Sharp threshold for rigidity of random graphs

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Observation: Global Rigidity $\implies$ min-degree at least $d + 1$

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Basic question: **Is a given framework rigid?**

- Cauchy's rigidity theorem (1813), Maxwell's criterion (1864).
- A hard problem in general (may depend on φ).
  E.g., *Abbot '08:* coNP-hard for $d = 2$.
- *Asimow and Roth '79:* For a generic φ, rigidity is a simpler linear-algebraic property of G.

Infinitesimal Rigidity (Asimow-Roth)

The rigidity matrix R of a framework (G, φ) is a $d|V| \times |E|$ matrix:

$$R(x, xy) = \varphi(x) - \varphi(y) \quad , \quad R(y, xy) = \varphi(y) - \varphi(x)$$

A graph is d-rigid if $\operatorname{rank}(R) = d|V| - \binom{d+1}{2}$ for a generic φ.

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$d \ge 3$:

- Király, Theran, and Tomioka ; Jordán and Tanigawa: $p_c = O(\log n/n)$.

Main results

Theorem

Let $\left(G(n,M) : 0 \leq M \leq \binom{n}{2}\right)$. *Then, a.a.s.,*

1. $\tau(d\text{-}\mathit{rigidity}) = \tau(\mathit{min\text{-}degree}\ d)$, *and consequently,*
2. $\tau(d\text{-}\mathit{global\text{-}rigidity}) = \tau(\mathit{min\text{-}degree}\ d+1)$.

Remarks:

- "$\geq$" holds deterministically.
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Remarks:

- "$\geq$" holds deterministically.
- Tanigawa, Jordán: A $(d+1)$-rigid graph is d-globally-rigid.
- $p_c(d\text{-rigidity}) = (\log n + (d-1)\log\log n + O(1))/n$.
- $p_c(d\text{-global-rigidity}) = (\log n + d\log\log n + O(1))/n$.
- Applies to all abstract rigidity matroids.

Proof idea 1: Large closure

- Key definition: (Matroid closure)

$$C_d(G) = \left\{ xy \in \binom{V}{2} \;:\; R(\cdot, xy) \in \mathrm{column-space}(R(G)) \right\},$$

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- Observation: in the process $G(n, M),\ 0 \leq M \leq \binom{n}{2}$,

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- Coupling: Use independent $U(0,1)$'s to decide if $(!)$ occurs:

$$\mathbb{P}\left(|C_d(G(n,M))| \le \alpha \binom{n}{2} \right) \le \mathbb{P}\left(\mathrm{Bin}(M, 1-\alpha) \le dn - \binom{d+1}{2} \right).$$

Proof idea 2: From Large closure to rigidity

Set $M := \tau(\text{min-deg} = d) = \frac{1}{2}(n\log n + (d-1)\log\log n + O_P(1))$.

1. $|C_d(G(n,M))| = (1 - o(1))\binom{n}{2}$.
2. $C_d(G(n,M))$ is K_{d+2}^--free.
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Lemma: A.a.s, the only graph satisfying 1,2,3 is

$$C_d(G(n, M)) = K_n.$$

Open problems:

- Emergence of a giant d-rigid component in $G(n, p)$?
  (Known: $p_c = O_d(1/n)$, precise constant for $d = 1, 2$)
- Higher dimensions: Prove that $G(n, 1/2)$ is rigid in $\mathbb{R}^{cn}$?
  (Krivelevich, Michaeli, Lew conjecture: $c \approx 1 - \sqrt{1 - 1/2}$).
- When is a random k-regular graph a.a.s d-rigid?
  (Plausible conjecture $\forall d \geq 2$: iff $k \geq 2d$. Known for $d = 2$.)