Homological Percolation on a Torus

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Percolation theory studies the large scale properties of graphs as vertices or edges are removed at random.
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Let $G$ be a finite graph. Bernoulli bond percolation with parameter $p$ is the random subgraph obtained by including each edge of $G$ independently at random with probability $p$, so the probability of a given subgraph is given by

$$\mu_p(H) = p^{|E(H)|} (1 - p)^{|E(G)| - |E(H)|}.$$
The random-cluster model is a dependent generalization that adds additional weight to the subgraphs depending on $c(H)$, the number of connected components of $H$, yielding

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The random-cluster model is related to the Ising/Potts models of magnetism, and can be extended to infinite graphs via limits of finite graphs, the main graph of interest being $\mathbb{Z}^d$. 
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We can think of this as the 2-dimensional generalization of bond percolation, since edges are the 1-cells of the cubical complex structure on $\mathbb{Z}^3$. 
The plaquette model also admits a dependent generalization, which is naturally expressed in terms of homology.
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The boundary of an oriented plaquette is an sum of the oriented edges that it contains.

\[
\partial \sigma = e_1 + e_2 + e_3 + e_4
\]
Homology

The boundary of a sum of cells is the formal sum of the boundaries.

\[ \partial(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4) = \]

\[ \begin{array}{cccc}
\sigma_1 & \sigma_2 \\
\sigma_3 & \sigma_4 \\
\end{array} \]

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The boundary of a sum of cells is the formal sum of the boundaries.

The \((i-1)\)-dimensional boundary group of a percolation subcomplex \(P\) with coefficients in a group \(G\), written \(B_{i-1}(P; G)\) is the group of \(G\)-linear sums of \((i-1)\)-cells that are boundaries of sums of \(i\)-plaquettes.
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The \((i - 1)\)-dimensional cycle group \(Z_{i-1}(P; G)\) is the group of sums of \((i - 1)\)-cells with zero boundary.
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The \((i-1)\)-dimensional homology group is then defined as \(H_{i-1}(P; G) := Z_{i-1}(P; G) / B_{i-1}(P; G)\).
One can easily check that $|H_0(P; \mathbb{Z}_q)| = q^{c(P)}$. 
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We therefore define the $i$-dimensional plaquette random-cluster model with parameter $q \in \mathbb{N}$ on a finite subcomplex $X \subset \mathbb{Z}^d$ by

$$\mu_{p, q, i}(P) = \frac{1}{Z} p^{|P|} (1 - p)^{|X^i|-|P|} |H_{i-1}(P; \mathbb{Z}_q)|,$$

where $|P|$ is the number of $i$-plaquettes in $P$ and $|X^i|$ is the number of $i$-cells in $X$. 
The study of classical percolation in $\mathbb{Z}^d$ centers around the phase transition for the appearance of an infinite component at the critical probability

$$p_c(q) = \inf \{ p : \mu_{p,q}(0 \leftrightarrow \infty) > 0 \}.$$
The Phase Transition

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$$p_c(q) = \inf \{ p : \mu_{p,q}(0 \leftrightarrow \infty) > 0 \} .$$

It is not clear how best to generalize the infinite component. However, better topological tools are available in compact spaces.
Homological Percolation

Instead of $\mathbb{Z}^d$, consider a compact space with the same local structure, namely the torus $\mathbb{T}^d_N$ of dimension $d$ and diameter $N$. 

Source: Bobrowski and Skraba
Homological Percolation

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First, we compare the usual 1-dimensional percolation in each setting.

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A “global” loop in the torus $\mathbb{T}_N^d$ for large $N$ is a natural analogue of an infinite path in $\mathbb{Z}^d$. 

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Giant Cycles

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More precisely, we say that a cycle is giant if it has a nonzero image under the map on homology $\phi_* : H_1(P) \to H_1(\mathbb{T}_N^d)$ induced by the inclusion $P \hookrightarrow \mathbb{T}_N^d$. 
In $i$-dimensional plaquette percolation, we instead look for the appearance of giant $i$-cycles.
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For example, a giant 2-cycle is closed "surface" of 2-plaquettes that is not the boundary of a sum of 3-cubes.
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Let \( A = A(\mathbb{F}) \) be the event that \( P \) has a giant cycle in homology with coefficients in a field \( \mathbb{F} \) and \( S = S(\mathbb{F}) \) be the event \( P \) has representatives for all equivalence classes of giant cycles (i.e. \( \phi_* \) is surjective).
Main Results

Theorem (D., Schweinhart, 22)

For every $d \geq 2$, $1 \leq i \leq d - 1$, $q \in \mathbb{N}$, and field $\mathbb{F}$ with $\text{char}(\mathbb{F}) \neq 2$, there is a function $\lambda = \lambda(q, d, i, N)$ so that for every $\epsilon > 0$

$$
\begin{cases}
\mu_{\lambda - \epsilon, q, i, N}(A) \to 0 \\
\mu_{\lambda + \epsilon, q, i, N}(S) \to 1
\end{cases}
$$

as $N \to \infty$. When $d$ is even, we also have

$$
\lambda(q, d, d/2, N) = \frac{\sqrt{q}}{1 + \sqrt{q}}.
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- Duality between random-cluster models in complementary dimensions
- The action of the symmetries of the torus on giant cycles
- Sharp thresholds in symmetric boolean functions
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The Dual Model

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The dual to the classical random-cluster model with parameters $p, q$ in the plane is also a random-cluster model, but with parameters $p^*, q$, where $p^* = \frac{(1-p)q}{p+(1-p)q}$. 
Analogously, the random complex on $\mathbb{T}_N^d$ distributed as $\mu_{p,q,i,N}$ has a dual complex that differs from $\mu_{p^*,q,d-i,N}$ by at most a constant factor that depends only on $p$, $q$, and $i$. 
Planar Duality

Our main topological tool is a generalization of the square crossing lemma used to prove the Harris-Kesten Theorem on Bernoulli bond percolation in $\mathbb{Z}^2$.

**Lemma (Kesten, 82)**

Let $V$ be the event that there is a vertical crossing of an $n \times (n + 1)$ rectangle $R$. Then $P_{1/2}(V) = 1/2$.

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Duality

Let $Z$ be the event that there are no giant cycles in $P$, i.e. $\phi_*$ is zero. Let $A^\bullet, S^\bullet, Z^\bullet$ be the corresponding events in the dual plaquette system $P^\bullet$ and let $\psi_*$ be the map on homology induced by the inclusion $P^\bullet \to \mathbb{T}_N^d$. 

Lemma (Duality) \[ \text{rank } \phi_* + \text{rank } \psi_* = D. \] In particular, at least one of the events $A$ and $A^\bullet$ occurs, $S^\bullet \iff Z$, and $Z^\bullet \iff S$. 

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**Lemma (Duality)**

$\text{rank } \phi_\ast + \text{rank } \psi_\ast = D$. In particular, at least one of the events $A$ and $A^\bullet$ occurs, $S^\bullet \iff Z$, and $Z^\bullet \iff S$. 
Plaquette Duality

Lemma

$\mathbb{T}^d_N \setminus P$ is homotopy equivalent to $P^\bullet$. 
The duality lemma follows from the above observation and a version of Alexander duality, which tells us that a decomposition of a manifold yields pieces with related topological properties.
Lemma

Suppose $\text{char}(\mathbb{F}) \neq 2$. Then there exist constants $b_0 = b_0(D) > 0$ and $b_1 = b_1(D) > 0$ that do not depend on $N$ so that

$$\mu_{p,q,i,N}(S) \geq b_0 \mu_{p,q,i,N}(A)^{b_1}.$$
Lemma

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By the FKG inequality, the probability of the existence of two given giant cycles is at least the product of the probabilities that each exist.
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The span of the orbit of a giant cycle under the symmetries of the torus is the entire giant cycle space when the characteristic of $\mathbb{F}$ is not 2.
The theory of boolean functions on the hypercube tells us that symmetric increasing events for FKG measures have sharp thresholds.

**Theorem (Graham, Grimmett 06)**

There exists a constant $0 < C < \infty$ so that the following holds. Let $N \geq 1$, $I = \{1, \ldots, N\}$, $\Omega = \{0, 1\}^N$, and let $\mathcal{F}$ be the set of subsets of $\Omega$. Let $A \in \mathcal{F}$ be an increasing event. Let $\mu$ be a positive monotonic probability measure on $(\Omega, \mathcal{F})$. Let $X_i = \omega(i)$ and set $p = \mu(X_i = 1)$. If there exists a subgroup $A$ of the symmetric group on $N$ elements $\Pi_N$ acting transitively on $I$ so that $\mu$ and $A$ are $A$-invariant, then

\[
\frac{d}{dp} \mu_p(A) \geq \frac{C \mu_p(X_1)(1 - \mu_p(X_1))}{p(1 - p)} \min \{\mu_p(A), 1 - \mu_p(A)\} \log N.
\]
Finishing the Proof

Take $\lambda$ so that $\mu_{\lambda,q,i,N}(A) = 1/2$
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Using the symmetries of the torus, we can find a $c > 0$ such that $\mu_{\lambda,q,i,N}(S) > c$. 
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By the Graham-Grimmett Theorem, the thresholds for \( A \) and \( S \) are sharp and coincide.
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By the Graham-Grimmett Theorem, the thresholds for $A$ and $S$ are sharp and coincide.

The duality lemma implies that dual models have complementary thresholds, so in particular

$$\lambda(q, d, d/2, N) = \frac{\sqrt{q}}{1 + \sqrt{q}}.$$
Thanks!
References


