Fast relaxation in the random field Ising model

Reza Gheissari
Northwestern University

Joint with A. El Alaoui (Cornell), R. Eldan (Microsoft Research), A. Piana (Weizmann)

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The underlying geometry

$n \times n$ box in $\mathbb{Z}^d$ with nearest-neighbor edges $v \sim w$

d = 2:

d = 3:
The Ising model

Ising model: probability of assignment $\sigma$ of $\{+, -\}$ to the vertices.

$$\pi_{\beta,n}(\sigma) \propto e^{-H(\sigma)} \quad \text{where} \quad H(\sigma) = \beta \sum_{v \sim w} 1_{\{\sigma_v \neq \sigma_w\}}$$

- $H$ is the Hamiltonian or energy of a configuration;
- $\beta$ is an inverse-temperature parameter.

$$H(\sigma) = 14\beta$$
Glauber dynamics for the Ising model

1. Assign every site a rate-1 Poisson clock.
2. If the clock at site $v$ rings at time $t$,
3. Resample $X_t(v)$ conditionally on its neighbors $(X_t(w))_{w \sim v}$.
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Equilibrium distribution is exactly $\pi$!
Mixing time:

\[ t_{mix} = \inf \{ t : \max_{X_0} \| \mathbb{P}_{X_0}(X_t \in \cdot) - \pi \|_{tv} < 1/4 \} . \]
Quantifying the rate of convergence

Mixing time:

\[ t_{mix} = \inf \{ t : \max_{X_0} \| P_{X_0}(X_t \in \cdot) - \pi \|_{tv} < 1/4 \} . \]

Poincaré inequality i.e., spectral gap: closely related to \( t_{mix} \)

\[ \text{Var}_\pi(f) \leq \lambda^{-1} \cdot \mathcal{E}(f, f) \]

 Governs exponential rate of convergence to \( \pi \):

\[ \| P^t f - \pi[f] \|_{2,\pi}^2 \leq \| f \|_{2,\pi}^2 e^{-\lambda t} . \]
Mixing time of the Ising model on $\mathbb{Z}^d$

Consider the mixing time $t_{\text{MIX}}(\square)$

$\beta < \beta_c(d)$

Optimal $O(\log n)$ mixing
"Weak/strong spatial mixing"
[Martinelli Oliveiri '94] [Cesi '99] [Lubetzky Sly '10, '13]

$\beta = \beta_c(d)$

d = 2: Poly $O(n^c)$ mixing
[Lubetzky Sly '12]
d = 3, 4: ???

$\beta > \beta_c(d)$

Slow mixing: $\exp(c_\beta n^{d-1})$

d = 2: [Chayes et al '87]
d ≥ 3: Pisztora '96, Bodineau '05

R. Gheissari Northwestern
The random-field Ising model

The random field \((h_v)_v\) are i.i.d. symmetric (e.g., \(\mathcal{N}(0, b^2)\))

Random field Ising model (RFIM): probability of \(\sigma \in \{+, -\}^V\),

\[
\pi(\sigma) \propto e^{-H(\sigma)} \quad \text{where} \quad H(\sigma) = \beta \sum_{v \sim w} 1_{\{\sigma_v \neq \sigma_w\}} - \sum_v h_v \sigma_v
\]
The RFIM: phase diagram at equilibrium

\( d = 2 \) Imry-Ma phenomenon: exponential decay of correlations (in expectation or for typical \((h_v)_v\)) at all temperatures in \( \mathbb{Z}^2 \).

[Chatterjee ’17, Aizenman Peled ’18, Aizenman Harel Peled ’19, Ding Xia ’19]
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\( d \geq 3 \) **Phase transition**: At \( \beta \) large, \( \text{Var}(h) \) small: long range order

[Imbrie ’85, Bricmont Kupiainen ’88, Ding Liu Xia ’19]
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The Griffiths phase

[Griffiths ’69]: Regime of $(\beta, \text{Var}(h))$ where we have:

- exponential decay of correlations on average over $(h_v)_v$;
- $O(\log n)$ size regions where $h_v \approx 0$; get low temperature behavior.
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This slows down dynamics somewhat, destroys many standard approaches to bounding Glauber dynamics mixing time.

Recently: [Helmuth et al ’21] showed on general graphs at sufficiently large $\text{Var}(h)$, there exists poly-time (approximate) sampling algorithm
Weak spatial mixing (WSM) in expectation

**Definition**

RFIM has weak spatial mixing (WSM) in expectation if

\[
\mathbb{E}_h[\|\pi^+_B(\sigma_o \in \cdot) - \pi^-_B(\sigma_o \in \cdot)\|_{TV}] \leq Ce^{-r/C}.
\]
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$$\mathbb{E}_h[\|\pi_{B_r}^+(\sigma_o \in \cdot) - \pi_{B_r}^-(\sigma_o \in \cdot)\|_{TV}] \leq Ce^{-r/C}.$$

E.g., $\mathbb{Z}^2$ for all $\beta > 0$ and $\text{Var}(h) > 0$ [Ding, Xia ’19]

$d \geq 3$: Expect to hold throughout non-critical uniqueness regime
Theorem (El Alaoui, Eldan, G., Piana ’23)

If WSM in expectation holds, whp \((h_v)_v\) is such that RFIM dynamics has a weak Poincaré inequality with constant \(n^C\), i.e., \(\exists p, q : \frac{1}{p} + \frac{1}{q} = 1\):

\[
\text{Var}_\pi(f) \leq n^C \|f\|^{1/q}_\infty \cdot \mathcal{E}(f, f)^{1/p}
\]

Test functions converge algebraically on polynomial timescales:

\[
\|P^t f - \pi[f]\|^{2}_{2, \pi} \leq \|f\|^{2}_\infty n^C_1 \cdot t^{-C_2}.
\]
WSM in expectation and weak Poincaré inequality

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Weak PI’s date e.g., to critical interacting particle systems [Liggett ’91]
Consequences under WSM in expectation

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$$\|P^t f - \pi[f]\|_{2,\pi}^2 \leq \|f\|_\infty^2 n^{C_1} \cdot t^{-C_2}.$$

Implications:

1. Poly($n$) mixing from warm starts following [Lovasz Siminovits ’93];
2. Markov chain based sampling algorithm
   [Repeatedly run Glauber on domains adding one vertex]
Strong spatial mixing (SSM) in expectation

**Definition**

RFIM has strong spatial mixing (SSM) in expectation if for all \( v \in \Lambda \),

\[
\mathbb{E}_h \left[ \max_{\xi \in \{\pm 1\}^{\partial \Lambda \setminus \{z\}}} \| \pi_\Lambda^{\xi, +} (\sigma_v \in \cdot) - \pi_\Lambda^{\xi, -} (\sigma_v \in \cdot) \|_{TV} \right] \leq Ce^{-d(v,z)/C}.
\]
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(1) holds for $\beta < \beta_c$ and arbitrary $h$ [Ding Song Sun ’22]
(2) holds for arbitrary $\beta$ for $\text{Var}(h)$ large enough (“not hard”)
SSM in expectation and full Poincare inequality

Theorem (El Alaoui, Eldan, G., Piana ’23)

If SSM in expectation holds, whp \((h_v)_v\) is such that the RFIM Glauber dynamics has Poincaré inequality with constant \(n^{o(1)}\), i.e.,

\[
\text{Var}_\pi(f) \leq n^{o(1)} \mathcal{E}(f, f)
\]

In particular, test functions converge exponentially on \(n^{o(1)}\) timescales:

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\|P^t f - \pi[f]\|_{2,\pi}^2 \leq \|f\|_{2,\pi}^2 \cdot e^{-t/n^{o(1)}}.
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**Note:** the \(n^{o(1)}\) is actually \(\exp((\log n)^{d-1\over d})\)

**Corresponding lower bound** of \(\exp((\log n)^{d-1\over 4d})\) when \(\beta > \beta_c\).
A "standard" argument giving polynomial in 2D

1. Tile \((\mathbb{Z}/n\mathbb{Z})^d\) by boxes \(B_R(v)\) for \(R = C \log n\)
2. Whp, \((h_v)_v\) s.t. for all \(v\), \(\partial B_R(v)\)-to-\(v\) influence less than \(N^{-10}\)
A “standard" proof of polynomial in 2D

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3. By monotonicity, sandwich marginal at \(v\) started from + and − by chains on \(B_R(v)\) with + and − boundary conditions
4. Deduce that if \(T \gg \max_v t_{mix}(B^{\pm}_R(v))\),

\[
||\mathbb{P}_+(X_t(v) \in \cdot) - \mathbb{P}_-(X_t(v) \in \cdot)||_{tv} \leq N^{-10}
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from which monotonicity and a union bound imply mixing.
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4. Deduce that if \(t \gg \max_v t_{\text{mix}}(B_R^\pm(v))\),

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from which monotonicity and a union bound imply mixing.

5. Mixing time of a \(B_{\log n}\) is \(\exp((\log n)^{d-1})\) (exponential in cut-width)

− Polynomial for \(d = 2\); super-polynomial for \(d \geq 3\)
**Question:** Why should I expect to be able to do better?

Largest low-field region has *volume* $\log n$, so cut-width $(\log n)^{\frac{d-1}{d}}$!
Going beyond 2D

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**Proof structure:** Combine two types of ideas:

1. When field variance is large (e.g., under SSM in expectation):
   - do a smarter coarsening, irregular $(h_v)_v$-dependent tiles
2. Use *stochastic localization* scheme to reduce RFIM with WSM in expectation to RFIM with large field.
Inverse gap with sufficiently large field

**Goal:** pick \((h_v)\)-dependent blocks \((B_v)_v\) to reduce mixing time on \(\Lambda_n\):

1. The mixing time on each block is at most \(n^C\)
2. Expected \# of discrepancies in \(B_v\) from a \(\partial B_v\)-discrepancy is \(O(1)\)
3. \# blocks containing \(v\) in interior \(\gg\) \# containing \(v\) on boundary
Coarse-graining the field

**Def:** Call a box $B_R(v)$ *good* if SSM holds with constant $C$ on $B_R(v)$.

![Diagram showing sites and bad boxes](image)
Coarse-graining the field

**Def:** Call a box $B_R(v)$ *good* if SSM holds with constant $C$ on $B_R(v)$

Take $R = \log \log n$ and take as *blocks*:
- single *good* boxes;
- union of the bad boxes in a *connected bad-box component*
Stochastic localization to boost the field

What is stochastic localization?

\[ \mu_t(\sigma) = e^{\langle y_t, \sigma \rangle} \mu_0(\sigma) \]

(i.e., a random linear tilt in the direction of \( y_t \)) where

\[ y_t = t\sigma_* + B_t \quad \sigma_* \sim \mu_0 \quad \text{indep.} \]
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Upshots:

- Dirichlet form is a supermartingale under SL
- Stays in family of RFIMs, but with a growing field variance as \( t \uparrow \)
- If you control variance decay along localization, then let’s us reduce PI of \( \mu_0 \) to that under \( \mu_t \) for which previous argument applies.
Decay of variance along localization process

\[
\frac{d}{dt} \mathbb{E}[\text{Var}_{\mu_t}(\varphi)] \geq -\mathbb{E}[\text{Var}_{\mu_t}(\varphi)\|\text{Cov}(\mu_t)\|_{op}]
\]

Under WSM in expectation, $\|\text{Cov}(\mu_0)\|_{op}$ isn’t badly behaved.
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Under WSM in expectation, \(\|\text{Cov}(\mu_0)\|_{op}\) isn’t badly behaved.

1. A new use of FKG + Ito: Point-to-point influences are super-martingales under the stochastic localization process,

\[
\mathbb{E}[\|\text{Cov}(\mu_t)\|_{op}] \text{ "controlled by" } \|\text{Cov}(\mu_0)\|_{op}
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\[ \mathbb{E}[\|\text{Cov}(\mu_t)\|_{op}] \text{ "controlled by" } \|\text{Cov}(\mu_0)\|_{op} \]

2. But only have probabilistic bound on this, not deterministic:
   – on bad event, covariance matrix can have order \( n^d \) operator norm.
Thank you!