Fast relaxation in the random field Ising model

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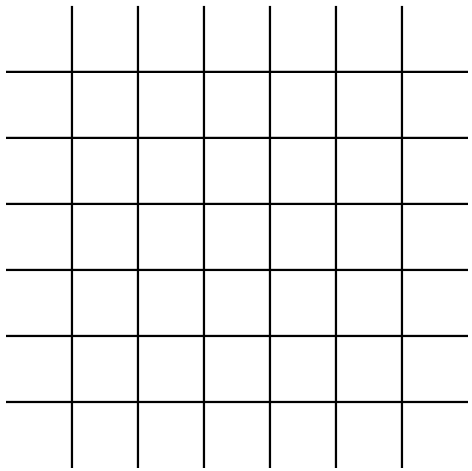
IPAM
Statistical mechanics beyond 2D
May 2024

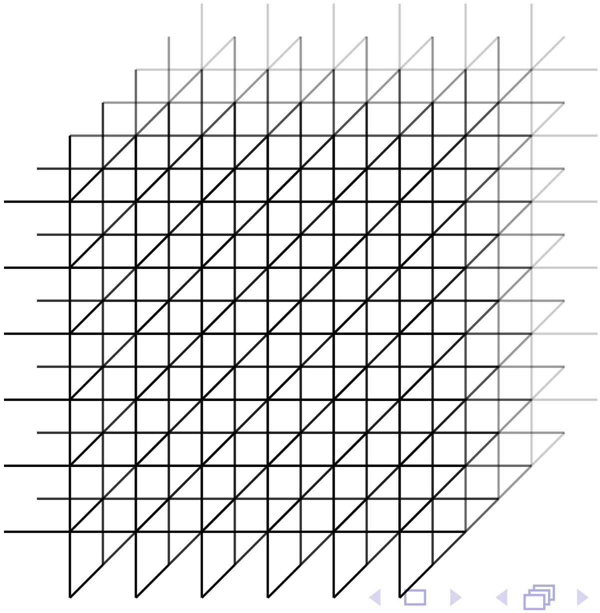
The underlying geometry

$n \times n$ box in $\mathbb{Z}^d$ with nearest-neighbor edges $v \sim w$

$d = 2$:



$d = 3$:

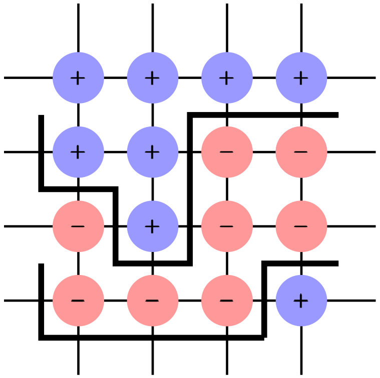


The Ising model

Ising model: probability of assignment σ of $\{+, -\}$ to the vertices.

$$\pi_{\beta,n}(\sigma) \propto e^{-H(\sigma)} \qquad \text{where} \qquad H(\sigma) = \beta \sum_{v \sim w} \mathbf{1}_{\{\sigma_v \neq \sigma_w\}}$$

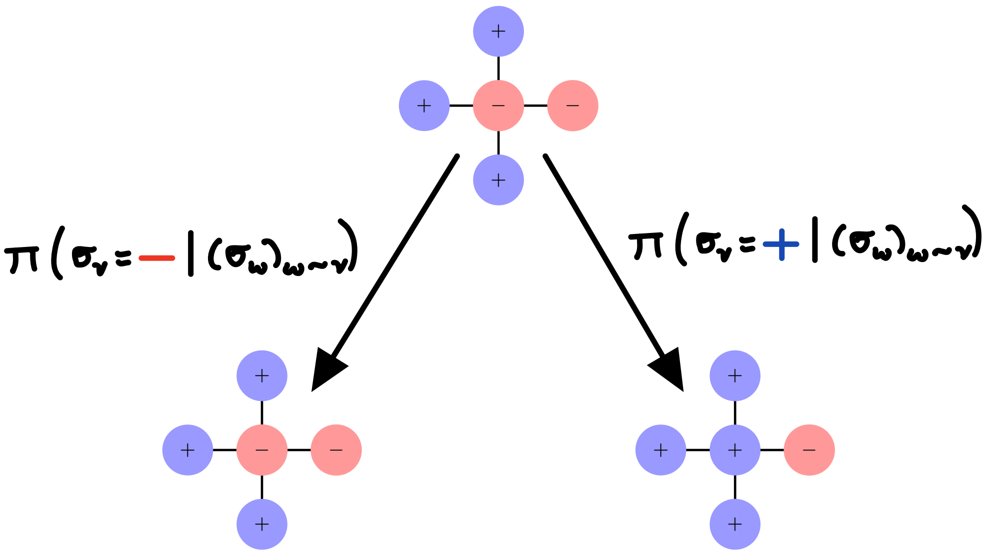
- H is the *Hamiltonian* or *energy* of a configuration;
- β is an *inverse-temperature* parameter.



$H(\sigma) = 14\beta$

Glauber dynamics for the Ising model

1. Assign every site a rate-1 Poisson clock.
2. If the clock at site v rings at time t,
3. Resample $X_t(v)$ conditionally on its neighbors $(X_t(w))_{w\sim v}$.



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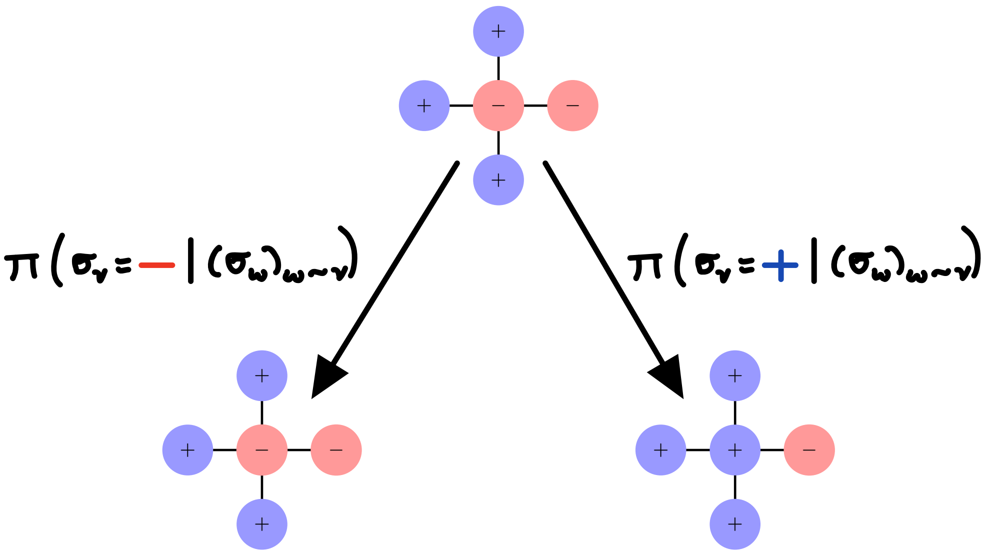

Equilibrium distribution is exactly π*!*

Quantifying the rate of convergence

Mixing time:

$$t_{mix} = \inf\{t : \max_{X_0} \|\mathbb{P}_{X_0}(X_t \in \cdot) - \pi\|_{tv} < 1/4\}\,.$$

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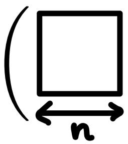
Poincaré inequality i.e., spectral gap: closely related to t_{mix}

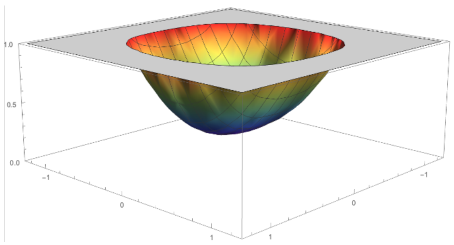
$$\mathrm{Var}_\pi(f) \leq \lambda^{-1} \cdot \mathcal{E}(f,f)$$

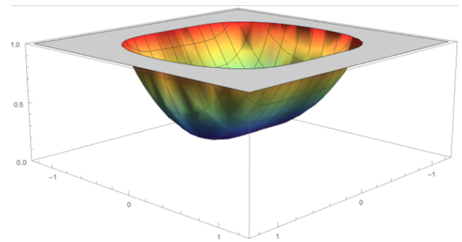
Governs exponential rate of convergence to π:

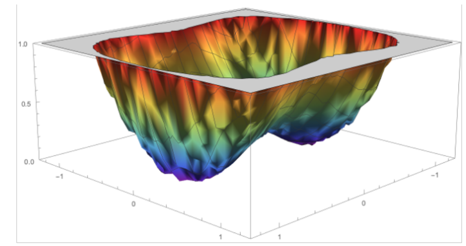
$$\big\|P^t f - \pi[f]\big\|_{2,\pi}^2 \leq \big\|f\big\|_{2,\pi}^2 e^{-\lambda t}\,.$$

Mixing time of the Ising model on $\mathbb{Z}^d$

Consider the mixing time t_{MIX}()







$\beta < \beta_c(d)$

Optimal $O(\log n)$ mixing

"Weak/strong spatial mixing"

[Martinelli Oliveiri '94]
[Cesi '99] [Lubetzky Sly '10, '13]

$\beta = \beta_c(d)$

$d = 2$: Poly $O(n^c)$ mixing
[Lubetzky Sly '12]

$d = 3, 4$: ???

$d \geq 5$: Poly $O(n^c)$ mixing
[Bauerschmidt Dagallier '22]

$\beta > \beta_c(d)$

Slow mixing: $\exp(c_\beta n^{d-1})$

$d = 2$: [Chayes et al '87]

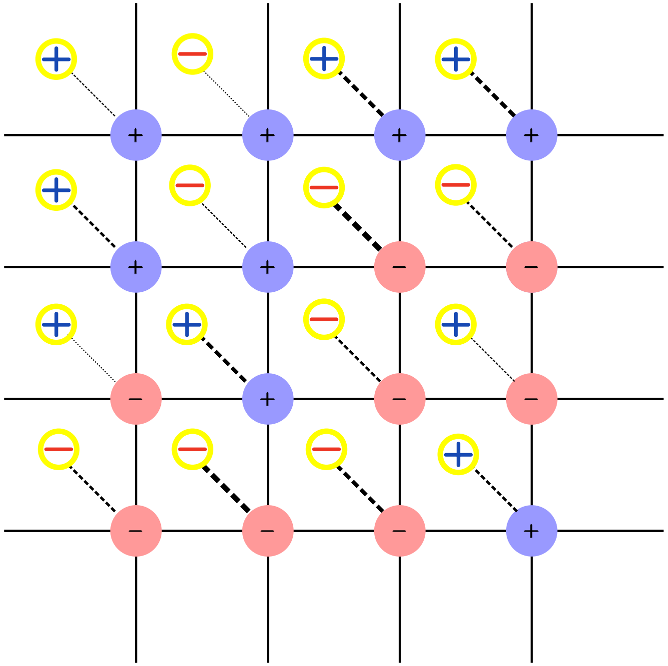
$d \geq 3$: Pisztora '96, Bodineau '05

The random-field Ising model

The random field $(h_v)_v$ are i.i.d. symmetric (e.g., $\mathcal{N}(0,b^2)$)

Random field Ising model (RFIM): probability of $\sigma \in \{+,-\}^V$,

$$\pi(\sigma) \propto e^{-H(\sigma)} \qquad \text{where} \qquad H(\sigma) = \beta \sum_{v\sim w} \mathbf{1}_{\{\sigma_v \neq \sigma_w\}} - \sum_v h_v \sigma_v$$



The RFIM: phase diagram at equilibrium

$\mathbf{d = 2}$ *Imry-Ma phenomenon*: exponential decay of correlations (in expectation or for typical $(h_v)_v$) at *all temperatures* in $\mathbb{Z}^2$.

[Chatterjee '17, Aizenman Peled '18, Aizenman Harel Peled '19, Ding Xia '19]

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d ≥ 3 *Phase transition*: At β large, $\mathrm{Var}(h)$ small: long range order
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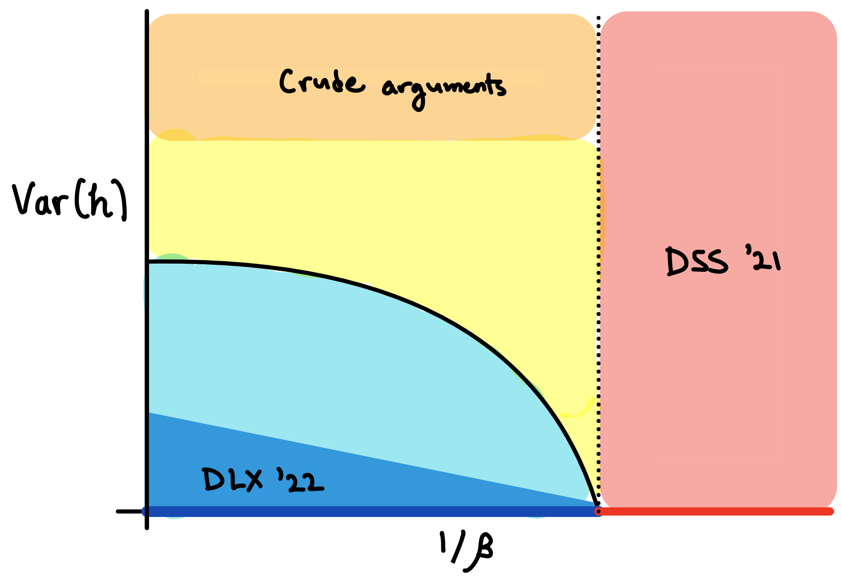
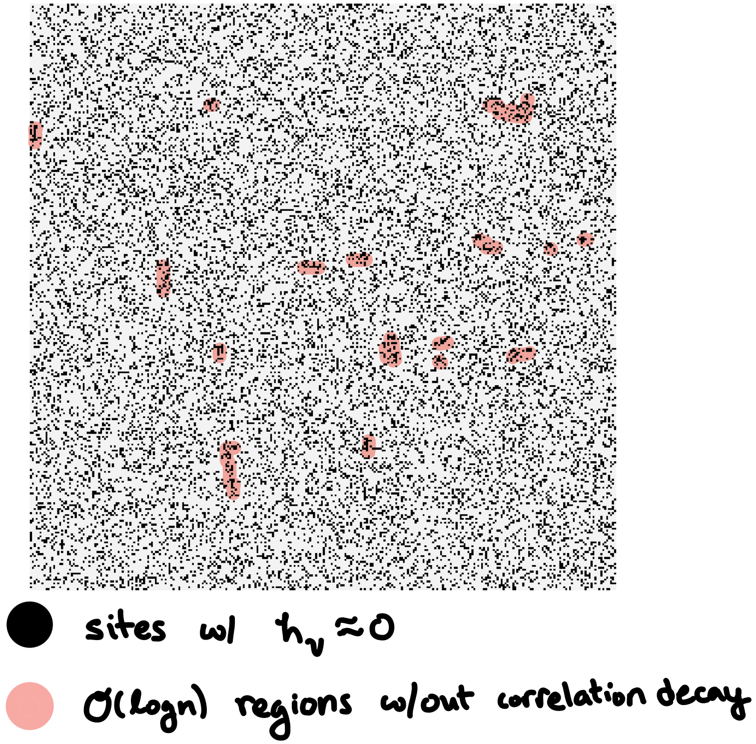


The Griffiths phase

[Griffiths '69]: Regime of $(\beta, \mathrm{Var}(h))$ where we have:

- exponential decay of correlations on average over $(h_v)_v$;
- $O(\log n)$ size regions where $h_v \approx 0$; get low temperature behavior.



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- $O(\log n)$ size regions where $h_v \approx 0$; get low temperature behavior.

This slows down dynamics somewhat, destroys many standard approaches to bounding Glauber dynamics mixing time.

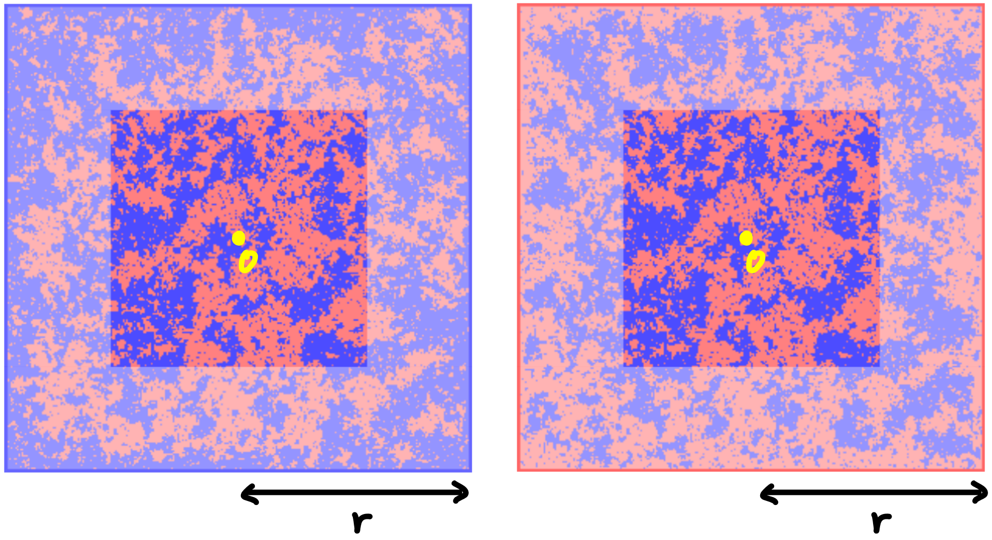
Recently: [Helmuth et al '21] showed on general graphs at sufficiently large $\mathrm{Var}(h)$, there exists poly-time (approximate) sampling algorithm

Weak spatial mixing (WSM) in expectation

Definition

RFIM has weak spatial mixing (WSM) in expectation if

$$\mathbb{E}_h[\|\pi^{+}_{B_r}(\sigma_o \in \cdot) - \pi^{-}_{B_r}(\sigma_o \in \cdot)\|_{\mathrm{TV}}] \le Ce^{-r/C}\,.$$



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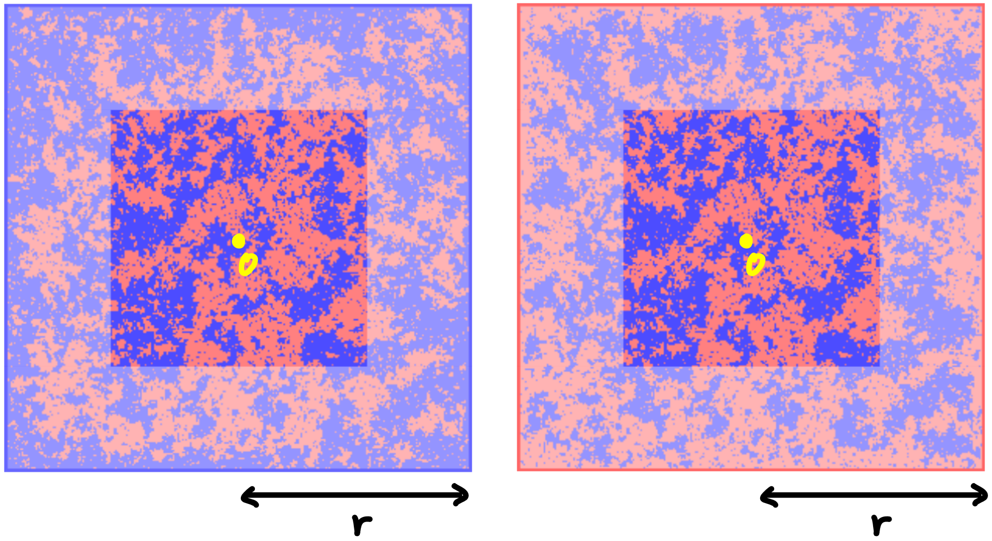

E.g., $\mathbb{Z}^2$ for all $\beta > 0$ and $\mathrm{Var}(h) > 0$ [Ding, Xia '19]

$d \geq 3$: Expect to hold throughout non-critical uniqueness regime

WSM in expectation and weak Poincaré inequality

Theorem (El Alaoui, Eldan, G., Piana '23)

If WSM in expectation holds, whp $(h_v)_v$ is such that RFIM dynamics has a weak Poincaré inequality with constant n^C, i.e., $\exists p, q : \frac{1}{p} + \frac{1}{q} = 1$:

$$Var_\pi(f) \le n^C \|f\|_\infty^{1/q} \cdot \mathcal{E}(f,f)^{1/p}$$

Test functions converge algebraically on polynomial timescales:

$$\|P^t f - \pi[f]\|_{2,\pi}^2 \le \|f\|_\infty^2 n^{C_1} \cdot t^{-C_2} \,.$$

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Weak PI's date e.g., to critical interacting particle systems [Liggett '91]

Consequences under WSM in expectation

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Implications:

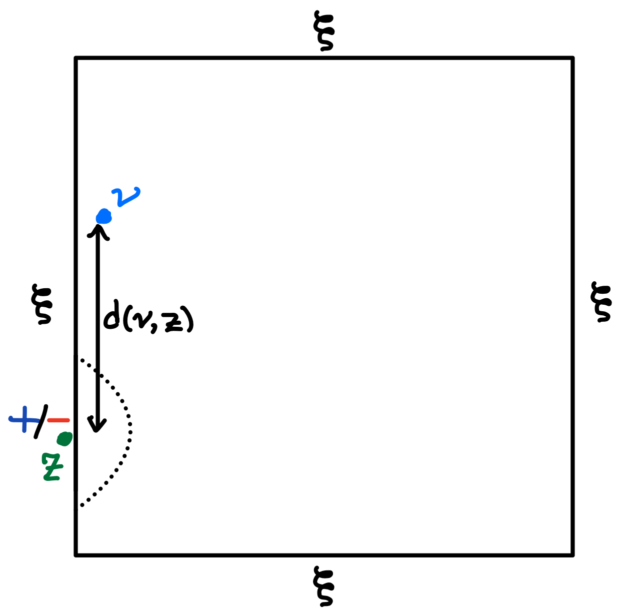
1. Poly(n) mixing from warm starts following [Lovasz Siminovits '93];
2. Markov chain based sampling algorithm
   [Repeatedly run Glauber on domains adding one vertex]

Strong spatial mixing (SSM) in expectation

Definition

RFIM has strong spatial mixing (SSM) in expectation if for all $v \in \Lambda$,

$$\mathbb{E}_h\Big[\max_{\xi \in \{\pm 1\}^{\partial\Lambda \setminus \{z\}}} \|\pi_\Lambda^{\xi,+}(\sigma_v \in \cdot) - \pi_\Lambda^{\xi,-}(\sigma_v \in \cdot)\|_{\mathrm{TV}}\Big] \le Ce^{-d(v,z)/C}\,.$$



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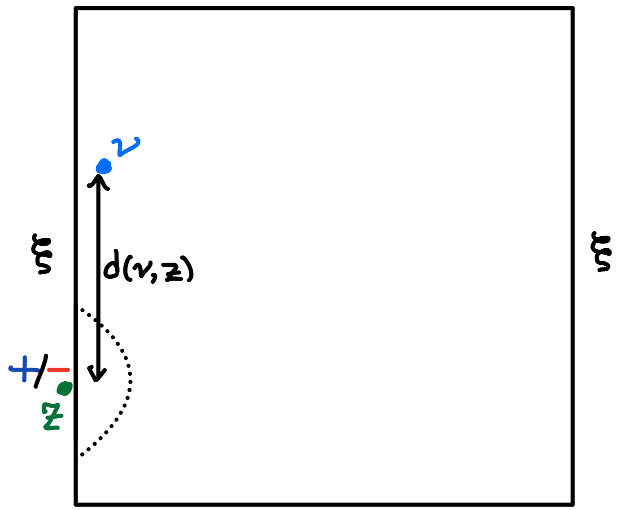

(1) holds for $\beta < \beta_c$ and arbitrary h [Ding Song Sun '22]
(2) holds for arbitrary β for Var(h) large enough ("not hard")

SSM in expectation and full Poincare inequality

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If SSM in expectation holds, whp $(h_v)_v$ is such that the RFIM Glauber dynamics has Poincaré inequality with constant $n^{o(1)}$, i.e.,

$$Var_\pi(f) \leq n^{o(1)} \mathcal{E}(f, f)$$

In particular, test functions converge exponentially on $n^{o(1)}$ timescales:

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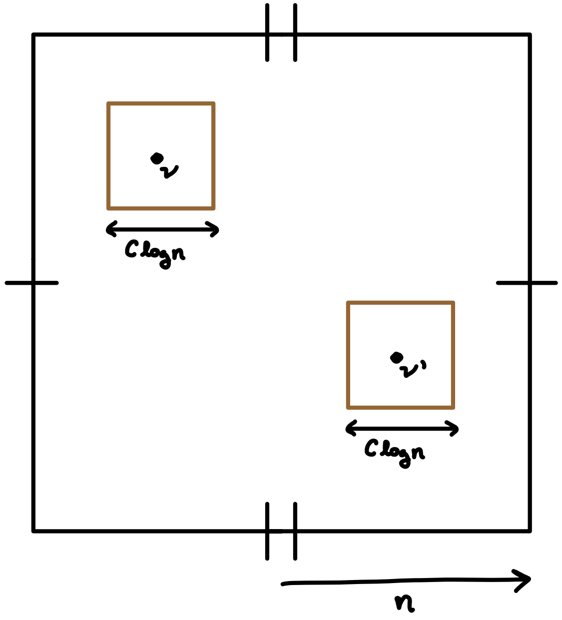
Note: the $n^{o(1)}$ is actually $\exp((\log n)^{\frac{d-1}{d}})$

Corresponding lower bound of $\exp((\log n)^{\frac{d-1}{4d}})$ when $\beta > \beta_c$.

A "standard" argument giving polynomial in 2D

1. Tile $(\mathbb{Z}/n\mathbb{Z})^d$ by boxes $B_R(v)$ for $R = C \log n$
2. Whp, $(h_v)_v$ s.t. for all v, $\partial B_R(v)$-to-v influence less than N^{-10}



A “standard" proof of polynomial in 2D

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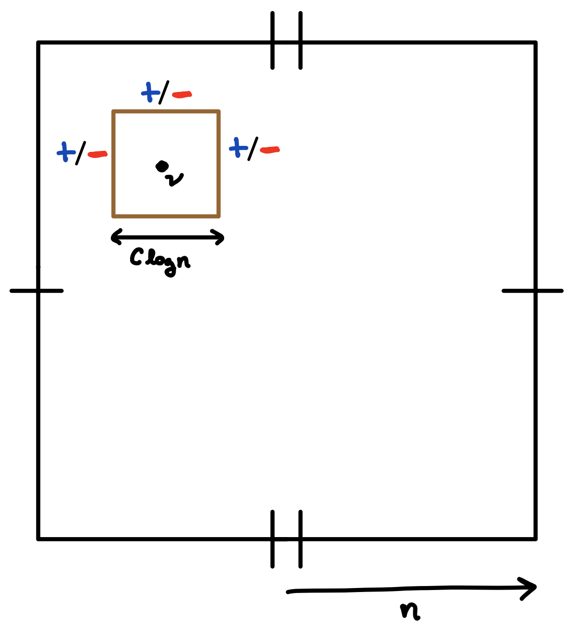

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4. Deduce that if $T \gg \max_v t_{mix}(B_R^{\pm}(v))$,

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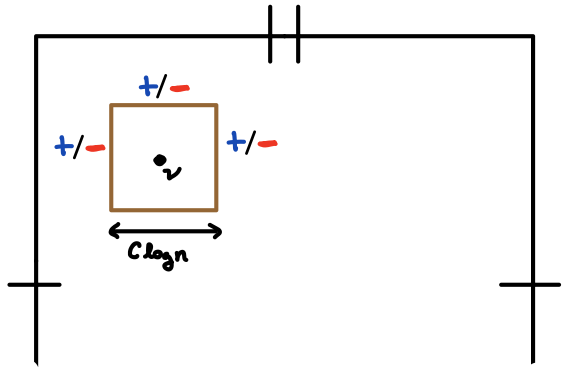
from which monotonicity and a union bound imply mixing.



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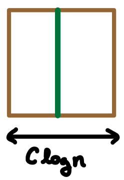
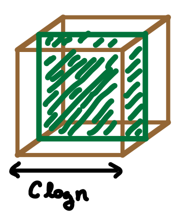


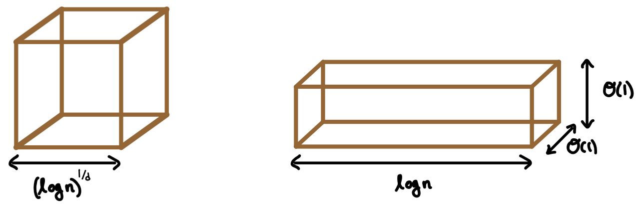


5. Mixing time of a $B_{\log n}$ is $\exp((\log n)^{d-1})$ (exponential in cut-width)
   – Polynomial for $d = 2$; super-polynomial for $d \geq 3$

Going beyond 2D

Question: Why should I expect to be able to do better?

Largest low-field region has *volume* $\log n$, so cut-width $(\log n)^{\frac{d-1}{d}}$!



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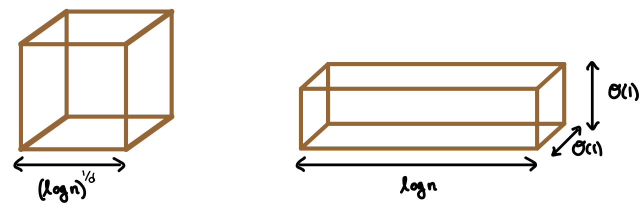


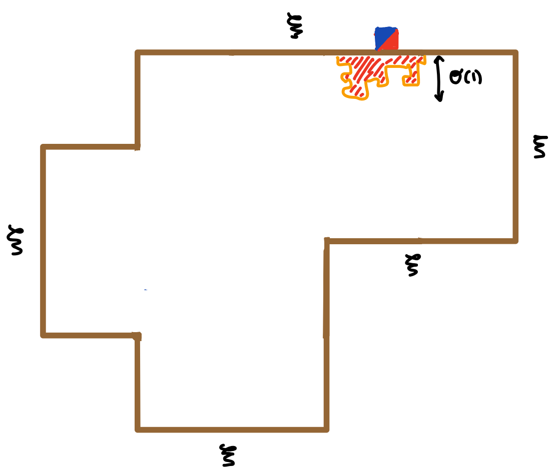
Proof structure: Combine two types of ideas:

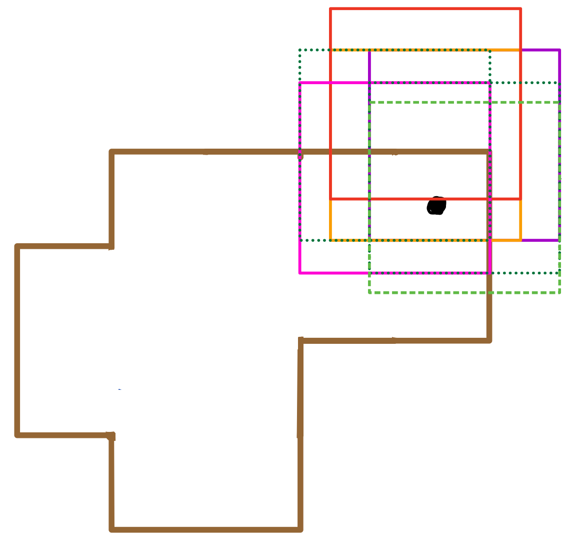
1. When field variance is large (e.g., under SSM in expectation):
   do a smarter coarsening, irregular $(h_v)_v$-dependent tiles
2. Use *stochastic localization* scheme to reduce RFIM with WSM in expectation to RFIM with large field.

Inverse gap with sufficiently large field

Goal: pick (h_v)-dependent blocks $(B_v)_v$ to reduce mixing time on Λ_n:

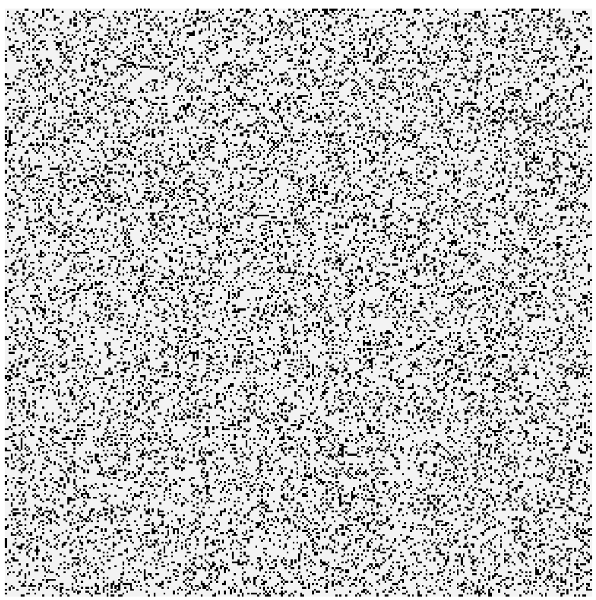
1. The mixing time on each block is at most n^C
2. Expected # of discrepancies in B_v from a ∂B_v-discrepancy is $O(1)$
3. # blocks containing v in interior $\gg$ # containing v on boundary



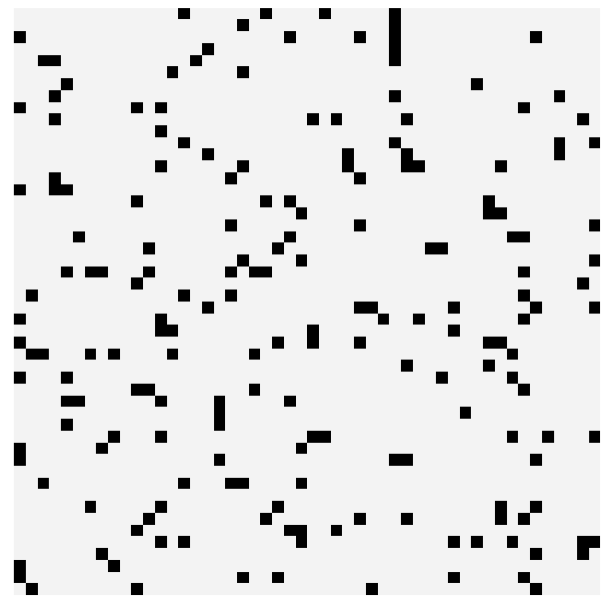


Coarse-graining the field

Def: Call a box $B_R(v)$ *good* if SSM holds with constant C on $B_R(v)$



● sites w/ $h_v \approx 0$



● bad boxes

Coarse-graining the field

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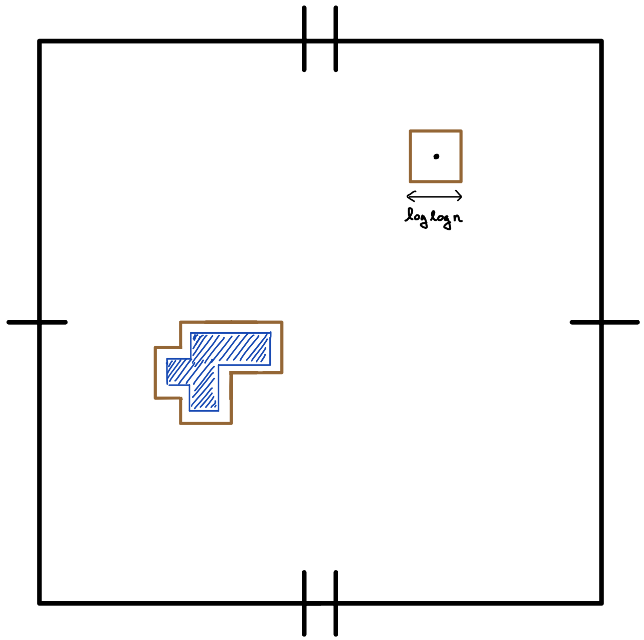
Take $R = \log\log n$ and take as *blocks*

- single *good* boxes;
- union of the bad boxes in a *connected bad-box component*



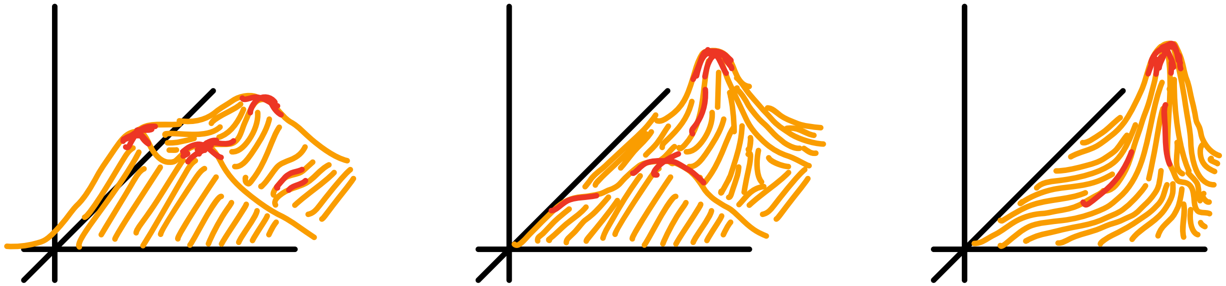
Stochastic localization to boost the field

What is stochastic localization?

$$\mu_t(\sigma) = e^{\langle y_t, \sigma \rangle} \mu_0(\sigma)$$

(i.e., a random linear tilt in the direction of y_t) where

$$y_t = t\sigma_* + B_t \qquad \sigma_* \sim \mu_0 \quad \text{indep.}$$



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Upshots:

- Dirichlet form is a supermartingale under SL
- Stays in family of RFIMs, but with a growing field variance as $t \uparrow$
- If you control variance decay along localization, then let's us reduce PI of μ_0 to that under μ_t for which previous argument applies.

Decay of variance along localization process

$$\frac{d}{dt}\mathbb{E}[\mathrm{Var}_{\mu_t}(\varphi)] \geq -\mathbb{E}[\mathrm{Var}_{\mu_t}(\varphi)\|\mathrm{Cov}(\mu_t)\|_{op}]$$

Under WSM in expectation, $\|\mathrm{Cov}(\mu_0)\|_{op}$ isn't badly behaved.

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1. A new use of FKG + Ito: Point-to-point influences are super-martingales under the stochastic localization process,

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2. But only have probabilistic bound on this, not deterministic:
–on bad event, covariance matrix can have order n^d operator norm.

Thank you!