1. Thus

$$A = \begin{pmatrix} b_1 & b_1(\kappa_0' - 1) \\ b_0(\kappa_0' + 1) & b_0(\kappa_0')^2 \end{pmatrix}$$

with DLU factorization:

$$D = \begin{pmatrix} b_1 & 0 \\ 0 & b_0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ \kappa_0' + 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \kappa_0' - 1 \\ 0 & 1 \end{pmatrix}.$$

Case s=3

$$I_0 = 1$$

$$I_1 = -b_2 - b_1 \left(\kappa_1'\right)^2 - b_0 \left[\left(\frac{\kappa_0''}{2} - \kappa_0'\right)^2 + 2\kappa_0' - 1\right] \qquad = -\operatorname{tr} A$$

$$I_2 = b_1 b_2 + b_0 b_2 \left(\kappa_0'\right)^2 + b_0 b_1 \left[1 - \kappa_0'(1 + \kappa_1') + \frac{\kappa_0''}{2}\right]^2 \qquad = \frac{1}{2}[(\operatorname{tr} A)^2 - \operatorname{tr} A^2]$$

$$I_3 = -b_0 b_1 b_2 \qquad = \det A$$

NB1: here first two identities of type I have been used

NB2: setting $b_0 = 0$ one recover the $s = 2$ case, modulo the replacement $b_0, b_1, k_0' \mapsto b_1, b_2, k_1'$, that is $N \mapsto N - 1$.

We may thus obtaine the top-left 2×2 block from the $s = 2$ case.

Completing the sudoku, we get $A = DLU$, with

$$D = \begin{pmatrix} b_2 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & b_0 \end{pmatrix}, \; L = \begin{pmatrix} 1 & 0 & 0 \\ \kappa_1' + 1 & 1 & 0 \\ \frac{1}{2}\kappa_0'' - 2\kappa_0' + 1 & \kappa_0' - 1 & 1 \end{pmatrix}, \; U = \begin{pmatrix} 1 & \kappa_1' - 1 & \frac{1}{2}\kappa_0'' - 1 \\ 0 & 1 & \kappa_0' + 1 \\ 0 & 0 & 1 \end{pmatrix}$$

NB3: I_0, I_1, I_3 are easily reproduced. But I_2 is only recovered modulo a term proportional to $k_1' - k_0' + \frac{1}{2}$, which however vanish, due to first identity of type II !

Case s=4

$$D = \mathrm{diag}(b_3, b_2, b_1, b_0),$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \kappa_2' + 1 & 1 & 0 & 0 \\ \frac{1}{2}\kappa_1'' - 2\kappa_1' + 1 & \kappa_1' - 1 & 1 & 0 \\ \frac{1}{6}\kappa_0''' - \frac{1}{2}\kappa_0'' - \kappa_0' + 1 & \frac{1}{2}\kappa_0'' - 1 & \kappa_0' + 1 & 1 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & \kappa_2' - 1 & \frac{1}{2}\kappa_1'' - 1 & \frac{1}{6}\kappa_0''' - \frac{3}{2}\kappa_0'' + 3\kappa_0' - 1 \\ 0 & 1 & \kappa_1' + 1 & \frac{1}{2}\kappa_0'' - 2\kappa_0' + 1 \\ 0 & 0 & 1 & \kappa_0' - 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Case s=4

$$D = \mathrm{diag}(b_3, b_2, b_1, b_0),$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \kappa_2' + 1 & 1 & 0 & 0 \\ \frac{1}{2}\kappa_1'' - 2\kappa_1' + 1 & \kappa_1' - 1 & 1 & 0 \\ \frac{1}{6}\kappa_0''' - \frac{1}{2}\kappa_0'' - \kappa_0' + 1 & \frac{1}{2}\kappa_0'' - 1 & \kappa_0' + 1 & 1 \end{pmatrix},$$

$$U = \begin{pmatrix} 1 & \kappa_2' - 1 & \frac{1}{2}\kappa_1'' - 1 & \frac{1}{6}\kappa_0''' - \frac{3}{2}\kappa_0'' + 3\kappa_0' - 1 \\ 0 & 1 & \kappa_1' + 1 & \frac{1}{2}\kappa_0'' - 2\kappa_0' + 1 \\ 0 & 0 & 1 & \kappa_0' - 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Case s=4

$$D = \mathrm{diag}(b_3, b_2, b_1, b_0),$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \kappa_2' + 1 & 1 & 0 & 0 \\ \frac{1}{2}\kappa_1'' - 2\kappa_1' + 1 & \kappa_1' - 1 & 1 & 0 \\ \frac{1}{6}\kappa_0''' - \frac{1}{2}\kappa_0'' - \kappa_0' + 1 & \frac{1}{2}\kappa_0'' - 1 & \kappa_0' + 1 & 1 \end{pmatrix},$$

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$$L_0(x) = 1 \qquad L_1(x) = -x + 1,$$

$$L_2(x) = \frac{x^2}{2} - 2x + 1 \qquad L_3(x) = -\frac{x^3}{6} + \frac{3x^2}{2} - 3x + 1$$

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$$L_0^{(-1)}(x) - L_{-1}^{(0)}(x) = 1 \qquad L_1^{(-1)}(x) - L_0^{(0)}(x) = -x - 1$$

$$L_2^{(-1)}(x) - L_1^{(0)}(x) = \frac{x^2}{2} - 1 \qquad L_3^{(-1)}(x) - L_2^{(0)}(x) = -\frac{x^3}{6} + \frac{x^2}{2} + x - 1$$

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$$D = \mathrm{diag}(b_3, b_2, b_1, b_0),$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \kappa_2' + 1 & 1 & 0 & 0 \\ \frac{1}{2}\kappa_1'' - 2\kappa_1' + 1 & \kappa_1' - 1 & 1 & 0 \\ \frac{1}{6}\kappa_0''' - \frac{1}{2}\kappa_0'' - \kappa_0' + 1 & \frac{1}{2}\kappa_0'' - 1 & \kappa_0' + 1 & 1 \end{pmatrix},$$

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$$L_0^{(-1)}(x) + L_{-1}^{(0)}(x) = 1 \qquad L_1^{(-1)}(x) + L_0^{(0)}(x) = -x + 1$$

$$L_2^{(-1)}(x) + L_1^{(0)}(x) = \frac{x^2}{2} - 2x + 1 \qquad L_3^{(-1)}(x) + L_2^{(0)}(x) = -\frac{x^3}{6} + \frac{3x^2}{2} - 3x + 1$$

The conjecture

For $t = 1$ and $\Delta = 1/2$, and for $r = N - s$, the EFP can be given as $\det_s(I - A)$ where the $s \times s$ matrix A is given as $A = DLU$ and

$$D_{ij} = h_{r+i}(0)\,\delta_{ij}$$

$$L_{ij} = \frac{(-1)^{i-j}}{h_{r+i}(0)} \left[L^{(-1)}_{i-j}(\partial_z) + (-1)^{i-1} L^{(0)}_{i-j-1}(\partial_z) \right] h_{r+i}(z) \Big|_{z=0}$$

$$U_{ij} = \frac{(-1)^{i-j}}{h_{r+j}(0)} \left[L^{(-1)}_{j-i}(\partial_z) + (-1)^{j} L^{(0)}_{j-i-1}(\partial_z) \right] h_{r+j}(z) \Big|_{z=0}$$

where functions $h_j(z)$ are the Gauss hypergeometric functions given above.

The conjecture

For $t = 1$ and $\Delta = 1/2$, and for $r = N - s$, the EFP can be given as $\det_s(I - A)$ where the $s \times s$ matrix A is given as $A = DLU$ and

$$
\begin{aligned}
D_{ij} &= h_{r+i}(0)\,\delta_{ij} \\
L_{ij} &= \frac{(-1)^{i-j}}{h_{r+i}(0)} \left[L^{(-1)}_{i-j}(\partial_z) + (-1)^{i-1} L^{(0)}_{i-j-1}(\partial_z) \right] h_{r+i}(z)\Big|_{z=0} \\
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\end{aligned}
$$

where functions $h_j(z)$ are the Gauss hypergeometric functions given above.

- note that dependence on parameter s is both via the size of the matrix, and the parameter $r = N - s$
- Appearance of Laguerre polynomials does not come as a surprise, if one recalls relations such as

$$
\int_{C_0} \frac{(1-z)^{n+\alpha}}{z^{n+1}} f(z) \frac{\mathrm{d}z}{2\pi \mathrm{i}} = (-1)^n L^{(\alpha)}_n(\partial_z) f(z)\Big|_{z=0}
$$

The conjecture

For $t = 1$ and $\Delta = 1/2$, and for $r = N - s$, the original MIR for EFP,

$$F_N^{(N-s,s)} = (-1)^s \oint_{C_0} \cdots \oint_{C_0} \prod_{j=1}^{s} \frac{1}{z_j^{N-s}(z_j-1)^{s-j+1}} \times \prod_{1\le j<k\le s} \frac{z_j - z_k}{z_j z_k - z_j + 1}\, h_{N,s}(z_1,\dots,z_s)\,\frac{\mathrm{d}^s z}{(2\pi\mathrm{i})^s}$$

can be given as $\det_s(I - A)$ where the $s \times s$ matrix $A = A(N,s)$ reads

$$A_{ij} = \oint_{C_0}\oint_{C_0} \frac{e_i^L(z) e_j^U(w)}{1 - z - w}\frac{\mathrm{d}z\mathrm{d}w}{(2\pi\mathrm{i})^2}, \qquad i,j = 1,\dots,s, \qquad (*)$$

with

$$e_i^L(z) := \frac{(1-z)^{i-1}}{z^i}\left(1 + (-1)^i z\right) h_{r+i}(z),$$

$$e_j^U(w) := \frac{(1-w)^{j-1}}{h_{r+j}(0) w^j}\left(1 + (-1)^{j+1} w\right) h_{r+j}(w).$$

The conjecture

- crucial in this derivation were our two sets of identities;
- and also our ansatz, fixing at step s, all entries of an $(s-1)\times(s-1)$ sub-block of A, so that s new conditions at each step were sufficient;
- however nice is the result, it is still just a guess;
- unable to proceed with our calculation beyond $s=4$;
- desperately seeking a proof.

$$A_{ij} = \oint_{C_0}\oint_{C_0} \frac{e_i^L(z)e_j^U(w)}{1-z-w}\frac{\mathrm{d}z\mathrm{d}w}{(2\pi\mathrm{i})^2}, \qquad i,j=1,\ldots,s, \qquad (*)$$

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Check

Check the $s = 5$ case: evaluate with Mathematica both our conjectural expression and the MIR, for $N = 7, \ldots, 13$:

N	*Determinant*	*MIR*
7	0	0
8	0	0
9	0	0
10	$\frac{61347}{43178090900}$	$\frac{61347}{43178090900}$
11	$\frac{49711519}{1636618150125}$	$\frac{49711519}{1636618150125}$
12	$\frac{54886057499}{221251085257500}$	$\frac{54886057499}{221251085257500}$
13	$\frac{3870965779057}{3266307568354500}$	$\frac{3870965779057}{3266307568354500}$

Integral form for matrix A

As said, the matrix A admits the following integral representation

$$A_{ij} = \oint_{C_0} \oint_{C_0} \frac{e_i^L(z) e_j^U(w)}{1 - z - w} \frac{\mathrm{d}z \mathrm{d}w}{(2\pi \mathrm{i})^2}, \qquad i, j \in \{1, \ldots, s\},$$

where

$$e_i^L(z) := \frac{(1-z)^{i-1}}{z^i} \left(1 + (-1)^i z\right) h_{r+i}(z),$$
$$e_j^U(w) := \frac{(1-w)^{j-1}}{h_{r+j}(0) w^j} \left(1 + (-1)^{j+1} w\right) h_{r+j}(w).$$

Or, equivalently,

$$A_{ij} = \oint_{C_0} \oint_{C_0} e_i^L(z) e_j^U(w) \int_0^\infty \mathrm{e}^{(z+w-1)t} \mathrm{d}t \frac{\mathrm{d}z \mathrm{d}w}{(2\pi \mathrm{i})^2}, \qquad \mathrm{Re}\,(z+w) < 1$$

Fredholm determinant

Let $\hat{K}_{[0,\infty)}$ be a linear integral operator acting on functions defined on $\mathbb{R}^+$ according to the rule

$$(\hat{K}_{[0,\infty)}f)(t_1) = \int_0^\infty K(t_1,t_2)f(t_2)\mathrm{d}t_2$$

with kernel

$$K(t_1,t_2) = \oint_{C_0}\oint_{C_0} \mathrm{e}^{\left(z-\frac{1}{2}\right)t_1+\left(w-\frac{1}{2}\right)t_2} \sum_{j=1}^{s} e_j^L(z)e_j^U(w)\frac{\mathrm{d}z\mathrm{d}w}{(2\pi\mathrm{i})^2}.$$

Proposition

Given matrix $A = A(N,s)$ as in $(*)$, for any finite integer s, we have

$$\det_s(I-A) = \det\left(1-\hat{K}_{[0,\infty)}\right).$$

Remark

The kernel $K(t_1,t_2)$ is not 'of integrable form' (in the sense of [Its-Izergin-Korepin-Slavnov'92]).

Scaling limit

We want to study the behaviour of the kernel $K(t_1, t_2)$ in the scaling limit, i.e. (recall that $r = N - s$)

$$s = \lceil yN \rceil, \qquad y \in (0, 1/2], \qquad N \to \infty$$

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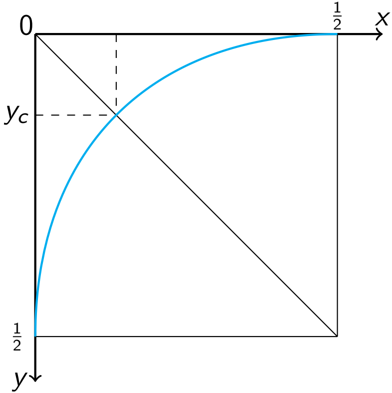

$$y_c := 1 - \frac{\sqrt{3}}{2}$$

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$$s = \lceil yN \rceil, \qquad y \in (0, 1/2], \qquad N \to \infty$$

In this limit

$$K(t_1, t_2) \sim \oint_{C_0} \oint_{C_0} \mathrm{e}^{\left(z - \frac{1}{2}\right) t_1 + \left(w - \frac{1}{2}\right) t_2} \, \mathrm{e}^{N[g(w) + g(z)]} \, f(z, w) \, \frac{\mathrm{d}z \mathrm{d}w}{(2\pi \mathrm{i})^2}$$

where

$$g(w) := y \log \frac{1 - w}{w} + \log \frac{(1 - 2w)(2 - w)(1 + w) + 2\left(1 - w + w^2\right)^{3/2}}{3\sqrt{3}(1 - w)^2}$$

while $f(z, w)$ is some complicate but explicit function.

Saddle points

Saddle-point equation

$$g'(w) = \frac{y}{w(w-1)} - \frac{1-\sqrt{1-w+w^2}}{w(w-1)} = 0$$

has two solutions

$$w_\pm = \frac{1 \pm \sqrt{1-8y+4y^2}}{2}$$

which collide when $y = y_c := 1 - \frac{\sqrt{3}}{2}$, recall, $y \in (0, \frac{1}{2}]$.

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which collide when $y = y_c := 1 - \frac{\sqrt{3}}{2}$, recall, $y \in (0, \frac{1}{2}]$.

- y_c happens to correspond to the intersection of the arctic curve with the main diagonal
- for values $y \in (0, y_c)$ i.e. outside the arctic curve (frozen region) $w_{\pm}$ are both real, with an exponential decay of the integrals, ruled by w_{-}
- for values $y \in (y_c, 1/2)$, i.e. inside the arctic curve (disordered region) $w_{\pm}$ are complex conjugate, and contribute both to the integrals, producing an oscillatory behaviour

in analogy with dimer models [Kenyon-Okounkov-Sheffield'06]

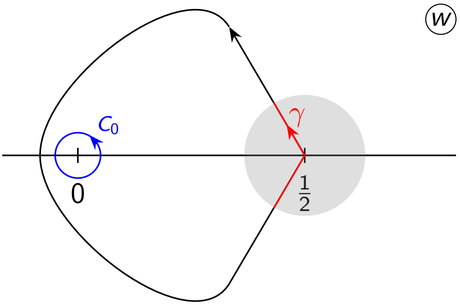
y close to y_c

Let us study $K(t_1, t_2)$ in the vicinity of $y = y_c$.

Let $y = y_c - \eta$, and $w = \frac{1}{2} + \lambda$, with η, λ small. We have

$$g(w)\Big|_{w=\frac{1}{2}+\lambda} = 4\eta\lambda - \frac{4}{3\sqrt{3}}\lambda^3 + O(\lambda^4)$$

which sets the scales $\lambda = O(N^{-1/3})$, $\eta = O(N^{-2/3})$.



and similarly for $z = \frac{1}{2} + \mu$, with $\mu = O(N^{-1/3})$.

y close to y_c

We now rescale

$$\tilde{\lambda} = q\lambda, \qquad \tilde{\mu} = q\mu, \qquad q = \frac{2^{2/3}}{3^{1/6}} N^{1/3},$$

and

$$\sigma = \frac{4N}{q}\eta = 2^{4/3} 3^{1/6} N^{2/3} \eta.$$

where $\tilde{\lambda}$, $\tilde{\mu}$, and σ are $O(N^0)$.

We also rescale the variables t_1 and t_2 and the kernel itself

$$\widetilde{K}(t_1, t_2) := q\, K(q\, t_1, q\, t_2), \qquad q > 0, \qquad t_1, t_2 \in [0, \infty)$$

obtaining

$$\widetilde{K}(t_1, t_2) = -\int_{\tilde{\gamma}} \int_{\tilde{\gamma}} \frac{\mathrm{e}^{\tilde{\mu} t_1 + \tilde{\lambda} t_2 + \sigma(\tilde{\lambda} + \tilde{\mu}) - (\tilde{\lambda}^3 + \tilde{\mu}^3)/3}}{\tilde{\lambda} + \tilde{\mu}} \frac{\mathrm{d}\tilde{\lambda} \mathrm{d}\tilde{\mu}}{(2\pi \mathrm{i})^2}.$$

Summing up

$$\lim_{N\to\infty}\left(\det{}_s(1-A)\Big|_{s=N\left(1-\frac{\sqrt{3}}{2}\right)-\frac{N^{1/3}}{2^{4/3}3^{1/6}}\sigma}\right)=\det\left(1-\hat{\widetilde{K}}_{[0,\infty)}\right)$$

with kernel

$$\widetilde{K}(t_1,t_2)=-\int_{\tilde\gamma}\int_{\tilde\gamma}\frac{\mathrm{e}^{\tilde\mu t_1+\tilde\lambda t_2+\sigma(\tilde\lambda+\tilde\mu)-(\tilde\lambda^3+\tilde\mu^3)/3}}{\tilde\lambda+\tilde\mu}\frac{\mathrm{d}\tilde\lambda\mathrm{d}\tilde\mu}{(2\pi\mathrm{i})^2}.$$

Summing up

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Proposition

Let $\hat{K}^{\mathrm{Ai}}$ the linear integral operator on the real line, with kernel

$$K^{\mathrm{Ai}}(t_1,t_2)=\frac{\mathrm{Ai}(t_1)\,\mathrm{Ai}'(t_2)-\mathrm{Ai}'(t_1)\,\mathrm{Ai}(t_2)}{t_1-t_2}.$$

One has

$$\det\left(1-\hat{\widetilde{K}}_{[0,\infty)}\right)=\det\left(1-\hat{K}^{\mathrm{Ai}}_{[\sigma,\infty)}\right)=:\mathcal{F}_2(\sigma),$$

Conclusions

Conjecture

At ice point, $\Delta = \frac{1}{2}$, $t = 1$, *the following holds*

$$F_N^{(N-s,s)} = \det_s(1 - A)$$

where $A = A(N, s)$ *is the* $s \times s$ *matrix given in* $(*)$.

Theorem

Given the $s \times s$ *matrix* $A = A(N, s)$, *see* $(*)$, *the following holds*

$$\lim_{N \to \infty} \left(\det_s(1 - A) \Big|_{s = N\left(1 - \frac{\sqrt{3}}{2}\right) - \frac{N^{1/3}}{2^{4/3} 3^{1/6}} \sigma} \right) = \mathcal{F}_2(\sigma).$$

The presented result is in full agreement with the conjecture in [Ayyer-Chhita-Johansson'23] and with the numerical simulations in [Korepin-Lyberg-Viti'23][Prauhofer-Spohn'24].