Frozen boundaries and their fluctuations in the square-ice model

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Joint work with A. Pronko (Steklov Mathematical Institute, St. Petersburg)
The six-vertex model

\[\text{[Lieb'67][Sutherland'67]}\]

\[\Delta = \frac{a^2 + b^2 - c^2}{2ab}\]
\[t = \frac{b}{a}\]

square ice:
\[a = b = c \quad \text{or} \quad \Delta = \frac{1}{2}, \quad t = 1\]
Phase diagram

- $\Delta > 1$
  - FE
- $|\Delta| < 1$
  - critical
- $\Delta = -\frac{1}{2}$ (Dual Ice)
- $\Delta = -\frac{1}{2}$ (Ice)
- $\Delta = 0$ (FF)
- $\Delta = -\infty$
  - AF
- $\Delta = 1/2$ (Ice)
The Domain Wall boundary conditions

[Korepin’82]
Arctic curves

$\Delta = -1/2$

Conjectural analytic expressions have been around for some time [Lyberg, Korepin, Viti'18].

Rigorous proof provided for the sole $\Delta = 1/2$ case [Aggarwal'19].
Arctic curves

\[ \Delta = -1/2 \]

\[ \Delta = -1 \]

- Conjectural analytic expressions have been around for some time
  [FC–Pronko’09]
- Rigorous proof provided for the sole \( \Delta = 1/2 \) case
  [Aggarwal’19]

[Lyberg, Korepin, Viti’18]
Interface fluctuations

- Two different statistics:
  - Intersection of most external path with diagonal
  - Maximum deviation of most external path

For $\Delta = 0$, the model is in correspondence with Airy process; the first statistic is governed by GUE TW [Johansson'00], and consequently [Corwin-Quastel-Remenik'13] the second statistic is governed by GOE TW.
two different statistics:
- intersection of most external path with diagonal
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for $\Delta = 0$, the model is in correspondence with Airy$_2$ process; first statistics is governed by GUE TW [Johansson’00], and consequently [Corwin-Quastel-Remenik’13] second statistic is governed by GOE TW
Interface fluctuations ($\Delta = 1/2$)

Strong numerical evidence that interface fluctuations follow GUE TW

[Praehofer-Spohn'19] (private communication)
[Korepin-Lyberg-Viti’23] [Prahofer-Spohn’24]

Moreover, indirect but strong hint from [Ayyer-Chhita-Johansson’23], where GOE TW was proven for the maximum of the most external path.
Interface fluctuations \((\Delta = 1/2)\)

Strong numerical evidence that interface fluctuations follow GUE TW

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\[ \text{[Korepin-Lyberg-Viti’23] [Prauhofer-Spohn’24]} \]

Moreover, indirect but strong hint from [Ayyer-Chhita-Johansson’23], where GOE TW was proven for the \textit{maximum} of the most external path.
Partition function

\[ Z_N := \sum_{\{C\}} a^{n_a} b^{n_b} c^{n_c} \]

\[ n_a + n_b + n_c = N^2 \]

\( Z_N \) evaluated as an I-K or Hankel determinant [Korepin’82][Izergin’87]
One-point boundary correlation function $H_N^{(r)}$
One-point boundary correlation function $H_N^{(r)}$

$H_N^{(r)}$ evaluated as an I-K or Hankel determinant with one modified column

[Bogoliubov-Pronko-Zvonarev’02]
Emptiness Formation Probability (EFP) $F_{N}^{(r,s)}$

$r$

$s$

$(r,s)$

Discriminates the transition between top-left ordered region and central disordered region of the curve.

Multiple Integral Representations (MIRs) provided

[FC-Pronko'08]['21]
Emptiness Formation Probability (EFP) $F_N^{(r,s)}$

- discriminates the transition between top-left ordered region and central disordered region of the curve
- expected stepwise behaviour in correspondence of the Arctic curve
- Multiple Integral Representations (MIRs) provided [FC-Pronko ’08] [’21]
Multiple Integral Representation for EFP

Generating function of the one-point boundary correlator:

\[ h_N(z) := \sum_{r=1}^{N} H_N^{(r)} z^{r-1}, \quad h_N(1) = 1 \]

Now define:

\[ h_{N,s}(z_1, \ldots, z_s) := \frac{1}{\Delta_s(z_1, \ldots, z_s)} \det \left[ (z_j - 1)^k z_j^{s-k} h_{N-s+k}(z_j) \right]_{j,k=1}^s \]

- symmetric polynomials of order \( N - 1 \).
- they provide a new, alternative representation (wrt Izergin-Korepin’one) for the partially inhomogeneous partition function \( Z_N(\lambda_1, \ldots, \lambda_s) \).
- two important properties:

\[ h_{N,s}(z_1, \ldots, z_{s-1}, 1) = h_{N,s-1}(z_1, \ldots, z_{s-1}) \]
\[ h_{N,s}(z_1, \ldots, z_{s-1}, 0) = h_N(0) h_{N-1,s-1}(z_1, \ldots, z_{s-1}) \]
Multiple Integral Representation for EFP

\[ F_N^{(r,s)} = (-1)^s \int_{C_0} \cdots \int_{C_0} \prod_{j=1}^{s} \left[ \frac{(t^2 - 2\Delta t)z_j + 1}{z_j^r(z_j - 1)^{s-j+1}} \right] \]

\[ \times \prod_{1 \leq j < k \leq s} \frac{z_j - z_k}{t^2 z_j z_k - 2\Delta t z_j + 1} \quad h_{N,s}(z_1, \ldots, z_s) \frac{d^sz}{(2\pi i)^s} . \]

**Remark:** Similar (but somewhat simpler) expressions occur for various correlation functions of ASEP [Tracy-Widom’08-11], or also of the six-vertex model (possibly with higher spin, or coloured), but only in its stochastic version [Borodin-Corwin-Gorin’14] [Borodin-Petrov’16] [Aggarwal-Borodin’16] [Borodin-Bufetov-Wheeler’16] [Borodin-Corwin-Ferrari’16] [Dimitrov’16] [Barraquand-Borodin-Corwin’20] [Borodin-Wheeler’20]...
1) restrict to \( t = 1 \), and change variables: \( z_j \mapsto z_j^{-1}, \ j = 1, \ldots, s \):

\[
F_N^{(r,s)} = \oint_{C_\infty} \cdots \oint_{C_\infty} J_N^{(r,s)}(z_1, \ldots, z_s) \, d^s z,
\]

where

\[
J_N^{(r,s)}(z_1, \ldots, z_s) = \frac{1}{(2\pi i)^s} \prod_{j=1}^{s} \frac{[1 - 2\Delta + z_j]^{s-j}}{z_j^{N-r}(z_j - 1)^{s-j+1}} \times \prod_{1 \leq j < k \leq s} \frac{z_j - z_k}{1 - 2\Delta z_k + z_j z_k} h_{N,s}(z_1, \ldots, z_s).
\]

2) deform integration contours. Miraculously, poles from double products give vanishing contribution \([FC-Di Giulio-Pronko'21]\). Thus

\[
F_N^{(r,s)} = \oint_{C_1 \cup C_0} \cdots \oint_{C_1 \cup C_0} J_N^{(s)}(z_1, \ldots, z_s) \, d^s z.
\]

that is:

\[
F_N^{(r,s)} = \sum_{k=0}^{s} I_k, \quad I_k := \sum_{|S|=k} \prod_{i \in S} \oint_{C_0} dz_i \prod_{j \in S^c} \oint_{C_1} dz_j J_N^{(r,s)}(z_1, \ldots, z_s)
\]
Two lemmas

Lemma
For arbitrary values of parameters \( r, s, \Delta, \)

\[
l_0 \equiv \res_{z_1=1} \ldots \res_{z_s=1} J_N^{(r,s)}(z_1, \ldots, z_s) = 1.
\]

(Actually holds for generic values of \( t \) as well).

Lemma
At the ice point, \( \Delta = 1/2, t = 1, \) and for \( r = N - s \) (square EFP)

\[
l_s \equiv \res_{z_1=0} \ldots \res_{z_s=0} J_N^{(N-s,s)}(z_1, \ldots, z_s) = (-1)^s h_N \cdots h_{N-s+1},
\]

where \( h_N \equiv h_N(0), \) etc.

Proof is elementary
And when $k \neq 0, s$?

Recall:

$$I_k := \sum_{|S|=k} \prod_{i \in S} \oint_{C_0} \frac{dz_i}{2\pi i} \prod_{j \in S^c} \oint_{C_1} \frac{dz_j}{2\pi i} \prod_{j=1}^{s} \frac{[1-2\Delta + z_j]^{s-j}}{z_j^{N-r}(z_j - 1)^{s-j+1}}$$

$$\times \prod_{1 \leq j < k \leq s} \frac{z_j - z_k}{1 - 2\Delta z_k + z_j z_k} h_{N,s}(z_1, \ldots, z_s)$$

where

$$h_{N,s} := \frac{1}{\Delta_s(z_1, \ldots, z_s)} \det \left[ (z_j - 1)^{k} z_j^{s-k} h_{N-s+k}(z_j) \right]_{j,k=1}^{s}$$
Two types of identities (type I)

\[ h'_{N-1}(1) = \frac{1}{1 - 2\Delta t + t^2} \left\{ \frac{h'_N}{h_N} - t^2 \right\}, \]

\[ h''_{N-2}(1) = \frac{1}{(1 - 2\Delta t + t^2)^2} \left\{ - \frac{h''_N}{h_N} + 2 \frac{h'_{N-1}h'_N}{h_{N-1}h_N} - 2 (1 - 2\Delta t + 2t^2) \frac{h'_{N-1}}{h_{N-1}} \right. \]

\[ \left. + 2 \frac{h'_N}{h_N} - 2t^2 + 2t^4 \right\}, \]

\[ h'''_{N-3}(1) = \frac{1}{(1 - 2\Delta t + t^2)^2} \left\{ \frac{h'''_N}{h_N} - 3 \frac{h'_{N-2}h'_N}{h_{N-2}h_N} - 3 \frac{h'_{N-1}h'_N}{h_{N-1}h_N} \right. \]

\[ \left. + 3 (2 + 3t^2 - 4t\Delta) \frac{h'_{N-1}}{h_{N-1}} - 6 \frac{h'_N}{h_N} + 6 \frac{h'_{N-2}h'_{N-1}h'_N}{h_{N-2}h_{N-1}h_N} \right. \]

\[ \left. - 6 (2 + 3t^2 - 4t\Delta) \frac{h'_{N-2}h'_N}{h_{N-2}h_N} + 6 \frac{h'_{N-2}h'_N}{h_{N-2}h_N} + 6 \frac{h'_{N-1}h'_N}{h_{N-1}h_N} \right. \]

\[ \left. + 6 (1 + 2t^2 + 3t^4 - 4t\Delta - 6t^3\Delta + 4t^2\Delta^2) \frac{h'_{N-2}}{h_{N-2}} \right. \]

\[ \left. - 6 (2 + 3t^2 - 4t\Delta) \frac{h'_{N-1}}{h_{N-1}} + 6 \frac{h'_N}{h_N} + 18t^4 - 6t^6 - 12t^3\Delta \right\}. \]

\[ h''''_{N-4}(1) = \ldots \]
Two types of identities (type I)

\[ h'_{N-1}(1) = \frac{1}{1 - 2\Delta t + t^2} \left\{ \frac{h'_N}{h_N} - t^2 \right\}, \]

\[ h''_{N-2}(1) = \frac{1}{(1 - 2\Delta t + t^2)^2} \left\{ -\frac{h''_N}{h_N} + 2 \frac{h'_{N-1}h'_N}{h_{N-1}h_N} - 2 \left(1 - 2\Delta t + 2t^2\right) \frac{h'_{N-1}}{h_{N-1}} \right. \]
\[ 
+ 2 \frac{h'_N}{h_N} - 2t^2 + 2t^4 \left. \right\} , \]

\[ h'''_{N-3}(1) = \frac{1}{(1 - 2\Delta t + t^2)^2} \left\{ \frac{h'''_N}{h_N} - 3 \frac{h'_{N-2}h'_N}{h_{N-2}h_N} - 3 \frac{h''_{N-1}h'_N}{h_{N-1}h_N} \right. \]
\[ 
- 6 \frac{h'_N}{h_N} + 1 + 2t^2 - 3t^4 - 4t\Delta - 6t^3\Delta + 4t^2\Delta^2 \left. \right\} . \]

- valid for any \( \Delta \) and \( t \)
- follows from availability of different MIR’s for EFP (see [FC-Di Giulio-Pronko’21] for details)
- relate sums over the set of functions \( H_N^{(r)}, \ r = 1, \ldots, N \) to the first few values of them (sum rules identities)
- allow to express the result of integration of our MIRs in terms of the sole value of \( h_N(z) \) (and derivatives) at the origin

\[ h''''_{N-4}(1) = \ldots \]
Two types of identities (type II)

When $\Delta = 1/2$ and $t = 1$ [Zeilberger’96]

$$h_N(z) = \frac{(N)_{N-1}}{(2N)_{N-1}} \, _2F_1 \left( \begin{array}{c} -N + 1, N \\ -2N + 2 \end{array} \bigg| z \right).$$

It is easy to derive:

$$\frac{h'_N}{h_N} - \frac{h'_{N-1}}{h_{N-1}} - \frac{1}{2} = 0$$

$$\frac{h''_N}{h_N} - \frac{h''_{N-1}}{h_{N-1}} - \frac{h'_N}{h_N} - 2 \frac{h_{N-2}}{h_{N-1}} + \frac{7}{2} = 0$$

$$\frac{h'''_N}{h_N} - \frac{h'''_{N-1}}{h_{N-1}} - \frac{3}{2} \left( \frac{h'_N}{h_N} \right)^2 - 2 \frac{1}{2} \left( \frac{h_{N-2}}{h_{N-1}} - \frac{7}{4} \right) = 0$$

$$\frac{h''''_N}{h_N} - \ldots = 0$$
Two types of identities (type II)

When $\Delta = 1/2$ and $t = 1$ [Zeilberger’96]

$$h_N(z) = \frac{(N)_{N-1}}{(2N)_{N-1}} \, _2F_1\left(\begin{array}{c}-N+1, \, N \\ -2N+2 \end{array} \bigg| z\right).$$

It is easy to derive:

$$\frac{h'_N}{h_N} - \frac{h'_{N-1}}{h_{N-1}} - \frac{1}{2} = 0$$

$$\frac{h''_N}{h'_{N-1}} + \frac{h''_{N-1}}{h'_N} + \frac{h''_{N-2}}{h'_{N-2} - \frac{7}{2}} = 0$$

- valid only at ice-point
- follow from standard relation for Gauss hypergeometric functions
- involves only functions $h_N(z)$ and derivatives, evaluated at $z = 0$
- allow to express the result of integration of our MIRs in terms of just $2s - 1$ formally independent objects, $h_{N-s+1}, \ldots, h_N, h'_N, \ldots, h^{(s-1)}_N$
Determinant structure

Inspired by [Tracy-Widom’08] [Saenz–Tracy-Widom’22] we assume that, for each $s$, an $s \times s$ matrix $A = A(N, s)$ exists, such that

$$\sum_{k=0}^{s} \lambda^k I_k = \det_s(I - \lambda A)$$

Clearly, from last lemma, $\det_s A = I_s$, for any $s$.

Below, we shall also observe that

- $A$ is such that by eliminating its last row and column, the reduction $s \mapsto s - 1$, $N \mapsto N - 1$ is made;

- $A$ can be given explicitly in a factorized form $A = DLU$. 
Determinant structure

Inspired by [Tracy-Widom’08] [Saenz-Tracy-Widom’22] we assume that, for each \( s \), an \( s \times s \) matrix \( A = A(N, s) \) exists, such that

\[
\sum_{k=0}^{s} \lambda^k l_k = \text{det}_s(I - \lambda A)
\]

Clearly, from last lemma, \( \text{det}_s A = I_s \), for any \( s \).

Below, we shall also observe that

- \( A \) is such that by eliminating its last row and column, the reduction \( s \mapsto s - 1, \ N \mapsto N - 1 \) is made;
- \( A \) can be given explicitly in a factorized form \( A = DLU \).

To proceed, it is convenient to introduce the abbreviated notations

\[
b_i \equiv h_{N-i}, \quad i = 0, 1, 2, \ldots, s - 1,
\]

and

\[
\kappa_i' = \frac{h'_{N-i}}{h_{N-i}}, \quad \kappa_i'' = \frac{h''_{N-i}}{h_{N-i}}, \quad \kappa_i''' = \frac{h'''_{N-i}}{h_{N-i}}, \quad \ldots
\]

Recall that \( h_N \equiv h_N(0), \ h'_N \equiv h'_N(0), \) etc.
Case $s=1$

\[ F_N^{(N-1,1)} = \int_{C_1 \cup C_0} \frac{1}{z(z-1)} h_N(z) \frac{dz}{2\pi i} \]

\[ = h_n(1) - h_N(0) \]

\[ = 1 - b_0 \]

That is

\[ l_0 = 1 \quad l_1 = -b_0 \]

as we already knew from our two lemmas.

We are looking for $1 \times 1$ matrix $A$ such that

\[ \det_1(1 - A) = 1 - b_0 \]

Thus:

\[ A = b_0 \]
Case $s=2$

\[
\begin{align*}
I_0 &= 1 \\
I_1 &= -b_0 k'_0 - b_1 - b_0 k'_0 h'_{N-1}(1) = -\text{tr} \, A \\
I_2 &= b_0 b_1 = \det A
\end{align*}
\]

Use first identity of type I, namely $h'_{N-1}(1) = k'_0 - 1$, and get

\[
\begin{align*}
I_0 &= 1 \\
I_1 &= -b_1 - b_0 (k'_0)^2 = -\text{tr} \, A \\
I_2 &= b_0 b_1 = \det A
\end{align*}
\]

If $2 \times 2$ matrix $A$ exists, it must be such that when $b_0 = 0$ its top-left entries is $b_1$. Thus

\[
A = \begin{pmatrix}
b_1 & b_1(k'_0 - 1) \\
b_0(k'_0 + 1) & b_0(k'_0)^2
\end{pmatrix}
\]

with DLU factorization:

\[
D = \begin{pmatrix} b_1 & 0 \\ 0 & b_0 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ k'_0 + 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & k'_0 - 1 \\ 0 & 1 \end{pmatrix}.
\]
Case $s=3$

\[ l_0 = 1 \]

\[ l_1 = -b_2 - b_1 (\kappa'_1)^2 - b_0 \left[ \left( \frac{\kappa''_0}{2} - \kappa'_0 \right)^2 + 2\kappa'_0 - 1 \right] = -\text{tr} \, A \]

\[ l_2 = b_1 b_2 + b_0 b_2 (\kappa'_0)^2 + b_0 b_1 \left[ 1 - \kappa'_0 (1 + \kappa'_1) + \frac{\kappa''_0}{2} \right]^2 = \frac{1}{2} [(\text{tr} \, A)^2 - \text{tr} \, A^2] \]

\[ l_3 = -b_0 b_1 b_2 = \det A \]

NB1: here first two identities of type I have been used

NB2: setting $b_0 = 0$ one recover the $s = 2$ case, modulo the replacement $b_0, b_1, k'_0 \mapsto b_1, b_2, k'_1$, that is $N \mapsto N - 1$.

We may thus obtaine the top-left $2 \times 2$ block from the $s = 2$ case.

Completing the sudoku, we get $A = DLU$, with

\[
D = \begin{pmatrix}
 b_2 & 0 & 0 \\
 0 & b_1 & 0 \\
 0 & 0 & b_0
\end{pmatrix}, \quad L = \begin{pmatrix}
 1 & 0 & 0 \\
 \kappa'_1 + 1 & 1 & 0 \\
 \frac{1}{2} \kappa''_0 - 2\kappa'_0 + 1 & \kappa'_0 - 1 & 1
\end{pmatrix}, \quad U = \begin{pmatrix}
 1 & \kappa'_1 - 1 & \frac{1}{2} \kappa''_0 - 1 \\
 0 & 1 & \kappa'_0 + 1 \\
 0 & 0 & 1
\end{pmatrix}
\]

NB3: $l_0, l_1, l_3$ are easily reproduced. But $l_2$ is only recovered modulo a term proportional to $k'_1 - k'_0 + \frac{1}{2}$, which however vanish, due to first identity of type II!
Case s=4

\[ D = \text{diag}(b_3, b_2, b_1, b_0), \]

\[ L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \kappa'_2 + 1 & 1 & 0 & 0 \\ \frac{1}{2} \kappa''_1 - 2 \kappa'_1 + 1 & \kappa'_1 - 1 & 1 & 0 \\ \frac{1}{6} \kappa'''_0 - \frac{1}{2} \kappa''_0 - \kappa'_0 + 1 & \frac{1}{2} \kappa''_0 - 1 & \kappa'_0 + 1 & 1 \end{pmatrix}, \]

\[ U = \begin{pmatrix} 1 & \kappa'_2 - 1 & \frac{1}{2} \kappa''_1 - 1 & \frac{1}{6} \kappa'''_0 - \frac{3}{2} \kappa''_0 + 3 \kappa'_0 - 1 \\ 0 & 1 & \kappa'_1 + 1 & \frac{1}{2} \kappa''_0 - 2 \kappa'_0 + 1 \\ 0 & 0 & 1 & \kappa'_0 - 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \]
Case $s=4$

\[ D = \text{diag}(b_3, b_2, b_1, b_0), \]

\[ L = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\kappa_2' + 1 & 1 & 0 & 0 \\
\frac{1}{2} \kappa_1'' - 2 \kappa_1' + 1 & \kappa_1' - 1 & 1 & 0 \\
\frac{1}{6} \kappa_0''' - \frac{1}{2} \kappa_0'' - \kappa_0' + 1 & \frac{1}{2} \kappa_0'' - 1 & \kappa_0' + 1 & 1
\end{pmatrix}, \]

\[ U = \begin{pmatrix}
1 & \kappa_2' - 1 & \frac{1}{2} \kappa_1'' - 1 & \frac{1}{6} \kappa_0''' - \frac{3}{2} \kappa_0'' + 3 \kappa_0' - 1 \\
0 & 1 & \kappa_1' + 1 & \frac{1}{2} \kappa_0'' - 2 \kappa_0' + 1 \\
0 & 0 & 1 & \kappa_0' - 1 \\
0 & 0 & 0 & 1
\end{pmatrix}. \]
Case $s=4$

\[ D = \text{diag}(b_3, b_2, b_1, b_0), \]

\[ L = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \kappa'_2 + 1 & 0 & 0 \\
\kappa'_2 - 2\kappa'_1 + 1 & 0 & \kappa'_1 - 1 & 0 \\
\frac{1}{6}\kappa'''' - \frac{1}{2}\kappa'' - \kappa'_0 + 1 & \frac{1}{2}\kappa'' - 1 & \kappa'_0 + 1 & 1
\end{pmatrix}, \]

\[ U = \begin{pmatrix}
1 & \kappa'_2 - 1 & \frac{1}{2}\kappa'' - 1 & \frac{1}{6}\kappa'''' - \frac{3}{2}\kappa'' + 3\kappa'_0 - 1 \\
0 & 1 & \kappa'_1 + 1 & \frac{1}{2}\kappa'' - 2\kappa'_0 + 1 \\
0 & 0 & 1 & \kappa'_0 - 1 \\
0 & 0 & 0 & 1
\end{pmatrix}. \]

\[ L_0(x) = 1 \]
\[ L_1(x) = -x + 1, \]
\[ L_2(x) = \frac{x^2}{2} - 2x + 1 \]
\[ L_3(x) = -\frac{x^3}{6} + \frac{3x^2}{2} - 3x + 1 \]
Case $s=4$

$$D = \text{diag}(b_3, b_2, b_1, b_0),$$

$$L = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\kappa'_2 + 1 & 1 & 0 & 0 \\
\frac{1}{2}\kappa''_1 - 2\kappa'_1 + 1 & \kappa'_1 - 1 & 1 & 0 \\
\frac{1}{6}\kappa'''_0 - \frac{1}{2}\kappa''_0 - \kappa'_0 + 1 & \frac{1}{2}\kappa''_0 - 1 & \kappa'_0 + 1 & 1
\end{pmatrix},$$

$$U = \begin{pmatrix}
1 & \kappa'_2 - 1 & \frac{1}{2}\kappa''_1 - 1 & \frac{1}{6}\kappa'''_0 - \frac{3}{2}\kappa''_0 + 3\kappa'_0 - 1 \\
0 & 1 & \kappa'_1 + 1 & \frac{1}{2}\kappa''_0 - 2\kappa'_0 + 1 \\
0 & 0 & 1 & \kappa'_0 - 1 \\
0 & 0 & 0 & 1
\end{pmatrix}. $$

$$L_0^{(-1)}(x) - L_0^{(0)}(x) = 1$$
$$L_2^{(-1)}(x) - L_2^{(0)}(x) = \frac{x^2}{2} - 1$$
$$L_1^{(-1)}(x) - L_0^{(0)}(x) = -x - 1$$
$$L_3^{(-1)}(x) - L_2^{(0)}(x) = -\frac{x^3}{6} + \frac{x^2}{2} + x - 1$$
Case $s=4$

\[ D = \text{diag}(b_3, b_2, b_1, b_0), \]

\[ L = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\kappa_2' + 1 & 1 & 0 & 0 \\
\frac{1}{2} \kappa_1'' - 2 \kappa_1' + 1 & \kappa_1' - 1 & 1 & 0 \\
\frac{1}{6} \kappa_0''' - \frac{1}{2} \kappa_0'' - \kappa_0' + 1 & \frac{1}{2} \kappa_0'' - 1 & \kappa_0' + 1 & 1
\end{pmatrix}, \]

\[ U = \begin{pmatrix}
1 & \kappa_2' - 1 & \frac{1}{2} \kappa_1'' - 1 & \frac{1}{6} \kappa_0''' - \frac{3}{2} \kappa_0'' + 3 \kappa_0' - 1 \\
0 & 1 & \kappa_1' + 1 & \frac{1}{2} \kappa_0'' - 2 \kappa_0' + 1 \\
0 & 0 & 1 & \kappa_0' - 1 \\
0 & 0 & 0 & 1
\end{pmatrix}. \]

\[ L_0^{(-1)}(x) + L_{-1}^{(0)}(x) = 1 \quad L_1^{(-1)}(x) + L_0^{(0)}(x) = -x + 1 \]

\[ L_2^{(-1)}(x) + L_1^{(0)}(x) = \frac{x^2}{2} - 2x + 1 \quad L_3^{(-1)}(x) + L_2^{(0)}(x) = -\frac{x^3}{6} + \frac{3x^2}{2} - 3x + 1 \]
The conjecture

For $t = 1$ and $\Delta = 1/2$, and for $r = N - s$, the EFP can be given as $\det_s(I - A)$ where the $s \times s$ matrix $A$ is given as $A = DLU$ and

$$D_{ij} = h_{r+i}(0) \delta_{ij}$$

$$L_{ij} = \frac{(-1)^{i-j}}{h_{r+i}(0)} \left[ L_{i-j}^{(-1)}(\partial z) + (-1)^{i-1} L_{i-j-1}^{(0)}(\partial z) \right] h_{r+i}(z) \bigg|_{z=0}$$

$$U_{ij} = \frac{(-1)^{i-j}}{h_{r+j}(0)} \left[ L_{j-i}^{(-1)}(\partial z) + (-1)^{j} L_{j-i-1}^{(0)}(\partial z) \right] h_{r+j}(z) \bigg|_{z=0}$$

where functions $h_j(z)$ are the Gauss hypergeometric functions given above.
The conjecture

For \( t = 1 \) and \( \Delta = 1/2 \), and for \( r = N - s \), the EFP can be given as \( \text{det}_s(I - A) \) where the \( s \times s \) matrix \( A \) is given as \( A = DLU \) and

\[
D_{ij} = h_{r+i}(0) \delta_{ij}
\]

\[
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\]

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\]

where functions \( h_j(z) \) are the Gauss hypergeometric functions given above.

- note that dependence on parameter \( s \) is both via the size of the matrix, and the parameter \( r = N - s \)
- Appearance of Laguerre polynomials does not come as a surprise, if one recalls relations such as

\[
\int_{C_0} \frac{(1 - z)^{n+\alpha}}{z^{n+1}} f(z) \frac{dz}{2\pi i} = (-1)^n L_n^{(\alpha)}(\partial_z)f(z) \bigg|_{z=0}
\]
The conjecture

For $t = 1$ and $\Delta = 1/2$, and for $r = N - s$, the original MIR for EFP,

$$F_{N}^{(N-s,s)} = (-1)^{s} \oint_{C_{0}} \cdots \oint_{C_{0}} \prod_{j=1}^{s} \frac{1}{z_{j}^{N-s}(z_{j} - 1)^{s-j+1}}$$

$$\times \prod_{1 \leq j < k \leq s} \frac{z_{j} - z_{k}}{z_{j}z_{k} - z_{j} + 1} h_{N,s}(z_{1}, \ldots, z_{s}) \frac{d^{s}z}{(2\pi i)^{s}}$$

can be given as $\det_{s}(I - A)$ where the $s \times s$ matrix $A = A(N, s)$ reads

$$A_{ij} = \oint_{C_{0}} \oint_{C_{0}} \frac{e_{i}^{L}(z)e_{j}^{U}(w)}{1 - z - w} \frac{dzdw}{(2\pi i)^{2}}, \quad i, j = 1, \ldots, s, \quad (*)$$

with

$$e_{i}^{L}(z) := \frac{(1 - z)^{i-1}}{z^{i}} \left(1 + (-1)^{i} z\right) h_{r+i}(z),$$

$$e_{j}^{U}(w) := \frac{(1 - w)^{j-1}}{h_{r+j}(0)w^{j}} \left(1 + (-1)^{j+1} w\right) h_{r+j}(w).$$
The conjecture

- crucial in this derivation were our two sets of identities;
- and also our ansatz, fixing at step $s$, all entries of an $(s - 1) \times (s - 1)$ sub-block of $A$, so that $s$ new conditions at each step were sufficient;
- however nice is the result, it is still just a guess;
- unable to proceed with our calculation beyond $s = 4$;
- desperately seeking a proof.

\[ A_{ij} = \oint_{C_0} \oint_{C_0} \frac{e_i^L(z)e_j^U(w)}{1 - z - w} \frac{dzdw}{(2\pi i)^2}, \quad i,j = 1, \ldots, s, \quad (*) \]

with

\[ e_i^L(z) := \frac{(1 - z)^{i-1}}{z^i} (1 + (-1)^i z) h_{r+i}(z), \]
\[ e_j^U(w) := \frac{(1 - w)^{j-1}}{h_{r+j}(0)w^j} (1 + (-1)^{j+1} w) h_{r+j}(w). \]
Check the \( s = 5 \) case: evaluate with Mathematica both our conjectural expression and the MIR, for \( N = 7, \ldots, 13 \):

<table>
<thead>
<tr>
<th>( N )</th>
<th>Determinant</th>
<th>MIR</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>( \frac{61347}{43178090900} )</td>
<td>( \frac{61347}{43178090900} )</td>
</tr>
<tr>
<td>11</td>
<td>( \frac{49711519}{1636618150125} )</td>
<td>( \frac{49711519}{1636618150125} )</td>
</tr>
<tr>
<td>12</td>
<td>( \frac{54886057499}{221251085257500} )</td>
<td>( \frac{54886057499}{221251085257500} )</td>
</tr>
<tr>
<td>13</td>
<td>( \frac{3870965779057}{3266307568354500} )</td>
<td>( \frac{3870965779057}{3266307568354500} )</td>
</tr>
</tbody>
</table>
Integral form for matrix $A$

As said, the matrix $A$ admits the following integral representation

$$A_{ij} = \oint_{C_0} \oint_{C_0} \frac{e_i^L(z)e_j^U(w)}{1 - z - w} \frac{dzdw}{(2\pi i)^2}, \quad i, j \in \{1, \ldots, s\},$$

where

$$e_i^L(z) := \frac{(1 - z)^{i-1}}{z^i} \left(1 + (-1)^i z\right) h_{r+i}(z),$$

$$e_j^U(w) := \frac{(1 - w)^{j-1}}{h_{r+j}(0)w^j} \left(1 + (-1)^{j+1} w\right) h_{r+j}(w).$$

Or, equivalently,

$$A_{ij} = \oint_{C_0} \oint_{C_0} e_i^L(z)e_j^U(w) \int_0^\infty e^{(z+w-1)t} dt \frac{dzdw}{(2\pi i)^2}, \quad \text{Re} (z + w) < 1$$
Fredholm determinant

Let \( \hat{K}_{[0, \infty)} \) be a linear integral operator acting on functions defined on \( \mathbb{R}^+ \) according to the rule

\[
(\hat{K}_{[0, \infty)} f)(t_1) = \int_0^\infty K(t_1, t_2)f(t_2)\,dt_2
\]

with kernel

\[
K(t_1, t_2) = \oint_{C_0} \oint_{C_0} e^{(z-\frac{1}{2})t_1 + (w-\frac{1}{2})t_2} \sum_{j=1}^s e_j^L(z)e_j^U(w)\frac{dz\,dw}{(2\pi i)^2}.
\]

Proposition

Given matrix \( A = A(N,s) \) as in (\( \ast \)), for any finite integer \( s \), we have

\[
\det_s(I - A) = \det (1 - \hat{K}_{[0, \infty)}).
\]

Remark

The kernel \( K(t_1, t_2) \) is not ‘of integrable form’ (in the sense of \( \text{[Its-Izergin-Korepin-Slaunov'92]} \)).
Scaling limit

We want to study the behaviour of the kernel $K(t_1, t_2)$ in the scaling limit, i.e. (recall that $r = N - s$)

$$s = \lceil yN \rceil, \quad y \in (0, 1/2], \quad N \to \infty$$
Scaling limit

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$$y_c := 1 - \frac{\sqrt{3}}{2}$$
Scaling limit

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\[
s = \lceil yN \rceil, \quad y \in (0, 1/2], \quad N \to \infty
\]

In this limit

\[
K(t_1, t_2) \sim \oint_{C_0} \oint_{C_0} e^{(z - \frac{1}{2})t_1 + (w - \frac{1}{2})t_2} e^{N[g(w) + g(z)]} f(z, w) \frac{dz dw}{(2\pi i)^2}
\]

where

\[
g(w) := y \log \frac{1 - w}{w} + \log \frac{(1 - 2w)(2 - w)(1 + w) + 2 (1 - w + w^2)^{3/2}}{3\sqrt{3}(1 - w)^2}
\]

while \( f(z, w) \) is some complicate but explicit function.
Saddle points

Saddle-point equation

\[ g'(w) = \frac{y}{w(w-1)} - \frac{1 - \sqrt{1 - w + w^2}}{w(w-1)} = 0 \]

has two solutions

\[ w_\pm = \frac{1 \pm \sqrt{1 - 8y + 4y^2}}{2} \]

which collide when \( y = y_c := 1 - \frac{\sqrt{3}}{2} \), recall, \( y \in (0, \frac{1}{2}] \).
Saddle points

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\[ w_{\pm} = \frac{1 \pm \sqrt{1 - 8y + 4y^2}}{2} \]

which collide when \( y = y_c := 1 - \frac{\sqrt{3}}{2} \), recall, \( y \in (0, \frac{1}{2}] \).

- \( y_c \) happens to correspond to the intersection of the arctic curve with the main diagonal
- for values \( y \in (0, y_c) \) i.e. outside the arctic curve (frozen region) \( w_{\pm} \) are both real, with an exponential decay of the integrals, ruled by \( w_- \)
- for values \( y \in (y_c, 1/2) \), i.e. inside the arctic curve (disordered region) \( w_{\pm} \) are complex conjugate, and contribute both to the integrals, producing an oscillatory behaviour in analogy with dimer models [Kenyon-Okounkov-Sheffield’06]
Let us study $K(t_1, t_2)$ in the vicinity of $y = y_c$.

Let $y = y_c - \eta$, and $w = \frac{1}{2} + \lambda$, with $\eta, \lambda$ small. We have

$$g(w)\big|_{w=\frac{1}{2}+\lambda} = 4\eta\lambda - \frac{4}{3\sqrt{3}}\lambda^3 + O(\lambda^4)$$

which sets the scales $\lambda = O(N^{-1/3}), \eta = O(N^{-2/3})$.

and similarly for $z = \frac{1}{2} + \mu$, with $\mu = O(N^{-1/3})$. 
We now rescale

\[ \tilde{\lambda} = q\lambda, \quad \tilde{\mu} = q\mu, \quad q = \frac{2^{2/3}}{3^{1/6}} N^{1/3}, \]

and

\[ \sigma = \frac{4N}{q} \eta = 2^{4/3} 3^{1/6} N^{2/3} \eta. \]

where \( \tilde{\lambda}, \tilde{\mu}, \) and \( \sigma \) are \( O(N^0) \).

We also rescale the variables \( t_1 \) and \( t_2 \) and the kernel itself

\[ \tilde{K}(t_1, t_2) := q K(q t_1, q t_2), \quad q > 0, \quad t_1, t_2 \in [0, \infty) \]

obtaining

\[ \tilde{K}(t_1, t_2) = - \int_{\tilde{\gamma}} \int_{\tilde{\gamma}} \frac{e^{\tilde{\mu} t_1 + \tilde{\lambda} t_2 + \sigma(\tilde{\lambda} + \tilde{\mu}) - (\tilde{\lambda}^3 + \tilde{\mu}^3)/3}}{\tilde{\lambda} + \tilde{\mu}} \frac{d\tilde{\lambda} d\tilde{\mu}}{(2\pi i)^2}. \]
Summing up

\[
\lim_{N \to \infty} \left. \left( \det_s (1 - A) \right) \right|_{s=N(1 - \frac{\sqrt{3}}{2}) - \frac{N^{1/3}}{2^{4/3} 3^{1/6}} \sigma} = \det \left( 1 - \hat{K}_{[0, \infty)} \right)
\]

with kernel

\[
\hat{K}(t_1, t_2) = - \int \int_{\tilde{\gamma}} \int \int \frac{e^{\tilde{\mu} t_1 + \tilde{\lambda} t_2 + \sigma (\tilde{\lambda} + \tilde{\mu}) - (\tilde{\lambda}^3 + \tilde{\mu}^3)/3}}{\tilde{\lambda} + \tilde{\mu}} \frac{d\tilde{\lambda} d\tilde{\mu}}{(2\pi i)^2}.
\]
Summing up

\[
\lim_{N \to \infty} \left( \frac{\det_s(1 - A)}{N} \right) \bigg|_{s=N(1-\frac{\sqrt{3}}{2})} = \det \left( 1 - \hat{K}_{[0,\infty)} \right)
\]

with kernel

\[
\tilde{K}(t_1, t_2) = - \int_{\tilde{\gamma}} \int_{\tilde{\gamma}} \frac{e^{\tilde{\mu}t_1 + \tilde{\lambda}t_2 + \sigma(\tilde{\lambda} + \tilde{\mu}) - (\tilde{\lambda}^3 + \tilde{\mu}^3)/3}}{\tilde{\lambda} + \tilde{\mu}} \frac{d\tilde{\lambda} d\tilde{\mu}}{(2\pi i)^2}.
\]

**Proposition**

Let \( \hat{K}^{\text{Ai}} \) the linear integral operator on the real line, with kernel

\[
K^{\text{Ai}}(t_1, t_2) = \frac{\text{Ai}(t_1) \text{Ai}'(t_2) - \text{Ai}'(t_1) \text{Ai}(t_2)}{t_1 - t_2}.
\]

One has

\[
\det \left( 1 - \hat{K}_{[0,\infty)} \right) = \det \left( 1 - \hat{K}^{\text{Ai}}_{[\sigma,\infty)} \right) =: F_2(\sigma),
\]
Conclusions

Conjecture
At ice point, $\Delta = \frac{1}{2}$, $t = 1$, the following holds

$$F_N^{(N-s,s)} = \det_s(1 - A)$$

where $A = A(N,s)$ is the $s \times s$ matrix given in $(\ast)$. 

Theorem
Given the $s \times s$ matrix $A = A(N,s)$, see $(\ast)$, the following holds

$$\lim_{N \to \infty} \left( \det_s(1 - A) \bigg|_{s = N \left( 1 - \frac{\sqrt{3}}{2} \right) - \frac{N^{1/3}}{2^{4/3}3^{1/6}2}} \right) = \mathcal{F}_2(\sigma).$$

The presented result is in full agreement with the conjecture in [Ayyer-Chhita-Johansson’23] and with the numerical simulations in [Korepin-Lyberg-Viti’23][Prauhofer-Spohn’24].