The Plaquette Random Cluster Model and Potts Lattice Gauge Theory

Ben Schweinhart (joint with Paul Duncan)

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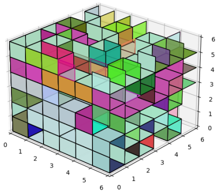
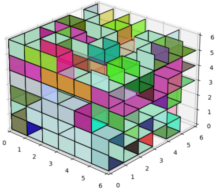
The Cubical Complex $\mathbb{Z}^d$



This talk will concern random subcomplexes (and random co-chains) on the cubical complex $\mathbb{Z}^d$: the collection of i-cubes whose vertices are integer lattice points for $0 \leq i \leq 3$. 0-faces are vertices, 1-faces are edges, 2-faces are plaquettes, and 3-faces are cubes, and so on.

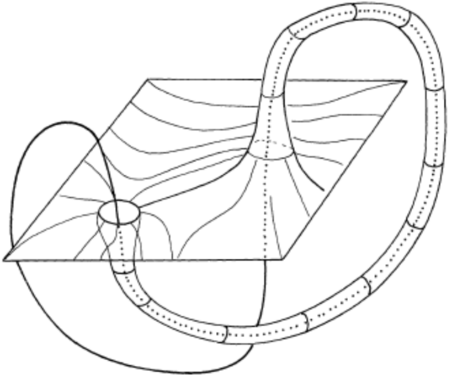
Bernoulli Plaquette Percolation



2-dimensional Bernoulli plaquette percolation on $\mathbb{Z}^3$ with probability p is the random subcomplex of $\mathbb{Z}^3$ including all vertices and edges, where 2-dimensional plaquettes are included independently with probability p.

On a Sharp Transition from Area Law to Perimeter Law in a System of Random Surfaces*

M. Aizenman[1,**], J. T. Chayes[2,***], L. Chayes[2,***], J. Fröhlich[3], and L. Russo[4]



For a 1-cycle γ let W_γ be the event that γ is "bounded by a surface of plaquettes" in two-dimensional plaquette percolation.

On a Sharp Transition from Area Law to Perimeter Law in a System of Random Surfaces*

M. Aizenman[1,**], J. T. Chayes[2,***], L. Chayes[2,***], J. Fröhlich[3], and L. Russo[4]

Theorem 1.1. *For rectangular* $(N \times M)$ *loops* γ, *in a lattice plane, the quantity* $\langle W_\gamma \rangle_p$ *has the following asymptotic behavior:*

$$\langle W_\gamma \rangle_p \sim \begin{cases} \exp[-\alpha(p)\,\mathrm{Area}(\gamma)] & \text{for} \quad p < 1 - \pi_c \\ \exp[-c(p)\,\mathrm{Per}(\gamma)] & \text{for} \quad p > 1 - \hat{\varrho}_c \end{cases} \tag{1.2}$$

with some $0 < \alpha(p),\ c(p) < \infty$.

The symbol $\langle W_\gamma \rangle \sim e^{-V(\gamma)}$ *means here that* $\lim_{M,N\to\infty} -\log\langle W_\gamma \rangle / V(\gamma) = 1$, *i.e. the constants* $\alpha(p)$ *and* $c(p)$ *are actually well defined.*

We generalize this theorem to a family of dependent plaquette percolation models, as well as to higher dimensional plaquette percolation in co-dimension one.

Commun. Math. Phys. 92, 19–69 (1983)

Of course, our interest in these pure stochastic-geometric effects is also motivated by discussions of "quark confinement" in *gauge models*. There, the quantity for which area, versus perimeter, law is of interest is the expected value of "Wilson loop" variables. It turns out that, at least for the abelian $\mathbb{Z}(2)$ gauge model, such a transition can be traced exactly [8] to a geometric effect of the type discussed here, albeit in a system of *interacting* plaquettes. These, and other, relations with gauge models are described in Sect. 7.

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Of course, our interest in these pure stochastic-geometric effects is also motivated by discussions of "quark confinement" in *gauge models*. There, the quantity for which area, versus perimeter, law is of interest is the expected value of "Wilson loop" variables. It turns out that, at least for the abelian $\mathbb{Z}(2)$ gauge model, such a transition can be traced exactly [8] to a geometric effect of the type discussed here, albeit in a system of *interacting* plaquettes. These, and other, relations with gauge models are described in Sect. 7.

Nuclear Physics B235 [FS11] (1984) 1–18

TOPOLOGICAL ANOMALIES IN THE *n* DEPENDENCE OF THE *n*-STATES POTTS LATTICE GAUGE THEORY

Michael AIZENMAN[1] Jurg FRÖHLICH

We will explain these anomalies, and show how to account for them.

Presentation Outline

Co-chains on $\mathbb{Z}^d$

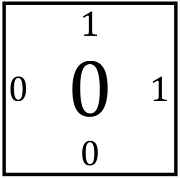
Let G be an abelian group, $X \subset \mathbb{Z}^d$ be a cubical complex, and $0 \leq i \leq d$.

Definition

The co-chain group $C^i(X; G)$ is the group of G-valued functions on the oriented i-dimensional faces of X, so that reversing the orientation of a face inverts its spin.

So an element of $C^0(X; G)$ is an assignment of spins to the vertices of X.

The Coboundary of a Cochain



$$
\begin{aligned}
\delta f(\sigma) &= f(\partial\sigma) \\
&= f(e_1)+f(e_2)+f(e_3)+f(e_4) \\
&= 0+0+1+1=0 \pmod 2.
\end{aligned}
$$

The coboundary of an i-cochain f is an $(i+i)$-cochain δf whose value on an $(i+1)$-plaquette σ is the sum of its values on the neighboring i-plaquettes in ∂f.

$$\delta f(\sigma) = f(\partial\sigma).$$

The q-state Potts Model

The q-state Potts model is a random assignment of elements of $\mathbb{Z}_q$ to the vertices of a graph X : a random element $f \in C^0(X; \mathbb{Z}_q)$.

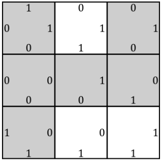
We can express the Potts Hamiltonian in terms of the coboundary operator. Let $f \in C^0(X; \mathbb{Z}_q)$ and let $e = (v, w)$ be an edge of X. Set

$$H(f) = -\sum_e I_{\delta f(e)=0} \, .$$

Then the q-**state Potts model** on X is the random 0-cochain $f \in C^0(X, \mathbb{Z}_q)$ so that

$$\mathbb{P}(f = f') \propto e^{-\beta H(f')} \, .$$

Potts Lattice Gauge Theory Hamiltonian



H(f)=-6

For $f \in C^1(X; \mathbb{Z}_q)$ set

$$H(f) = -\sum_{\sigma \in X^{(2)}} I_{\delta f(\sigma)=0}$$

where $X^{(2)}$ is the set of all 2-plaquettes of X.

Potts Lattice Gauge Theory

The q-**state Potts lattice gauge theory** on a finite cell complex X is the random-cochain $f \in C^1(X, \mathbb{Z}_q)$ where

$$\mathbb{P}(f = f') \propto e^{-\beta H(f')}\,.$$

(We will sweep all details about infinite volume limits and boundary conditions under the rug.)

Potts Lattice Gauge Theory

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 12, NUMBER 10 OCTOBER 1971

Duality in Generalized Ising Models and Phase Transitions without Local Order Parameters*

Franz J. Wegner†

PHYSICAL REVIEW D VOLUME 22, NUMBER 10 15 NOVEMBER 1980

Z_N and N-state Potts lattice gauge theories: Phase diagrams, first-order transitions, β functions, and $1/N$ expansions

J. B. Kogut R. B. Pearson J. Shigemitsu D. K. Sinclair

Ising lattice gauge theory was introduced by Wegner (1971), and Potts lattice gauge theory was defined by Kogut et al. in 1980

PHYSICAL REVIEW D VOLUME 10, NUMBER 8 15 OCTOBER 1974

Confinement of quarks*

Kenneth G. Wilson

Lattice gauge theories were introduced as a discretization of Euclidean Yang–Mills theory. They assign random elements of a complex matrix group G to the edges of a cell complex. When $d = 4$ the cases $G = U(1)$, $G = SU(2)$, and $G = SU(3)$ are models of the electromagnetic, weak nuclear, and strong nuclear forces, respectively.

The case where G is the multiplicative group of second (or third) complex roots of unity is 2(3)-state Potts lattice gauge theory.

The asymptotic behavior of a class of random variables called Wilson loop variables is believed to be related to the phenomenon of quark confinement.

Wilson Loop Variables

The **Wilson loop variable** associated with an 1-cycle γ is the random variable $W_\gamma : C^1(X, \mathbb{Z}_q) \to \mathbb{C}$ given by

$$W_\gamma = (f(\gamma))^{\mathbb{C}},$$

where for $g \in \mathbb{Z}_q$, $g^{\mathbb{C}}$ is the corresponding q-th root of unity in $\mathbb{C}$.

Sharpness for Potts Lattice Gauge Theory?

Conjecture

There exists a $\beta_c(q) > 0$ *and constants*

$$0 < c_1(\beta, q), c_2(\beta, q) < \infty$$

so that, for rectangular γ *in* $\mathbb{Z}^d$,

$$\mathbb{E}(W_\gamma) \sim \begin{cases} \exp(-c_1(\beta, q)\mathrm{Area}(\gamma)) & \beta < \beta_c(q) \\ \exp(-c_2(\beta, q)\mathrm{Perimeter}(\gamma)) & \beta > \beta_c(q) \end{cases}.$$

Sharpness for Potts Lattice Gauge Theory?

This is the $i = 1$ case of a more general conjecture for which the $i = 0$ case is sharpness for the Potts model in $\mathbb{Z}^d$.

It's easy to show that area law and perimeter law phases exist when the inverse temperature β is sufficiently low/high.

A proof of the area law for $q = 2$, $d = 3$ follows from a theorem of Lebowitz and Pfister (1981) by arguments of Bricmont, Lebowitz, and Pfister (1980).

Laanait, Messager, Ruiz showed that the conjecture holds for sufficiently large q when $d = 4$.

Potts Lattice Gauge Theory in Codimension Two

Theorem (Duncan and S., 2023)

Consider Potts lattice gauge theory on $\mathbb{Z}^3$*. For rectangular boundaries* γ

$$\mathbb{E}(W_\gamma) \sim \begin{cases} \exp(-c_1(\beta,q)\mathrm{Area}(\gamma)) & \beta < \beta^*(\beta_{\mathrm{slab}}(q)) \\ \exp(-c_2(\beta,q)\mathrm{Perimeter}(\gamma)) & \beta > \beta^*(\beta_c(q)) \end{cases}.$$

where

$$\beta^*(\beta) = \log\left(\frac{e^\beta + q - 1}{e^\beta - 1}\right).$$

Presentation Outline

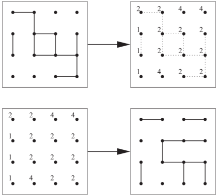
The Random Cluster Model

The **random cluster model** with parameters $p \in [0,1]\,, q \geq 0$ on a finite graph X is the random subgraph P so that

$$\mathbb{P}_{p,q}(P = \hat{P}) \propto p^{\#\text{edges}}(1-p)^{\#\text{non}-\text{edges}} q^{\beta_0(\hat{P})}$$

where $b_0(P)$ is the number of connected components.

The Coupling



We can couple the random cluster model P with parameters $p = 1 - e^{-\beta}$ and $q \in \mathbb{N}_{\geq 2}$ with Potts model f with parameters β and q so that:

- The conditional measure of P given f is Bernoulli percolation with probability p on edges where f is constant.
- The conditional measure of f given P assigns independent spins to each component (that is, f is a random uniform co-cycle in $Z^0(P; \mathbb{Z}_q) = \operatorname{Ker} \delta$).

Connection/Correlation Theorem

The spin correlation function for the q-state Potts model is

$$\tau_{\beta,q}(x,y) = \mathbb{P}(f(x) = f(y)) - \frac{1}{q}\,.$$

Theorem

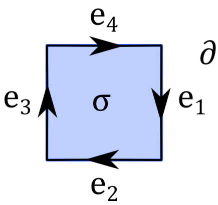
$$\tau_{\beta,q}(x,y) = \frac{1}{q}\mathbb{P}_{p,q}(x \leftrightarrow y)$$

where the probability is taken with respect to the random cluster model on X with parameters $p = 1 - e^{-\beta}$ and q.

Presentation Outline

Chains

- Given a cubical complex X, define the chain group $C_i(X; G)$ to be the group of formal linear combinations of i-dimensional faces of X (with coefficients in an abelian group G).
- Given an i-face α, define $\partial_i(\alpha)$ to be a signed sum of the $(i-1)$-faces contained in α, and extend ∂_i linearly to give a linear function $\partial_i : C_i(X; G) \to C_{i-1}(X; G)$.



$$\partial\sigma = e_1 + e_2 + e_3 + e_4$$

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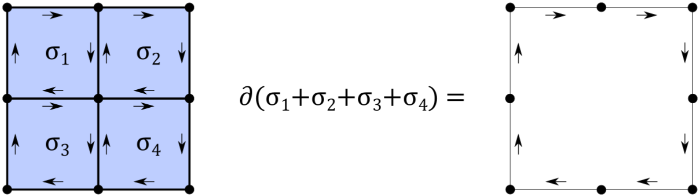
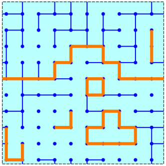
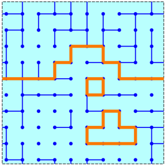


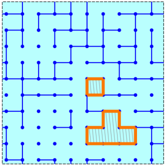
Cycles and Boundaries



Chains



Cycles



Boundaries

Let $Z_i(X) = \ker \partial_i$ be the group of **cycles**, and $B_i(X; G) = \operatorname{im} \partial_{i+1}$ be the group of i-chains that are **boundaries** of an $(i+1)$-chain.

Definition

Define the i^{th} **homology group** as the quotient $H_i(X; G) = Z_i(X; G)/B_i(X; G)$.

Co-chains on $\mathbb{Z}^d$

We can linearly extend a cochain $f \in C^i(X;G)$ to define a G-linear function on $C_i(X;G)$. That is, $C^i(X;G)$ is identified with with $\mathrm{Hom}(C_i(X;G),G)$.

In particular, if $f \in C^1(X;G)$ and $\gamma = \sum_{i=1}^n a_i e_i$ then $f(\gamma) = \sum_{i=1}^n a_i f(e_i)$.

Cochains

Recall that the coboundary map $\delta^i : C^i(X;G) \to C^{i+1}(X;G)$ is defined by

$$\delta^i f(\sigma) = f(\partial\sigma)\,.$$

Let $Z^i(X;G) = \ker \delta_i$ be the group of **cocycles**, and $B^i(X;G) = \operatorname{im} \delta_{i-1}$ be the group of i-cochains that are **coboundaries** of an $i+1$-cochain.

Definition

Define the i^{th} **cohomology group** as the quotient $H^i(X;G) = Z^i(X;G)/B^i(X;G)$.

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Presentation Outline

The Idea

We'd like to couple Potts lattice gauge theory with a dependent 2-dimensional plaquette percolation $P(p, q)$ so that Wilson loop expectations equal the probability that the loop is bounded by a "surface of plaquettes."

Attempts towards this end in the 80s were stymied by the discovery of so-called topological anomalies.

Nuclear Physics B235 [FS11] (1984) 1–18

TOPOLOGICAL ANOMALIES IN THE n DEPENDENCE OF THE n-STATES POTTS LATTICE GAUGE THEORY

Michael AIZENMAN[1] Jurg FRÖHLICH

While $b_0(P) = \operatorname{rank} H_0(P; G)$ does not depend on G, the "number of independent loops" $b_1(P) = \operatorname{rank} H_1(P; G)$ and the ''number of independent closed surfaces" $b_2(P) = \operatorname{rank} H_2(P; G)$ do!

Maritan and Omero (1982) defined a random two complex weighted by the "number of independent closed surfaces of plaquettes." The coupling failed because by not accounting for the dependence on G. A different attempt by Ginsparg, Goldschmidt, and Zuber (1980) ran into similar difficulties.

The Plaquette Random Cluster Model

A second (more minor) topological anomaly is that when q is not prime, $\left|H^{i-1}(X;\mathbb{Z}_q)\right| \neq q^{\mathrm{rank}(H_{i-1}(X;\mathbb{Z}_q))}$.

Definition (S. and Duncan (2023))

The i-**Random Cluster Model** $P(p,q)$ on a finite cell complex X is the random set of i-plaquettes so that

$$\mathbb{P}_{p,q}(P(p,q) = P) \propto p^{|P|}(1-p)^{|X|-|P|}\left|H^{i-1}(X;\mathbb{Z}_q)\right|.$$

The definition for prime q was discovered by Hiraoka and Shirai (2016). The current definition (rather, one equivalent to it) was suggested in our 2022 paper, and details of the coupling were worked our 2023 paper and independently by Shklarov (2023).

The Coupling, Generalized

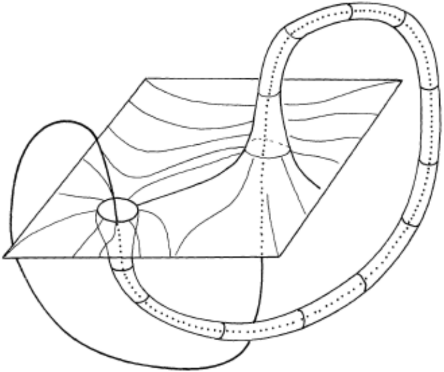
Theorem (Duncan and S., 2023)

Let $q \geq 2$, $\beta > 0$, and $p = 1 - e^{-\beta}$. The 2-dimensional plaquette random cluster model $P = P(p, q, \mathbb{Z}_q)$ can be coupled with q-state Potts lattice gauge theory with inverse temperature β so that

- *The conditional measure of P given f is Bernoulli plaquette percolation with probability p on the 2-plaquettes satisfying $\delta f = 0$.*
- *The conditional measure of f given P is the uniform measure on $Z^1(P; \mathbb{Z}_q)$.*

The special case of prime q is due to to Hiraoka and Shirai (2016). The case of general q was considered by Duncan and S. (2023) and Shklarov (2023).

V_γ



Let P be a 2-dimensional cubical complex. For a 1-cycle in $\mathbb{Z}^d$, define $V_\gamma = V_\gamma(q)$ to be the event that $[\gamma] = 0$ in $H_1(X; \mathbb{Z}_q)$ (that is, that γ is the boundary of a 2-chain.)

Image credit: Aizenman et al.

Wilson Loops and Homology

Theorem (Duncan and S. (2022, 2023))

Let γ be an 1-cycle in $\mathbb{Z}^d$. Then

$$\mathbb{E}_{\beta,q}(W_\gamma) = \mathbb{P}_{p,q}(V_\gamma(q))\,,$$

where the expectation in the right is taken with respect to q-state Potts lattice gauge theory and the probability on the right is evaluated for the corresponding plaquette random cluster model.

This is false if we take homology in a group other than $\mathbb{Z}_q$. This was the third topological anomaly observed by Aizenman and Fröhlich.

Consequences of the Coupling

- We can use the previous result together with a comparison to plaquette percolation to find new proofs of “area law” and “perimeter law” regimes for Potts lattice gauge theory when β is sufficiently low/high.
- The coupling allows us to generalize the Swendsen–Wang Algorithm (see Anthony’s poster!).
- For odd primes q, we prove that the i-dimensional RCM undergoes a sharp phase transition in the sense of homological percolation (Paul’s talk!). This implies a phase transition in the qualitative behavior of the Swendsen–Wang Algorithm.

Theorem

There exist constants and $0 < \tau(p,q), \upsilon(p,q) < \infty$ *so that for any suitable family of rectangular boundaries* $\{\gamma_l\}$,

$$
\begin{aligned}
-\frac{\log(\mathbb{P}_{p,q}(V_{\gamma_l}))}{\mathrm{Area}(\gamma_l)} &\to \tau(p,q) & \qquad p < p^*(p_{\mathrm{slab}}(q)) \\
-\frac{\log(\mathbb{P}_{p,q}(V_{\gamma_l}))}{\mathrm{Perimeter}(\gamma_l)} &\to \upsilon(p,q) & \qquad p > p^*(p_c(q))\,,
\end{aligned}
$$

where $p_c(q)$ *and* $p_{\mathrm{slab}}(q)$ *are the critical thresholds for the classical (one-dimensional) random-cluster model on* $\mathbb{Z}^3$ *and in slabs in* $\mathbb{Z}^3$, *respectively, and*

$$
p^* = p^*(p) = \frac{(1-p)q}{(1-p)q+p}\,.
$$

Note: The ACCFR theorem is the special case $q = 1$. Also, this implies a phase transition for Wilson loop variables in the corresponding Potts Lattice Gauge Theory.

- Aizenman, Chayes, Chayes, Fröhlich, and Russo (1983). On a sharp transition from area law to perimeter law in a system of random surfaces. Communications in Mathematical Physics.
- Aizenman and Fröhlich (1984). Topological anomalies in the n-dependence of the n-states Potts lattice gauge theory. Nuclear Physics B.
- Bricmont, Lebowitz, and Pfister (1980). On the surface tension of lattice systems. Annals of the New York Academy of Sciences.
- Chatterjee, S. Yang–Mills for probabilists (2016). In International Conference in Honor of the 75th Birthday of SRS Varadhan.
- Duncan, P. and Schweinhart, B. (2022). Topological phases in the plaquette random cluster model and Potts lattice gauge theory. arXiv:2207.08339.
- Duncan, P. and Schweinhart, B. (2023). A sharp deconfinement transition for Potts lattice gauge theory in codimension one. arXiv:2308.07534.
- Hiraoka, Y. and Shirai, T. (2016). Tutte polynomials and random-cluster models in Bernoulli cell complexes. RIMS Kokyuroku Bessatsu.

Presentation Outline

Proof Overview

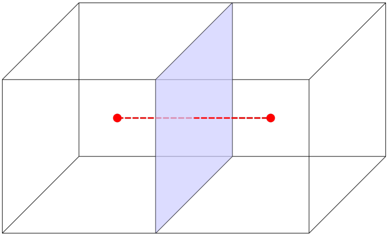
- With the PRCM in hand and a couple results about its duality properties, the proof isn't difficult.
- The perimeter law argument is nearly identical to ACCFR. Showing there is a well-defined constant requires a bit more work.
- The area law argument is similar to ones of ACCFR and Bricmont, Lebowitz, and Pfister.

Proof Ingredient 0: Trivial Bounds

Let γ be the boundary of a rectangle r in $\mathbb{Z}^3$. V_γ occurs if all plaquettes in r are occupied, yielding an area law lower bound on $\mathbb{P}_{p,q}(V_\gamma)$.

On the other hand, the absence of all plaquettes adjacent to γ is incompatible with V_γ. This leads to a perimeter law upper bound.

Proof Ingredient 1: Duality

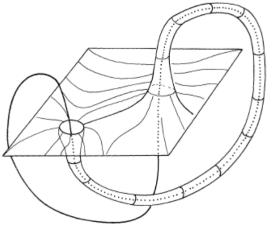


Proposition

The dual of the (free) i-random cluster model with parameters q and p is the (wired) $(d-i)$-random cluster model with parameters q and $p^*(p)$.

This follows from Alexander duality and the Euler–Poincaré formula

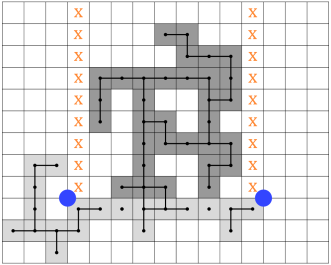
Proof Ingredient 2: Duality in Codimension One



Proposition

If $i = d - 1$ and γ is a $(d-1)$-boundary in $\mathbb{Z}^d$, then V_γ does not occur if and there is a loop of dual edges whose linking number with γ is non-zero modulo q.

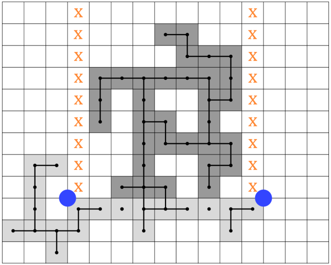
Proof Sketch: Perimeter Law



The ACCFR argument is easily modified. Let $p > p^*(p_c(q))$ so the dual random cluster model is subcritical.

Let C be the connected component of all vertices "below" the rectangle (shown in gray), and let $C' \subset C$ include all vertices contained in the cylinder above the rectangle (dark gray). With positive probability, we can block C' from leaving the cylinder (orange x's).

Proof Sketch: Perimeter Law



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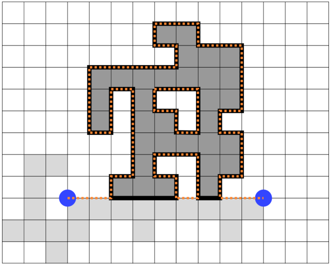


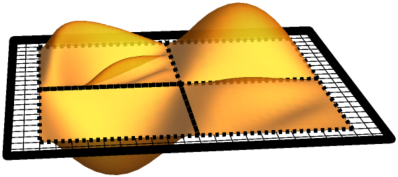
Let $\tau_1, ldots, \tau_j$ be the cubes centered at the dual vertices of C. We can write

$$\partial \sum_{i=1}^{j} \tau_j = \alpha_0 + \alpha_1$$

where α_1 consists of all boundary plaquettes “above” the rectangle. Then

$$\partial \alpha_1 = (-1)^{d-1} \gamma \, .$$

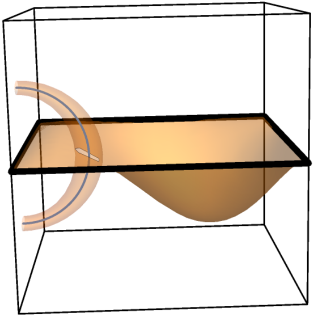
Proof Sketch: Area Law



$$\lim_{l\to\infty} -\frac{\log(\mathbb{P}_{p,q}(V_{\gamma_l}))}{\mathrm{Area}(\gamma_l)}$$

exists and is independent of the sequence $\{\gamma_l\}$. This is shown by tiling γ_l with m translates of $\gamma'_k := [0,K]^2 \times \{0\}$, and comparing the probability of the events V_{γ_l} with that of m copies of $V_{\gamma'_k}$.

Proof Sketch: Area Law



Next, if p is such that the PRCM has a unique Gibbs measure, the area law coefficient is the same if we take γ'_k to be the "equator" of $\Lambda = [0, K]^2 \times [-K, K]$. Duality relates this to the decay of the probability of a dual connection between $\partial\Lambda \cap \{\vec{e}_d > 0\}$ and $\partial\Lambda \cap \{\vec{e}_d < 0\}$; Bodineau proved this notion of surface tension for the RCM is non-vanishing above the slab percolation threshold.

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