The Radon–Nikodym topography of an amenable equivalence relation in an acyclic graph

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The setting

- (X, μ) denotes a standard probability space.
- E is a countable Borel equivalence relation (cBer) on X, i.e., each E-class is countable, and $E \subseteq X \times X$ is Borel measurable.
- A treeing of E is a Borel graph T on X that is acyclic, and whose connected component equivalence relation, E_T, coincides with E.
- E is called treeable if it admits a treeing.

Example (Boundary of the free group)

$X = \partial F_2 = \{$(right-)infinite reduced words in $a, a^{-1}, b, b^{-1}\}$
$\mu = \mu_{\partial F_2} =$ hitting measure for the simple random walk on F_2.
The action $F_2 \curvearrowright \partial F_2$ (via concatenate then reduce) gives rise to a cBer:

$$E_{\partial F_2} = \{(x, y) \in \partial F_2 \times \partial F_2 \,:\, F_2.x = F_2.y\}$$

A treeing $T_{\partial F_2}$ of $E_{\partial F_2}$ is given by connecting each $x \in \partial F_2$ to its shift $S(x)$, i.e., $S(x)$ is obtained by "chopping off" the first letter of x.

The Radon–Nikodym cocycle

On E we have the left and right fiber measures, μ_ℓ^E and μ_r^E:

$$\int_E h\, d\mu_\ell^E = \int_X \sum_{y\in[x]_E} h(x,y)\; d\mu(x)$$
$$\int_E h\, d\mu_r^E = \int_X \sum_{y\in[x]_E} h(y,x)\; d\mu(x).$$

E is called measure class preserving (mcp) if μ_ℓ^E and μ_r^E are equivalent.
The function $\rho := \frac{d\mu_\ell^E}{d\mu_r^E}$ is called the Radon–Nikodym cocycle of (E, X, μ).
It satisfies $\rho(z,x) = \rho(z,y)\rho(y,x)$ for all E-related points $x, y, z \in X$.
Interpretation: $\rho(y,x) = \rho^x(y) =$ "the weight of y from x's perspective."
The mass transport principle: for all Borel functions $h : E \to [0,\infty)$

$$\int_X \sum_{y\in[x]_E} h(x,y)\; d\mu(x) = \int_X \sum_{y\in[x]_E} h(y,x)\rho^x(y)\; d\mu(x)$$

Example (Boundary of F_2)

One computes that $\rho^x(S(x)) = 3$ for all $x \in \partial F_2$. In general, if $S^m(x) = S^n(y)$ then $\rho^x(y) = 3^{m-n}$.

Amenability

Definition

An mcp cBer E on (X, μ) is called amenable if there exists a measurable way of assigning to each $x \in X$ a sequence $(\lambda_n^x)_{n=0}^{\infty}$ of probability measures on $[x]_E$ such that $\|\lambda_n^x - \lambda_n^y\|_1 \xrightarrow{n \to \infty} 0$ for a.e. $(x, y) \in E$.

The assignment being measurable means that $(y, x) \mapsto \lambda_n^x(y)$ is Borel measurable for each n.

Example (Boundary of F_2)

$E_{\partial F_2}$ is amenable: for each $x \in \partial F_2$ and $n \geq 0$, take λ_n^x be the uniform distribution on $\{x, S(x), S^2(x), \ldots, S^n(x)\}$.

The Adams Dichotomy

Theorem (S. Adams)

Let E be an ergodic cBer on (X, μ) that is measure preserving (i.e., $\rho \equiv 1$), and treed by T.

1. *If E is amenable then a.e. T-component has at most two ends.*
2. *If E is nonamenable then the space of ends of a.e. T-component is nonempty and perfect.*

As stated, this fails miserably in the general mcp setting: $E_{\partial F_2}$ is amenable, but the treeing $T_{\partial F_2}$ is four-regular!

Vanishing and Nonvanishing Ends

Let E be an mcp cBer on (X, μ) that is treed by T.
For each T-component $[x]_E$, we equip its end completion $\overline{[x]_E} := [x]_E \cup \partial_T [x]_E$ with the topology generated by all half-spaces.

- An end $\xi \in \partial_T [x]_E$ is called vanishing if

$$\lim_{y \to \xi} \rho^x(y) = 0,$$

  where the limit is over $y \in [x]_E$. Otherwise ξ is called nonvanishing.
- These notions do not depend on the choice of representative x, i.e., if $[z]_E = [x]_E$ then $\lim_{y \to \xi} \rho^z(y) = 0$ iff $\lim_{y \to \xi} \rho^x(y) = 0$.

In the measure preserving setting, where $\rho \equiv 1$, **every** end is nonvanishing.

Example (Boundary of free group)

For each $x \in \partial F_2$, the "forward end" $\xi_{[x]} := \lim_{n \to \infty} S^n(x)$ is nonvanishing since $\rho^x(S^n(x)) = 3^n$. Every other end of this component is vanishing.

Generalized Adams Dichotomy

Theorem (Tserunyan, Tucker-Drob)

Let E be an ergodic mcp cBer on (X, μ) that is treed by T.

1. *If E is amenable, then a.e. T component has at most two nonvanishing ends.*
2. *If E is nonamenable then the space of nonvanishing ends of a.e. T-component is nonempty and perfect.*

Theorem (Tserunyan, Tucker-Drob)

Let E be an ergodic mcp cBer on (X, μ) that is treed by T. Then for a.e. $x \in X$, every vanishing end $\eta \in \partial_T [x]_E$ is geodesically RN-finite, i.e.

$$\sum_{y \in [x, \eta)_T} \rho^x(y) < \infty.$$

The Paddle-ball Lemma

The crux of the amenability half of the proof comes down to the case where the treeing T is generated by a single function f (like how $T_{\partial F_2}$ is generated by S).
For this, we use a (apparently new) combinatorial lemma:

Paddle-ball Lemma

Let T be a tree on a vertex set V with a distinguished end $\xi^+ \in \partial_T V$. Suppose that $\Sigma \subseteq \mathrm{Sym}(V)$ is a collection of permutations of V satisfying, for every $v \in V$:

(i) $v \in (\sigma^{-1}(v), \xi^+)_T$ for each $\sigma \in \Sigma$;
(ii) $v \in (\sigma^{-1}(v), \tau^{-1}(v))_T$ for all distinct $\sigma, \tau \in \Sigma$.

Then Σ freely generates a free subgroup $\langle \Sigma \rangle$ of $\mathrm{Sym}(V)$, and the action of $\langle \Sigma \rangle$ on V is free.

Maximal amenable subrelations

Our methods allow us to prove structural results for treeable equivalence relations that were previously only known to hold in the measure preserving setting.

An mcp cBer E is called (a.e.) smooth if $\rho^x([x]_E) < \infty$ almost surely, and nowhere smooth if $\rho^x([x]_E) = \infty$ almost surely.

Theorem (Adams (mp), Tserunyan-Tucker-Drob (mcp))

Let E be an mcp, treeable cBer on (X, μ) that is ergodic and nonamenable.

1. *Each nowhere smooth amenable subrelation of E is contained in a unique maximal amenable subrelation of E.*
2. *Every maximal amenable subrelation F of E is almost malnormal in E, i.e., if $\phi : X \to X$ is such that $(x, \phi(x)) \in E - F$, then $(\phi \times \phi)(F) \cap F$ is a.e. smooth.*

Thank you!