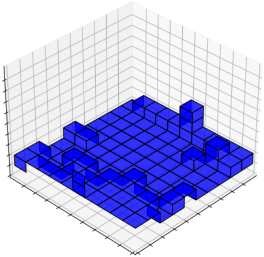
Homological Percolation on a Torus

Paul Duncan (joint with Matt Kahle and Ben Schweinhart)

May 7, 2024



Bond Percolation

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Let G be a finite graph. Bernoulli bond percolation with parameter p is the random subgraph obtained by including each edge of G independently at random with probability p, so the probability of a given subgraph is given by

$$\mu_p(H) = p^{|E(H)|}(1-p)^{|E(G)|-|E(H)|}.$$

The Random-Cluster Model

The random-cluster model is a dependent generalization that adds additional weight to the subgraphs depending on $c(H)$, the number of connected components of H, yielding

$$\mu_{p,q}(H) = \frac{1}{Z} p^{|E(H)|} (1-p)^{|E(G)|-|E(H)|} q^{c(H)} .$$

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The random-cluster model is related to the Ising/Potts models of magnetism, and can be extended to infinite graphs via limits of finite graphs, the main graph of interest being $\mathbb{Z}^d$.

Plaquette Percolation

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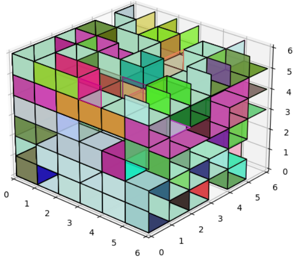
We can think of this as the 2-dimensional generalization of bond percolation, since edges are the 1-cells of the cubical complex structure on $\mathbb{Z}^3$.



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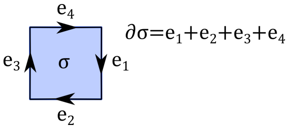
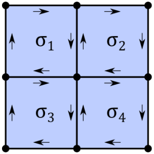
The boundary of an oriented plaquette is an sum of the oriented edges that it contains.

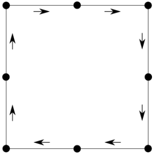


Homology

The boundary of a sum of cells is the formal sum of the boundaries.



$$\partial(\sigma_1+\sigma_2+\sigma_3+\sigma_4) =$$



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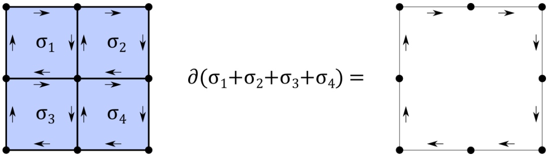

The $(i-1)$-dimensional boundary group of a percolation subcomplex P with coefficients in a group G, written $B_{i-1}(P;\ G)$ is the group of G -linear sums of $(i-1)$-cells that are boundaries of sums of i-plaquettes.

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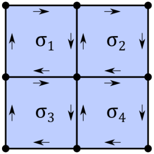

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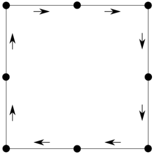

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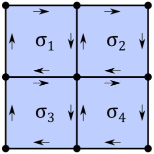

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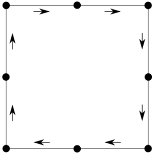

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The $(i-1)$-dimensional homology group is then defined as $H_{i-1}(P;\ G) := Z_{i-1}(P;\ G)/B_{i-1}(P;\ G)$.

The Plaquette Random-Cluster Model

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We therefore define the i-dimensional plaquette random-cluster model with parameter $q \in \mathbb{N}$ on a finite subcomplex $X \subset \mathbb{Z}^d$ by

$$\mu_{p,q,i}(P) = \frac{1}{Z} p^{|P|} (1-p)^{\left|X^i\right| - |P|} \left|H_{i-1}(P;\ \mathbb{Z}_q)\right| ,$$

where $|P|$ is the number of i-plaquettes in P and $\left|X^i\right|$ is the number of i-cells in X.

The Phase Transition

The study of classical percolation in $\mathbb{Z}^d$ centers around the phase transition for the appearance of an infinite component at the critical probability

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It is not clear how best to generalize the infinite component. However, better topological tools are available in compact spaces.

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Instead of $\mathbb{Z}^d$, consider a compact space with the same local structure, namely the torus $\mathbb{T}^d_N$ of dimension d and diameter N.

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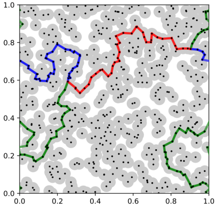
A "global" loop in the torus $\mathbb{T}_d^N$ for large N is a natural analogue of an infinite path in $\mathbb{Z}^d$.



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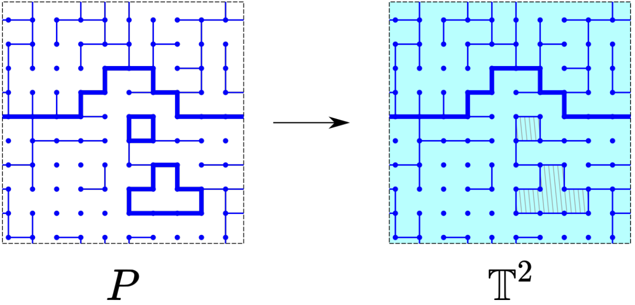
Giant Cycles



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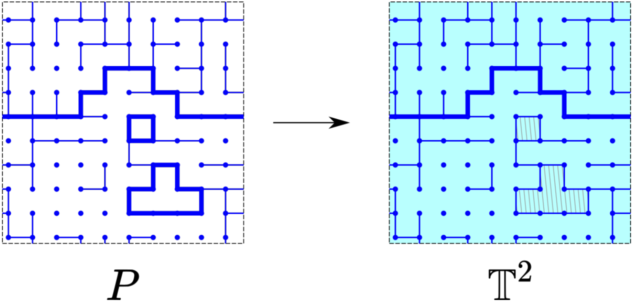

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More precisely, we say that a cycle is giant if it has a nonzero image under the map on homology $\phi_* : H_1(P) \to H_1(\mathbb{T}_N^d)$ induced by the inclusion $P \hookrightarrow \mathbb{T}_N^d$.

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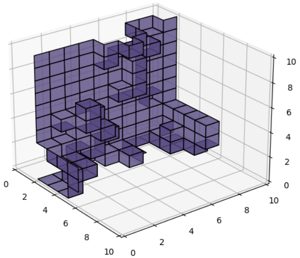
For example, a giant 2-cycle is closed "surface" of 2-plaquettes that is not the boundary of a sum of 3-cubes.



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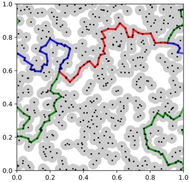
Let $A = A(\mathbb{F})$ be the event that P has a giant cycle in homology with coefficients in a field $\mathbb{F}$ and $S = S(\mathbb{F})$ be the event P has representatives for all equivalence classes of giant cycles (i.e. ϕ_* is surjective).



Main Results

Theorem (D., Schweinhart, 22)

For every $d \geq 2$, $1 \leq i \leq d-1$, $q \in \mathbb{N}$, *and field* $\mathbb{F}$ *with* $\mathrm{char}(F) \neq 2$, *there is a function* $\lambda = \lambda(q,d,i,N)$ *so that for every* $\epsilon > 0$

$$\begin{cases} \mu_{\lambda-\epsilon,q,i,N}(A) \to 0 \\ \mu_{\lambda+\epsilon,q,i,N}(S) \to 1 \end{cases}$$

as $N \to \infty$. *When* d *is even, we also have*

$$\lambda(q,d,d/2,N) = \frac{\sqrt{q}}{1+\sqrt{q}}.$$

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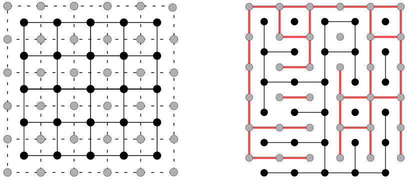
- Duality between random-cluster models in complementary dimensions
- The action of the symmetries of the torus on giant cycles
- Sharp thresholds in symmetric boolean functions

The Dual Model

In the plane, the is a correspondence between percolation subgraphs and subgraphs of the dual graph.



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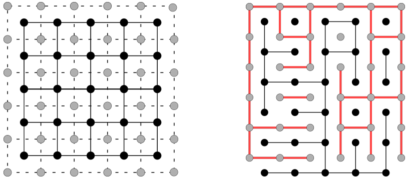
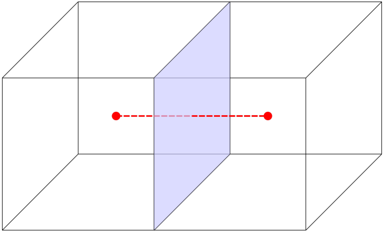


The dual to the classical random-cluster model with parameters p, q in the plane is also a random-cluster model, but with parameters p^*, q, where $p^* = \frac{(1-p)q}{p+(1-p)q}$.

The Dual Model

Analogously, the random complex on $\mathbb{T}_N^d$ distributed as $\mu_{p,q,i,N}$ has a dual complex that differs from $\mu_{p^*,q,d-i,N}$ by at most a constant factor that depends only on p, q, and i.



Planar Duality

Our main topological tool is a generalization of the square crossing lemma used to prove the Harris-Kesten Theorem on Bernoulli bond percolation in $\mathbb{Z}^2$.

Lemma (Kesten, 82)

Let V be the event that there is a vertical crossing of an $n \times (n+1)$ rectangle R. Then $\mathbb{P}_{1/2}(V) = 1/2$.

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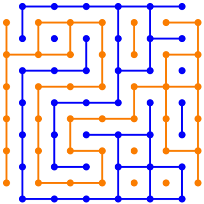

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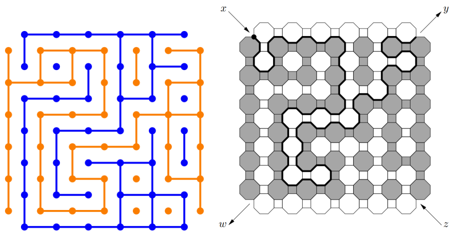

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Duality

Let Z be the event that there are no giant cycles in P, i.e. ϕ_* is zero. Let $A^\bullet, S^\bullet, Z^\bullet$ be the corresponding events in the dual plaquette system $P^\bullet$ and let ψ_* be the map on homology induced by the inclusion $P^\bullet \to \mathbb{T}_N^d$.

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Lemma (Duality)

$\operatorname{rank} \phi_* + \operatorname{rank} \psi_* = D$. *In particular, at least one of the events* A *and* $A^\bullet$ *occurs,* $S^\bullet \iff Z$, *and* $Z^\bullet \iff S$.

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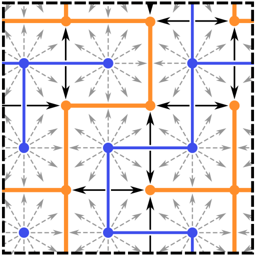

The duality lemma follows from the above observation and a version of Alexander duality, which tells us that a decomposition of a manifold yields pieces with related topological properties.

Giant Cycles via Symmetries

Lemma

Suppose $\mathrm{char}\left(\mathbb{F}\right) \neq 2$. *Then there exist constants* $b_0 = b_0\left(D\right) > 0$ *and* $b_1 = b_1\left(D\right) > 0$ *that do not depend on* N *so that*

$$\mu_{p,q,i,N}\left(S\right) \geq b_0 \mu_{p,q,i,N}\left(A\right)^{b_1} \ .$$

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By the FKG inequality, the probability of the existence of two given giant cycles is at least the product of the probabilities that each exist.

The span of the orbit of a giant cycle under the symmetries of the torus is the entire giant cycle space when the characteristic of $\mathbb{F}$ is not 2.

Sharp Thresholds

The theory of boolean functions on the hypercube tells us that symmetric increasing events for FKG measures have sharp thresholds.

Theorem (Graham, Grimmett 06)

There exists a constant $0 < C < \infty$ so that the following holds. Let $N \geq 1$, $I = \{1, \ldots, N\}$, $\Omega = \{0,1\}^N$, and let $\mathcal{F}$ be the set of subsets of Ω. Let $A \in \mathcal{F}$ be an increasing event. Let μ be a positive monotonic probability measure on $(\Omega, \mathscr{F})$. Let $X_i = \omega(i)$ and set $p = \mu(X_i = 1)$. If there exists a subgroup $\mathcal{A}$ of the symmetric group on N elements Π_N acting transitively on I so that μ and A are $\mathcal{A}$-invariant, then

$$\frac{d}{dp}\mu_p(A) \geq \frac{C\mu_p(X_1)(1-\mu_p(X_1))}{p(1-p)} \min\{\mu_p(A), 1-\mu_p(A)\} \log N.$$

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Using the symmetries of the torus, we can find a $c > 0$ such that $\mu_{\lambda,q,i,N}(S) > c$.

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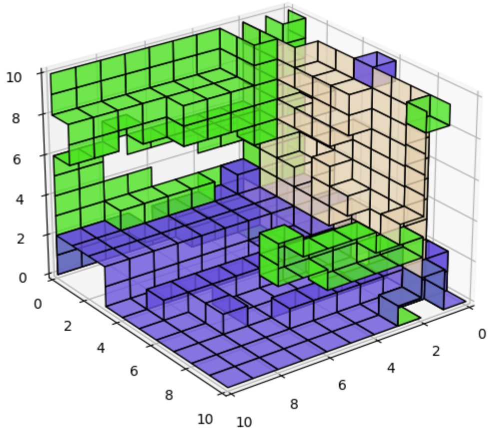
The duality lemma implies that dual models have complementary thresholds, so in particular

$$\lambda(q, d, d/2, N) = \frac{\sqrt{q}}{1 + \sqrt{q}}.$$

Thanks!



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