Dimers on a Riemann surface and compactified free field

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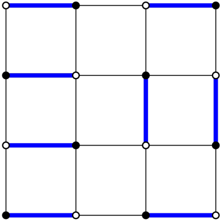
(University of Helsinki)

28.03.2024

Dimer model on a planar graph

- **Dimer model on a bipartite G:** perfect matching D, sampled with probability

$\mathbb{P}[D] \sim \mathbf{w}(D) = \prod_{bw \in D} \mathbf{w}(bw)$.



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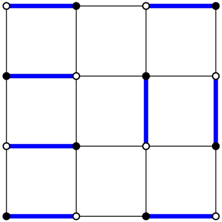
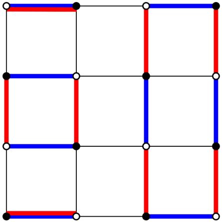


- **Height function (planar G):**

– choose a reference cover D_0



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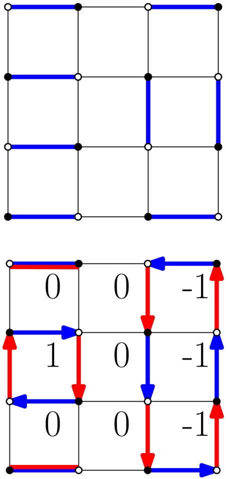

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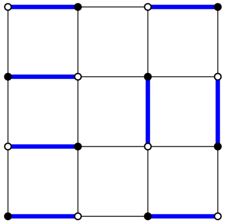

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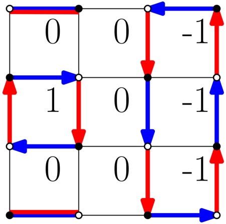

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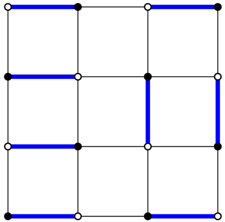

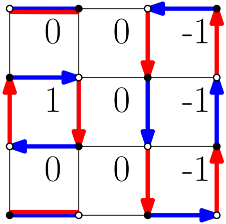

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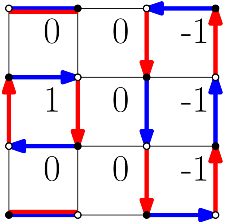

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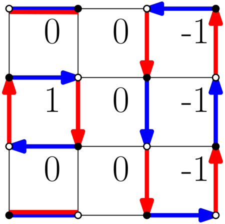

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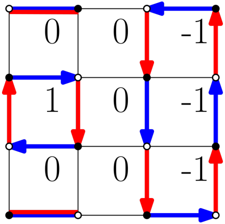

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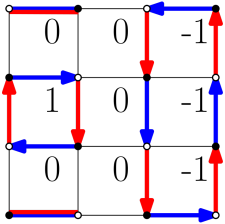

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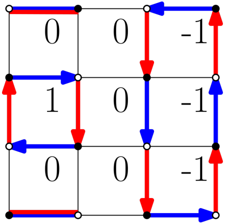

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- The case of Temperley graphs is well-understood:

[Kenyon'00], [Berestycki–Laslier–Ray'16]

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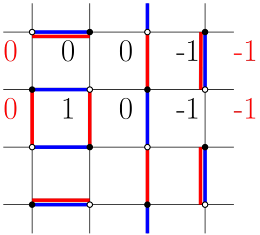

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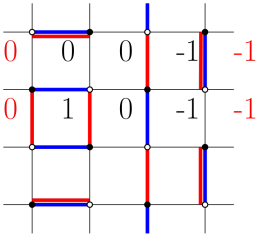

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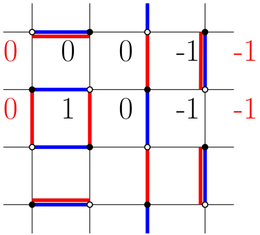

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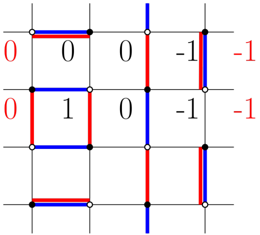

- NB! height changes between components of $\partial\Sigma$ are also random

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Φ — local fluctuations	Ψ — global behavior

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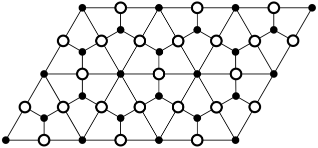
- Compactified free field $\mathfrak{m}^u = d\phi + \psi^u$ s.t. ϕ, ψ^u — independent

scalar component	instanton component
ϕ — GFF on Σ	ψ^u — random harmonic differential such that $[\psi^u - u] \in H^1(\Sigma, \mathbb{Z})$ a.s. and $\mathbb{P}[\psi^u = v] \sim \exp(-\frac{\pi}{2} \int_\Sigma v \wedge *v)$.

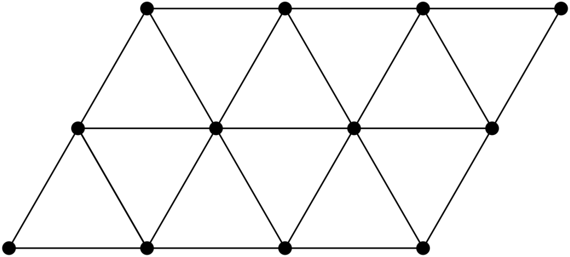
Universality result of Beresticky, Laslier and Ray

- Temperley graph is superposition G of primal graph Γ and dual $\Gamma^{\times}$.

Black vertices = vertices of $\Gamma \cup \Gamma^{\times}$,
White vertices = midpoints of edges



Primal graph Γ



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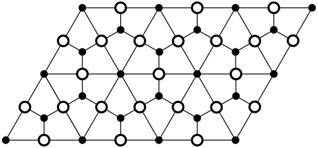
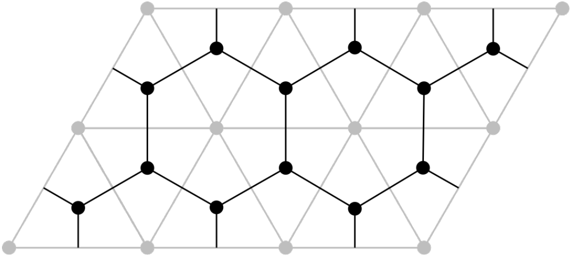


Dual graph $\Gamma^{\times}$



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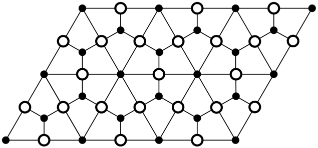

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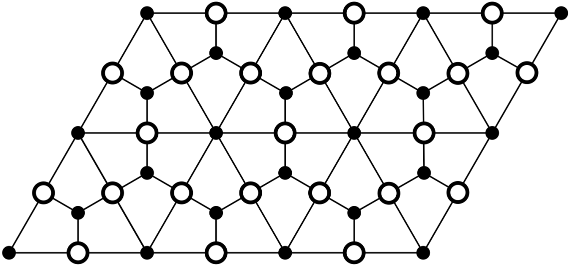
Bipartite graph $G = \Gamma \cup \Gamma^\times$



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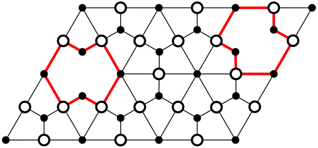
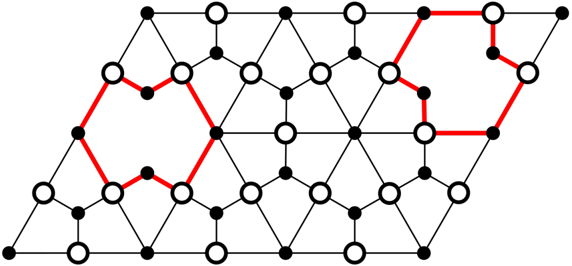


On a Riemann surface Σ: remove $-\chi(\Sigma)$ white vertices



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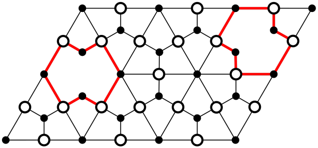

- Dimer covers of G $\leftrightsquigarrow$ cycle rooted spanning forests of Γ
- Height function increments $\leftrightsquigarrow$ winding of branches of trees

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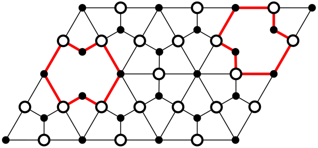

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– RW on Γ_δ converges to the Brownian motion on Σ and satisfies uniform crossing estimates up to any scale
– removed white vertices converge to $p_1, \ldots, p_{2g-2+n} \in \Sigma$

Then dh_δ has a limit which depends only on $\Sigma, p_1, \ldots, p_{2g-2+n}$.

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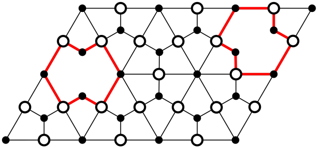

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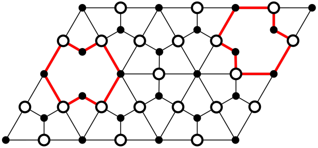

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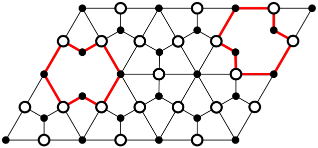

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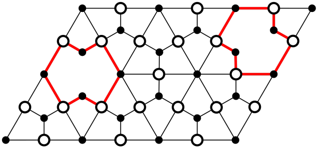

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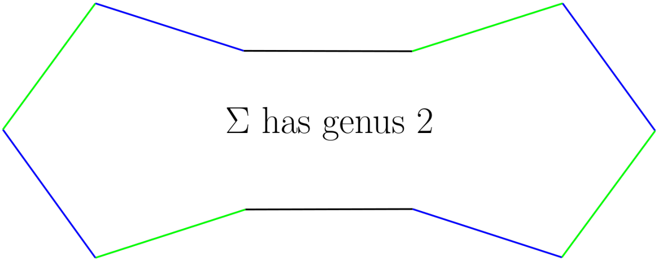
Problem: to identify the limit with the compactitied free field

Riemann surfaces with a flat metric with conical singularities

- We assume that $\partial\Sigma = \varnothing$ (otherwise we consider the double)



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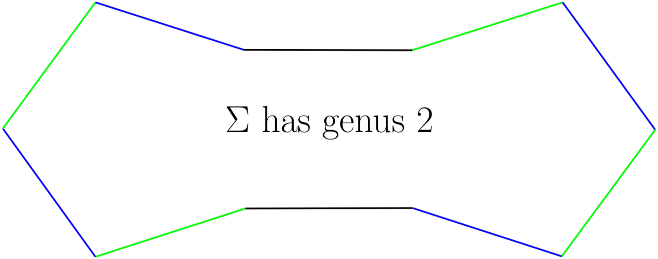
  ds^2 is flat on $\Sigma \smallsetminus \{p_1, \ldots, p_{2g-2}\}$



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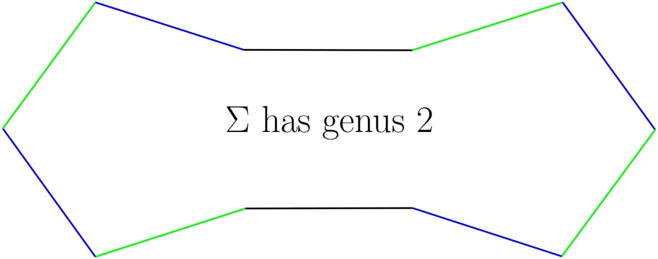
  ds^2 has conical singularities at p_i with cone angles 4π



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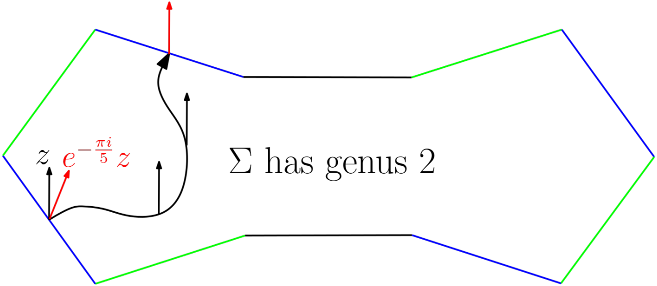
- We assume that $\partial\Sigma = \varnothing$ (otherwise we consider the double)
- ds^2 may have a holonomy: parallel transport of a vector along γ results in multiplication by $\exp(-2\pi i \int_\gamma u)$, u — harmonic 1-form
- We call u a holonomy 1-form.



winding of the curve is $\frac{\pi}{5}$

Nice graphs on (Σ, ds^2)

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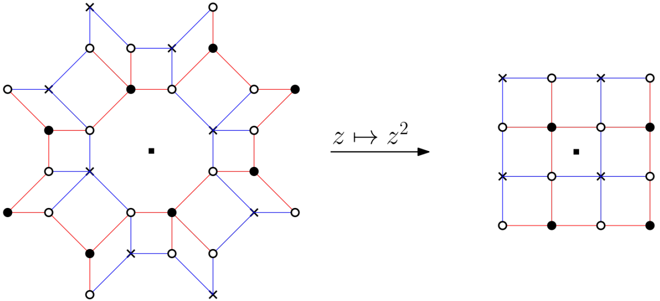
- We approximate (Σ, ds^2) with graphs G_δ on it s.t.



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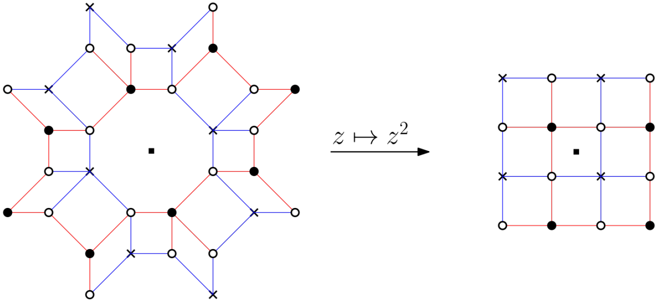
  (i) G_δ is 'nice' t-embedding in the bulk of $\Sigma \setminus \{p_1, \ldots, p_{2g-2}\}$



Nice graphs on (Σ, ds^2)

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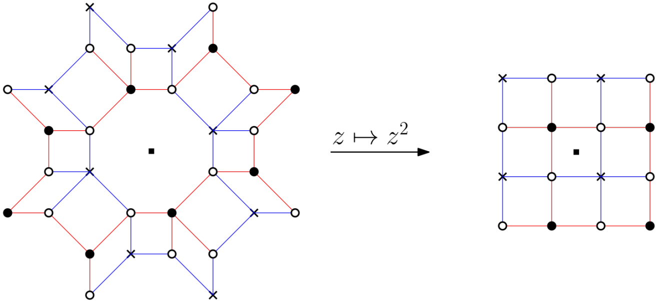
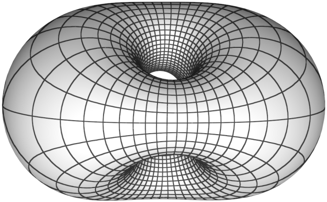


Example: pillow surface

- $f : \Sigma \to \mathbb{C}/\mathbb{Z}^2$ — branched cover ramified over the origin
- G is the preimage of the square lattice $\frac{1+i}{4N} + \frac{1}{2N}\mathbb{Z}^2$
- Any Σ can be approximated by pillow surfaces (but can't control positions of conical singularities)



Convergence result

Theorem(B). Let $\Sigma, p_1, \ldots, p_{-\chi(\Sigma)}$ be given, G_δ approximate Σ.

– If $\{dh_\delta - \mathbb{E}dh_\delta\}_{\delta>0}$ is tight, then any subsequential limit of $dh_\delta - \mathbb{E}dh_\delta$ is the compactified free field $\mathfrak{m}^u - \mathbb{E}\mathfrak{m}^u$ where u is some harmonic 1-form.

– If G_δ are Temperley and removed white vertices converge to $p_1, \ldots, p_{-\chi(\Sigma)}$, then u is the holonomy 1-form.

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To link with the result of Beresticky, Laslier and Ray: need to generalize the latter to metrics with conical singularities (work in progress)

Proof ideas

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Resembles Quillen's variation identity for $\det_\zeta(\bar\partial + \alpha)^*(\bar\partial + \alpha)$!

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Express $(\bar\partial + \alpha)^{-1}$ via theta functions to link Ψ_δ and ψ^u.

Relation between combinatorial setups

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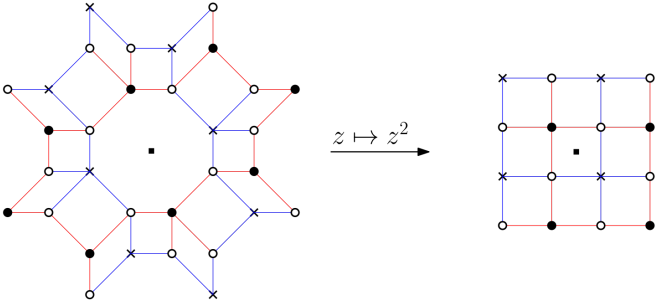

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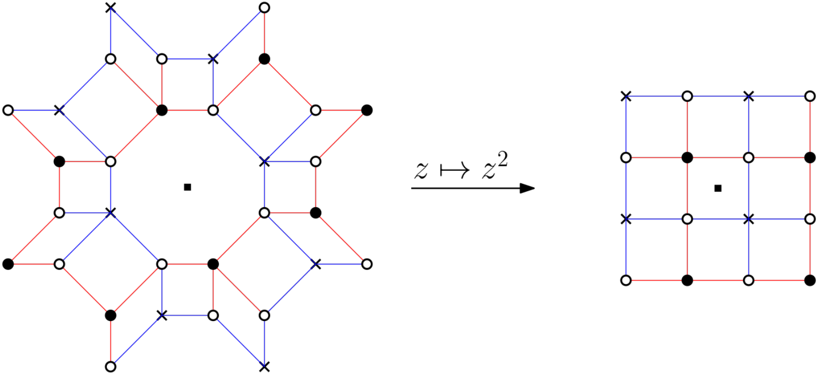

On the left the graph is still isoradial,

On the right the graph is Temperley: superposition of red primal and blue dual

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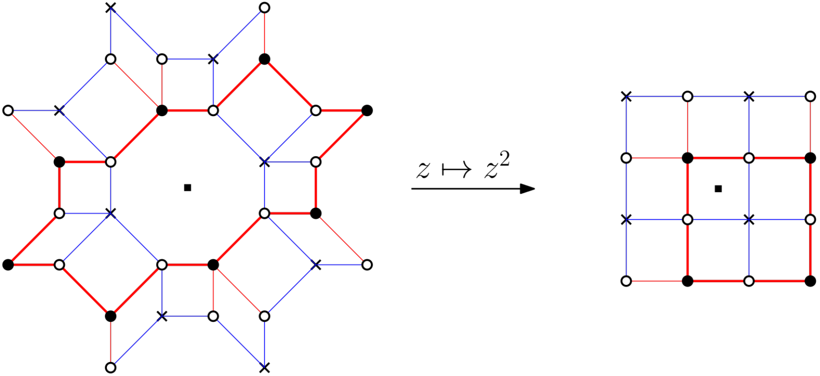

On the left the graph is still isoradial, but not Temperley anymore

On the right the graph is Temperley: superposition of red primal and blue dual

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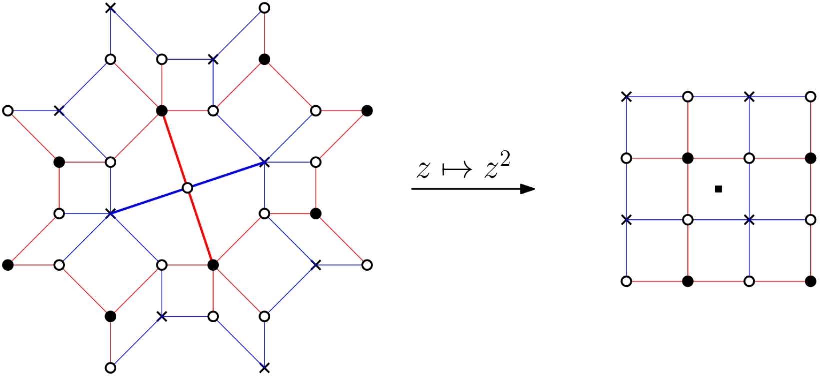

It is Temperley with one white vertex removed!

On the right the graph is Temperley: superposition of red primal and blue dual

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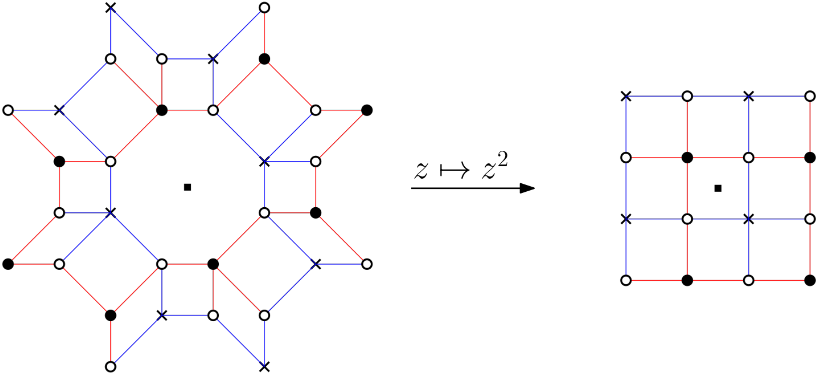
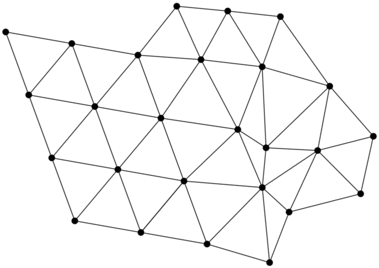


Conclusion: Temperley graph with removed white vertices is embedded naturally into a surface with conical singularities.

How to approximate a general $\Sigma, p_1, \ldots, p_{2g-2+n}$?

- $\Gamma \subset \Sigma$ is δ-separated $C\delta$-net, triangular lattice pattern near p_i's
- $\Gamma^\times$ — the associated Voronoi diagram, G — the Temperley graph
- The dual graph G^* is t-embedded — discrete complex analysis tools available (Chelkak–Laslier–Russkikh)



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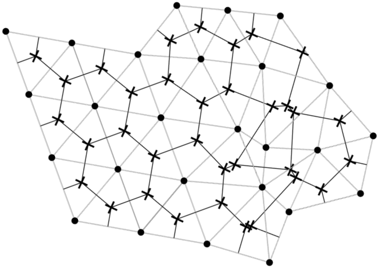

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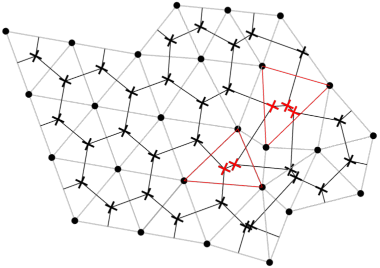

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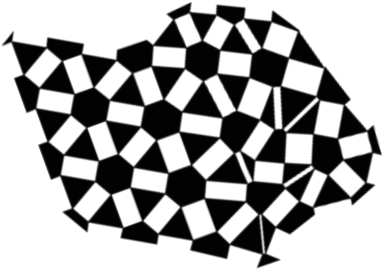

Thank you for your attention!